A Different Method of Solving a Problem of IMO

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Abstract: The IMO performs once a year, and has become an important activity in the field of mathematics. Because the problems in IMO are very difficult, and in general needs two days to finish the test of only six problems, therefore, it is significant to study how to solve and solve those IMO problems with various methods. With respect to question (a) of the problem of discussing, at first, using the so-called "exhaustive method" and the mathematical induction, the paper gets the conclusion of that if n is the integral multiple of 3, subtracting 1 from the nth power of 2 must be divisible by 7. Furthermore, it also proves by use of the disprove method that if n is not the integral multiple of 3, subtracting 1 from the nth power of 2 is impossible to be divisible by 7. The way of solving question (b) is similar to that of solving (a), in order to use the result of question (a) for the third step of the mathematical induction, the paper firstly consider the third power of that 1 added to (k+1)th power of 2 and applying the disprove method proves that it and hence that 1 added to the (k+1)th power of 2 are not divisible by 7, namely the question (b) is true.

Keywords: IMO, The Mathematical Induction, Algebraic Equation, Disprove Method

1. Introduction

The international mathematical Olympiads (IMO) has performed 57 sessions from 1959 to 2016 years, it attracts the interesting of many people who know the elementary mathematics, especially those pupils in high schools, and it is an important topic in the fields of mathematics [1-12]. Later, The IMO orderly and continuously performed once every year, the 60th IMO performed in the recent of 2019, the Chinese team won the championship. In general, the problems of IMO are very difficult to solve, some famous mathematical theorems and conjectures occur in the problems, for example, the famous Fermat theorem. Only very few top pupils in elementary mathematics can attend the contest. Therefore, studying the technique and different methods of solving the problems of IMO is a very valuable work. Solving mathematical problems with different methods has become one of the important contents of a lot of periodicals in the mathematics.

Nowadays, all of the problems of IMO can be found from the computer internetwork and some books about IMO. The problem of present studying selected from the book on IMO from 1959 to 2009 published by the famous Springer publisher [13]. The 6th IMO performed in previous Soviet Union, the first problem of the contest was supported by the Czechoslovakia, including (a), (b) two questions. It is a problem of the number theory about the divisibility [14]. Up till now, the solution can be only given by use of the remainder theorem [15]. However, pupils in high schools even some students in universities who don't specialize in the mathematics did not know the number theory, and not easy to understand the solution. The present work intends to offer a different method of solving this problem merely by use of the mathematical induction and the theory of elementary algebraic equation.

2. Solution

The first problem of the 6th IMO is given by the following two questions [13]:

a) Find all positive integers n for which $2^n - 1$ is divisible by 7.

b) Prove that there is no positive integer n such that $2^n + 1$ is divisible by 7.

Solution (a): If $2^n - 1$ is divisible by 7, supposing p is a positive integer, thus

$$2^n = 7p + 1 \qquad (1)$$

In terms of "the exhaustive method", when n=1, 2, p<1; when n=3, p=1, when n=4, or 5, p can not be an integer; when n=6, p=9, when n=7 or 8, p is not an integer; when n=9, p=73, when n=10, or 11, p is not an integer; when n=12, p=585,……. It can be found that when n is the integral multiple of 3, $2^n - 1$ is divisible by 7. Thus, whether all of n=3k (k=1, 2, 3……) can make $2^n - 1$ be divisible by 7? According to the mathematical induction,

When k=1, $2^3 - 1 = 7$, divisible by 7,

Supposing when k=m (m is any positive integer), it is also divisible by 7, thus when k=m+1,

$$2^{3k} - 1 = 2^{3(m+1)} - 1 = 2^3 \times 2^{3m} - 1 = 7 \times 2^{3m} + (2^{3m} - 1) \qquad (2)$$

In eq. (2), in terms of the supposition, either of 7×2^{3m} and $(2^{3m} - 1)$ is divisible by 7, therefore, when n is the integral multiple of 3, any of $2^n - 1$ is divisible by 7.

However, there are infinite big numbers which have not been verified by eq. (1), it is necessary to discuss whether there is a big prime that can not be verified by "the exhaustive method" to make $2^n - 1$ be divisible by 7? Supposing N is an arbitrary number of impossibly verifying by "the exhaustive method", it may be a composite number or a prime, but it is not the integral multiple of 3, considering

$$(2^N - 1)^3 = 2^{3N} - 3 \times 2^{2N} \times 1 + 3 \times 2^N \times 1^2 - 1 = (2^{3N} - 1) - 3 \times (2^N - 1)^2 - 3 \times (2^N - 1) \qquad (3)$$

in accord with the prior conclusion, $(2^{3N} - 1)$ in eq. (3) is divisible by 7, using the disprove method for the other part with the assumption of that it is also divisible by 7, namely,

$$-3 \times (2^N - 1)^2 - 3 \times (2^N - 1) = 7u \qquad (4)$$

where u is a positive integer, hence

$$3 \times (2^N - 1)^2 + 3 \times (2^N - 1) + 7u = 0 \qquad (5)$$

regarding $(2^N - 1)$ as a variable to solve the algebraic equation, the significant solution of eq. (5) requires

$$3^2 - 4 \times 3 \times 7u \geq 0 \qquad (6)$$

it results in

$$u \leq \frac{3}{28} < 1 \qquad (7)$$

this directly contradicts the assumption, therefore, either of $(2^N - 1)^3$ and $(2^N - 1)$ is not divisible by 7.

In conclusion, n=3k (k=1, 2,……) are all the positive integers to make $2^n - 1$ divisible by 7.

Solution (b):

According to the mathematical induction, when n=1, $2^n + 1 = 3$, not divisible by 7,

Supposing when n=k, $2^k + 1$ is also not divisible by 7, then when n=k+1, considering

$$(2^{k+1} + 1)^3 = 2^{3(k+1)} + 3 \times 2^{2(k+1)} + 3 \times 2^{(k+1)} + 1 = 8 \times 2^{3k} + 12 \times 2^{2k} + 6 \times 2^k + 1 = [7 \times 2^{3k} + (2^{3k} - 1) + 14 \times 2^{2k} + 7 \times 2^k] - 2 \times (2^k + 1)^2 + 3 \times (2^k + 1) + 1 \qquad (8)$$

Obviously, the part
$[7 \times 2^{3k} + (2^{3k} - 1) + 14 \times 2^{2k} + 7 \times 2^k]$
in eq. (8) is divisible by 7, using the disprove method to discuss the other part of eq. (8), assuming the other part is divisible by 7, thus

$$-2 \times (2^k + 1)^2 + 3 \times (2^k + 1) + 1 = 7q \qquad (9)$$

where q is a positive integer, namely,

$$2 \times (2^k + 1)^2 - 3 \times (2^k + 1) + (7q - 1) = 0 \qquad (10)$$

regarding $2^k + 1$ in eq. (10) as a variable to solve the equation, the requirement of existing the significant solution is

$$(-3)^2 - 4 \times 2 \times (7q - 1) \geq 0 \qquad (11)$$

it results in

$$q \leq \frac{17}{56} < 1 \qquad (12)$$

Eq. (12) evidently contradicts the assumption of that q is a positive integer, so the other part is not divisible by 7. Therefore, (b) is true.

3. Conclusion

In consideration of the first problem of the 6th IMO, without using the number theory which overpass the level of the mathematics of high school as ref.[15], this paper solved (a), (b) merely used the mathematical induction and the theory of algebraic equation. With respect to (a), using "the exhaustive method" to eq. (1), but it only verified finite positive integers, in order to get all the positive integers to make $2^n - 1$ divisible by 7, it requires to prove if n is a big number, especially a big prime which can not be verified by "the exhaustive method" but not the integral multiple of 3, $2^n - 1$ is not divisible by 7. The process of present solution is easy to understand for more readers. Moreover, the paper changes a problem of the number theory into using the simple

mathematical induction and the theory of algebraic equation to deal with, this is very important for researching the mathematical theory and the technique of solving mathematical problems

References

[1] A. C. McBride. The 2001 International Mathematical Olympiad [J]. Scottish Mathematical Council Journal, 2002, 31: 13-15.

[2] Cecil Rousseau and Gregg Patruno. The International Mathematical Olympiad Training Session [J]. College Mathematics Journal, 1985, 16 (5): 362-365. Online: 30n Jan. 2018.

[3] Titu Andreescu and Zuming Feng. News and Letters [J]. Mathematics Magazine, 2003, 76 (3): 242-247.

[4] M. Shifman. You failed your math test, Comrade Einstein: Adventures and misadventures of young mathematicians or test your skills in almost recreational mathematics [M]. USA: World Scientific, 2005, pp 232.

[5] Steve Olson. Beautiful Solutions at the International Mathematical Olympiad [J]. The Mathematics Teacher, 2006, 99 (7): 527-528.

[6] LLko Brnetic. Iequalities at the International Mathematical Olympiad [J]. Osjecki Matematicki List, 2008, 8: 1; 15-18.

[7] Kyong Mi Choi. Influences of Formal Schooling on International Mathematical Olympiad Winners from Korea [J]. Roeper Review, 2013, 35 (3): 187-196.

[8] Stan Dolan. 55th International Mathematical Olympiad, Cape Town, 3-13 [J]. The Mathematical Gazette, 2014, 98 (543): 546.

[9] Alison Higgs and Mary Twomey. Editorial [J]. Ethics and Social Wolfare, 2015, 9 (3): 223-224.

[10] C. P. Shao, H. B. Li and L. Huang. Chanllenging Theorem Proves with Mathematical Olympiad Problems in Solid Geometry [J]. Mathematics in Computer Science, 2016, 10 (1): 75-96.

[11] S. Agievich, A. Gorodiliva, V. Idrisova, N. Kolomeec, G. Shushuev and N. Tokreva. Mathematical Problems of the Second International Students' Olympiad in Cryptography [J]. Cryptologia, 2017, 41 (6): 534-565.

[12] A. Gorodilova, S. Agievich, C. Carlet et al. Problems and Solutions from the Fourth International Students' Olympiad in Cryptography [J]. Cryptologia, 2019, 43 (2): 138-174.

[13] Dusan Djukic, Vladimir Jankovic, Ivan Matic, and Nikola Petrovic. The IMO Compendium-A Collection of Problems Suggested for The International Olympiads: 1959-2009 (Second Edition) [M]. New York: Springer, 2011: pp 33.

[14] ROSEN, KENNETH H. Elementary Number Theory and Its Applications [M]. Massachusetts: ADDISON- WESLEY PUBLISHING COMPANY, 1984: PP 18-24.

[15] Dusan Djukic, Vladimir Jankovic, Ivan Matic, and Nikola Petrovic. The IMO Compendium-A Collection of Problems Suggested for The International Olympiads: 1959-2004 (First Edition) [M]. New York: Springer, 2006: pp 333-370.