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A Coupling Method of Homotopy Perturbation and Aboodh Transform for Solving Nonlinear Fractional Heat - Like Equations

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Abstract: In this paper, we present the solution of nonlinear fractional Heat - Like equations by using Aboodh transform homotopy perturbation method (ATHPM). The proposed method was derived by combining Aboodh transform and homotopy perturbation method. This method is seen as a better alternative method to some existing techniques for such realistic problems. The results showed the efficiency and accuracy of the combined Aboodh transform and homotopy perturbation method.

Keywords: Homotopy Decomposition Method, Nonlinear Fractional Heat - Like Equation, Aboodh Transform

1. Introduction

Nonlinear fractional partial differential equations (FPDEs) [1-3] are generalizations of classical differential equations of integer order. A great number of crucial phenomena in physics, chemistry, biology, biomedical sciences, signal processing, systems identification, control theory, viscoelastic materials and polymers are well described by fractional ordinary differential equations and nonlinear FPDEs. In the resent years many researchers mainly had paid attention to studying the solution of nonlinear fractional partial differential equations by using various methods. Among these are the Variational Iteration Method (VIM) [27-28], Adomian Decomposition Method (ADM) [16-17], projected differential transform method [25], and the Differential Transform Method (ADM) [26], are the most popular ones that are used to solve differential and integral equations of integer and fractional order. The Homotopy Perturbation Method (HPM) [4-6] is a universal approach which can be used to solve both fractional ordinary differential equations FODEs as well as fractional partial differential equations FPDEs. This method was originally proposed by He [7, 8]. The HPM is a coupling of homotopy and the perturbation method. Recently, Khalid Aboodh, has introduced a new integral transform, named the Aboodh transform [18-24], and it has further applied to the solution of ordinary and partial differential equations. In this article, we use Aboodh transform and homotopy perturbation method together to solve Nonlinear Fractional Heat -Like Equations.

2. Fundamental Facts of the Fractional Calculus

In this section, some definitions and properties of the fractional calculus that will be used in this work are presented.

Definition 1:

The Gamma function is intrinsically tied in fractional calculus. The simplest interpretation of the gamma function is simply the generalization of the fraction for all real numbers. The definition of the gamma function is given by:

$$\Gamma(\mu) = \int_0^\infty e^{-t}\, t^{\mu-1} dt, \mu > 0 \qquad (1)$$

Definition 2:

A real function $f(x), x > 0$, is said to be in the space C_μ,

$\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(x) = x^p h(x)$, where $h(x) \in [0, \infty)$ and it is said to be in space C_μ^m if $f^{(m)} \in C_m, m \in \mathbb{N}$.

Definition 3:

The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_m, \mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-1)^{\alpha-1} f(t)dt, \alpha > 0, x > 0 \quad (2)$$

$$J^\alpha f(x) = f(x)$$

Some Properties of the operator:

For $f \in C_m, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x), J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$$

$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$$

Lemma 1:

If $m-1 < \alpha \leq m, m \in \mathbb{N}$ and $f \in C_m$, $\mu \geq -1$ then $D^\alpha J^\alpha f(x) = f(x)$ and,

$$J^\alpha D_0^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{x^k}{k!}, x > 0 \quad (3)$$

Definition 3: (Partial Derivatives of Fractional order)

Assume now that $f(x)$ is a function of n variables x_i, $i = 1, \ldots, n$ also of class C on $D \in \mathbb{R}_n$. As an extension of definition 2 we define partial derivative of order α for $f(x)$ respect to x_i

$$a\partial_{\underline{x}}^\alpha f = \frac{1}{\Gamma(m-\alpha)} \int_0^{x_i} (x_i - 1)^{m-\alpha-1} \partial_{x_i}^\alpha f(x_j)\Big|_{x_j=t} dt \quad (4)$$

If it exists, where $\partial_{x_i}^\alpha$ is the usual partial derivative of integer order m.

3. Fundamental Facts of the Aboodh Transformation Method

A new transform called the Aboodh transform defined for function of exponential order we consider functions in the set **A,** defined by:

$$\mathrm{A} = \{\mathrm{f(t)}: \exists\, \mathrm{M}, \mathrm{k}_1, \mathrm{k}_2 > 0, |\mathrm{f(t)}| < M\mathrm{e}^{-\mathrm{vt}} \quad (5)$$

For a given function in the set M must be finite number, k_1, k_2 may be finite or infinite. Aboodh transform which is defined by the integral equation

$$A[f(t)] = K(v) = \frac{1}{v} \int_0^\infty f(t) e^{-vt} dt, t \geq 0, \mathrm{k}_1 \leq v \leq \mathrm{k}_2 \quad (6)$$

The following results can be obtained from the definition and simple calculations

1) $A[t^n] = \frac{n!}{v^{n+2}}$
2) $A[f'(t)] = vK(v) - \frac{f(0)}{v}$
3) $A[f''(t)] = v^2 K(v) - \frac{f'(0)}{v} - f(0)$.
4) $A[f^{(n)}(t)] = v^n K(v) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{2-n+k}}$.

Theorem 1:

If $K(v)$ is Aboodh transform of $f(t)$, we Knows that Aboodh transform of derivative with integral order is given as follows:

$$A[f'(t)] = vK(v) - \frac{f(0)}{v}$$

Proof:

Let us take the Aboodh transform $f'(t) = \frac{d}{dt} f(t)$, use integration by parts as follows:

$$A\left[\frac{d}{dt} f(t)\right] = \frac{1}{v} \int_0^\infty \frac{d}{dt} f(t) e^{-vt} dt = \lim_{p\to\infty} \frac{1}{v} \int_0^p \frac{d}{dt} f(t) e^{-vt} dt$$

$$= \lim_{p\to\infty} \left\{ \left[\frac{1}{v} f(t) e^{-vt}\right]_0^p + \frac{1}{v} \int_0^p f(t) e^{-vt} dt \right\}$$

$$= vK(v) - \frac{f(0)}{v} \quad (7)$$

Equation (7) gives us the proof of Theorem 1. When we continue in the same manner, we get the Aboodh transform of the second order derivative as follows

$$A\left[\frac{d^2}{dt^2} f(t)\right] = A\left[\frac{d}{dt}\left(\frac{d}{dt} f(t)\right)\right] = vA\left[\frac{d}{dt} f(t)\right] - \left.\frac{\frac{d}{dt} f(t)}{v}\right|_{t=0}$$

$$= vA\left[vK(v) - \frac{f(0)}{v}\right] - \left.\frac{\frac{d}{dt} f(t)}{v}\right|_{t=0}$$

$$= v^2 K(v) - \frac{f'(0)}{v} - f(0)$$

If we go on the same way, we get the Aboodh transform of the nth order derivative as follows:

$$A[f^{(n)}(t)] = v^n K(v) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{2-n+k}} \text{ for } n \geq 1 \quad (8)$$

or

$$A[f^{(n)}(t)] = v^n \left[K(v) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{2+k}}\right] \quad (9)$$

Theorem 2:

If $K(v)$ is Aboodh transform of $f(t)$, one can take into consideration the Aboodh transform of the Riemann-Liouville derivative as follow:

$$A[D^\alpha f(t)] = v^\alpha \left[K(v) - \sum_{k=1}^n \frac{D^{\alpha-k} f(0)}{v^{\alpha-k+2}}\right]; -1 < n-1 \leq \alpha < n \quad (10)$$

Proof:

$$A[D^\alpha f(t)] = v^\alpha K(v) - \sum_{k=0}^{n-1} v^k [D^{\alpha-k-1} f(0)]$$

$$= v^{\alpha}K(v) - \sum_{k=0}^{n} v^{k-1}[D^{\alpha-k}f(0)] = v^{\alpha}K(v) - \sum_{k=1}^{n} v^{k-2}[D^{\alpha-k}f(0)]$$

$$= v^{\alpha}K(v) - \frac{1}{v^{-k+2}}\sum_{k=1}^{n}[D^{\alpha-k}f(0)] = v^{\alpha}K(v) - \sum_{k=0}^{n}\frac{1}{v^{\alpha-k+2-\alpha}}[D^{\alpha-k}f(0)]$$

$$= v^{\alpha}K(v) - \sum_{k=1}^{n} v^{\alpha}\frac{1}{v^{\alpha-k+2}}[D^{\alpha-k}f(0)]$$

Therefore, we get the Aboodh transformation of fractional order of $f(t)$ as follows:

$$A[D^{\alpha}f(t)] = v^{\alpha}\left[K(v) - \sum_{k=1}^{n}\left(\frac{1}{v}\right)^{\alpha-k+2}[D^{\alpha-k}f(0)]\right] \quad (11)$$

Definition 4:

The Aboodh transform of the Caputo fractional derivative by using Theorem 2 is defined as follows:

$$A[D_t^{\alpha}f(t)] = v^{\alpha}A[f(t)] - \sum_{k=0}^{m-1} v^{k-\alpha-2}f^{(k)}(0), m-1<\alpha<m \quad (12)$$

4. Basic Idea of Aboodh Transform Homotopy Perturbation Method (ATHPM)

To illustrate the basic idea of this method, we consider a general form of nonlinear non homogeneous partial differential equation as the follow:

$$D_t^{\alpha}u(x,t) = L(u(x,t)) + N(u(x,t)) + f(x,t), \alpha > 0 \quad (13)$$

with the following initial conditions

$$D_0^k u(x,0) = g_k, k = 0,\dots,n-1, D_0^n u(x,0) = 0 \text{ and } n = [\alpha]$$

Where D_t^{α} denotes without loss of generality the Caputo fraction derivative operator, f is a known function, N is the general nonlinear fractional differential operator and L represents a linear fractional differential operator.

Taking Aboodh transform on both sides of equation (13), to get:

$$A[D_t^{\alpha}u(x,t)] = A[L(u(x,t))] + A[N(u(x,t))] + A[f(x,t)] \quad (14)$$

Using the differentiation property of Aboodh transform and above initial conditions, we have:

$$A[u(x,t)] = v^{-\alpha}A[L(u(x,t))] + v^{-\alpha}A[N(u(x,t))] + g(x,t) \quad (15)$$

Operating with the Aboodh inverse on both sides of equation (15) gives:

$$u(x,t) = G(x,t) + A^{-1}\left[v^{-\alpha}A[L(u(x,t))] + v^{-\alpha}A[N(u(x,t))]\right] \quad (16)$$

Where $G(x,t)$ represents the term arising from the known function $f(x,t)$ and the initial condition.

Now, we apply the homotopy perturbation method

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t). \quad (17)$$

And the nonlinear term can be decomposed as:

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \quad (18)$$

Where $H_n(u)$ are He's polynomial and given by:

$$H_n(u_0,u_1,u_2\dots u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}[N(\sum_{i=0}^{\infty} p^i u_i(x,t))]_{p=0}, n = 0,1,2,\dots \quad (19)$$

Substituting equations. (18) and (17) in equation (16) we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) + p\left[A^{-1}\left[v^{-\alpha}A[L(\sum_{n=0}^{\infty} p^n u_n(x,t))] + v^{-\alpha}A[N(\sum_{n=0}^{\infty} p^n u_n(x,t))]\right]\right] \quad (20)$$

Which is the coupling of the Aboodh transform and the homotopy perturbation method using He's polynomials. Comparing the coefficient of like powers of p, the following approximations are obtained:

$$p^0 : u_0(x,t) = G(x,t),$$

$$p^1 : u_1(x,t) = A^{-1}[v^{-\alpha}A[L(u_0(x,t)) + H_0(u)]],$$

$$p^2 : u_2(x,t) = A^{-1}[v^{-\alpha}A[L(u_1(x,t)) + H_1(u)]],$$

$$p^3 : u_3(x,t) = A^{-1}[v^{-\alpha}A[L(u_2(x,t)) + H_2(u)]],$$

$$p^n : u_n(x,t) = A^{-1}[v^{-\alpha}A[L(u_{n-1}(x,t)) + H_{n-1}(u)]], \quad (21)$$

Then the solution is;

$$u(x,t) = \lim_{p\to 1} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \quad (22)$$

The above series solution generally converges very rapidly.

5. Applications

Example 5.1:

Let consider the following one dimensional fractional heat- like equation:

$$D_t^{\alpha}u(x,t) = \frac{1}{2}x^2 u_{xx}(x,t), 0<x<1, 0<\alpha\le 1, t>0 \quad (23)$$

with initial condition

$$u(x,0) = x^2 \quad (24)$$

Applying the Aboodh transform of both sides of Eq. (23),

$$A[D_t^{\alpha}u(x,t)] = A\left[\frac{1}{2}x^2 u_{xx}(x,t)\right] \quad (25)$$

Using the differential property of Aboodh transform Eq. (25) can be written as:

$$v^{\alpha}\left(A[u(x,t)] - v^{-2}u(x,0)\right) = A\left[\frac{1}{2}x^2 u_{xx}(x,t)\right] \quad (26)$$

Using initial condition (24), Eq. (26) can be written as:

$$A[u(x,t)] = v^{-2}x^2 + v^{-\alpha}A\left[\frac{1}{2}x^2 u_{xx}(x,t)\right] \quad (27)$$

The inverse Aboodh transform implies that:

$$\mathrm{u(x,t)} = \mathrm{x}^2 + \mathrm{A}^{-1}\left[\mathrm{v}^{-\alpha}\mathrm{A}\left[\frac{1}{2}\mathrm{x}^2\mathrm{u_{xx}(x,t)}\right]\right] \quad (28)$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x^2 + pE^{-1}\left[v^{-\alpha}E\left[\frac{1}{2}x^2\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xx}\right]\right] \quad (29)$$

Comparing the coefficient of like powers of p, the following approximations are obtained;

$$p^0 : u_0(x,t) = x^2$$

$$p^1 : u_1(x,t) = A^{-1}\left[v^{-\alpha}A\left[\frac{1}{2}x^2 u_0(x,t)_{xx}\right]\right] = A^{-1}\left[v^{-\alpha}A[x^2]\right] = A^{-1}\left[x^2 v^{-\alpha+2}\right] = \frac{x^2 t^{\alpha}}{\alpha!} = \frac{x^2 t^{\alpha}}{\Gamma(\alpha+1)},$$

$$p^2 : u_2(x,t) = A^{-1}\left[v^{-\alpha}A\left[\frac{1}{2}x^2 u_1(x,t)_{xx}\right]\right] = A^{-1}\left[v^{-\alpha}A\left[\frac{x^2 t^{\alpha}}{\Gamma(\alpha+1)}\right]\right] = \frac{x^2 t^{2\alpha}}{\Gamma(2\alpha+1)},$$

Proceeding in a similar manner, we have:

$$p^3 : u_3(x,t) = A^{-1}\left[v^{-\alpha}A\left[\frac{1}{2}x^2 u_2(x,t)_{xx}\right]\right] = \frac{x^2 t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$p^n : u_n(x,t) = A^{-1}\left[v^{-\alpha}A\left[\frac{1}{2}x^2 u_n(x,t)_{xx}\right]\right] = \frac{x^2 t^{n\alpha}}{\Gamma(n\alpha+1)},$$

Therefore the series solution $u(x,t)$ is given by:

$$u(x,t) = x^2\left(1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \cdots\right) \quad (30)$$

This equivalent to the exact solution in closed form:

$$u(x,t) = x^2 E_{\alpha}(t^{\alpha}) \quad (31)$$

where $E_{\alpha}(t^{\alpha})$ is the Mittag-Leffler function.

Example 5.2:

Consider the following tow - dimensional fractional heat like equation:

$$D_t^{\alpha}u = u_{xx} + u_{yy}, 0 < x, y < 2\pi, 0 < \alpha \le 2, t > 0 \quad (32)$$

With the initial conditions

$$u(x,y,0) = \sin x \sin y \quad (33)$$

Applying the Aboodh transform of both sides of Eq. (32),

$$A[D_t^{\alpha}u(x,y,t)] = A\left[u_{xx} + u_{yy}\right] \quad (34)$$

Using the differential property of Aboodh transform Eq. (34) can be written as:

$$v^{\alpha}\left(A[u(x,y,t)] - v^{-2}u(x,y,0)\right) = A\left[u(x,y,t)_{xx} + u(x,y,t)_{yy}\right] \quad (35)$$

Using initial condition (33), Eq. (35) can be written as:

$$A[u(x,y,t)] = v^{-2}\sin x \sin y + v^{-\alpha}A\left[u(x,y,t)_{xx} + u(x,y,t)_{yy}\right] \quad (36)$$

The inverse Aboodh transform implies that:

$$u(x,y,t) = \sin x \sin y + A^{-1}\left[v^{-\alpha}A\left[u(x,y,t)_{xx} + u(x,y,t)_{yy}\right]\right] \quad (37)$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,y,t) = \sin x \sin y + pA^{-1}\left[v^{-\alpha}A\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,y,t)\right)_{xx} + \left(\sum_{n=0}^{\infty} p^n u_n(x,y,t)\right)_{yy}\right]\right] \quad (38)$$

Comparing the coefficient of like powers of p, the following approximations are obtained;

$$p^0 : u_0(x,y,t) = \sin x \sin y$$

$$p^1 : u_1(x,y,t) = A^{-1}\left[v^{-\alpha}A\left[u_{0_{xx}} + u_{0_{yy}}\right]\right] = \frac{-2\sin x \sin y\ t^{\alpha}}{\Gamma(\alpha+1)}$$

$$p^2 : u_2(x,y,t) = A^{-1}\left[v^{-\alpha}A\left[u_{1_{xx}} + u_{1_{yy}}\right]\right] = \frac{4\sin x \sin y\ t^{2\alpha}}{\Gamma(2\alpha+1)}$$

Proceeding in a similar manner, we have:

$$p^3 : u_3(x,y,t) = \frac{-8\sin x \sin y\ t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$p^n : u_n(x,y,t) = \frac{(-2)^n \sin x \sin y\ t^{n\alpha}}{\Gamma(n\alpha+1)},$$

Therefore the series solution $u(x,t)$ is given by:

$$u(x,y,t) = \sin x \sin y\left(1 - \frac{(2t^{\alpha})}{\Gamma(\alpha+1)} + \frac{(2t^{\alpha})^2}{\Gamma(2\alpha+1)} - \frac{(2t^{\alpha})^3}{\Gamma(3\alpha+1)} + \cdots + \frac{(-2t^{\alpha})^n}{\Gamma(n\alpha+1)} + \cdots\right) \quad (39)$$

For the special case when $\alpha = 1$, we can get the solution in a closed form

$$u(x,y,t) = e^{-2t}\sin x \sin y \quad (40)$$

Example 5.3:

Consider the following three dimensional fractional heat-like equation:

$$D_t^{\alpha}u(x,y,z,t) = x^4y^4z^4 + \frac{1}{36}\left(x^2u_{xx} + y^2u_{yy} + z^2u_{zz}\right), 0 < x;\ y, z < 1, 0 < \alpha \le 1 \quad (41)$$

With the initial condition;

$$u(x,y,z,t) = 0 \quad (42)$$

Applying the Aboodh transform of both sides of Eq. (41),

$$A[D_t^{\alpha}u(x,y,t)] = A[x^4y^4z^4] + A\left[\frac{1}{36}\left(x^2u_{xx} + y^2u_{yy} + z^2u_{zz}\right)\right] \quad (43)$$

Using the differential property of Aboodh transform Eq.(43), and using initial condition (42), Eq. (43) can be written as:

$$A[u(x,y,z,t)] = v^{-2}x^4y^4z^4 + v^{-\alpha}A\left[\frac{1}{36}\left(u(x,y,z,t)_{xx} + \right.\right.$$

$$u(x,y,z,t)_{yy} + u(x,y,z,t)_{xx}\Big)\Big] \quad (44)$$

The inverse Aboodh transform implies that:

$$u(x,y,z,t) = x^4y^4z^4 + A^{-1}\left[v^{-\alpha}A\left[\frac{1}{36}\big(u(x,y,z,t)_{xx} + u(x,y,z,t)_{yy} + u(x,y,z,t)_{xx}\big)\right]\right] \quad (45)$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,y,z,t) = x^4y^4z^4 + pA^{-1}\left[v^{-\alpha}A\left[(\sum_{n=0}^{\infty} p^n u_n(x,y,z,t))_{xx} + (\sum_{n=0}^{\infty} p^n u_n(x,y,z,t))_{yy} + (\sum_{n=0}^{\infty} p^n u_n(x,y,z,t))_{yy}\right]\right] \quad (46)$$

Comparing the coefficient of like powers of p, the following approximations are obtained;

$$p^0 : u_0(x,y,z,t) = x^4y^4z^4$$

$$p^1 : u_1(x,y,z\ t) = A^{-1}\left[v^{-\alpha}A\left[\frac{1}{36}\Big(x^2 u_{0_{xx}} + y^2 u_{0_{yy}} + z^2 u_{0_{zz}}\Big)\right]\right] = \frac{x^4y^4z^4t^\alpha}{\Gamma(\alpha+1)},$$

$$p^2 : u_2(x,y,z\ t) = A^{-1}\left[v^{-\alpha}A\left[\frac{1}{36}\Big(x^2 u_{1_{xx}} + y^2 u_{1_{yy}} + z^2 u_{1_{zz}}\Big)\right]\right] = \frac{x^4y^4z^4t^{2\alpha}}{\Gamma(2\alpha+1)}$$

Proceeding in a similar manner, we have:

$$p^3 : u_3(x,y,z,t) = \frac{x^4y^4z^4t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$p^n : u_n(x,y,z,t) = \frac{x^4y^4z^4t^{n\alpha}}{\Gamma(n\alpha+1)},$$

Therefore the series solution $u(x,t)$ is given by:

$$u(x,t) = x^4y^4z^4\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \cdots\right) \quad (47)$$

Therefore the approximate solution of equation for the first N is given below as:

$$u_n(x,y,z,t) = \sum_{n=1}^{N} \frac{(x^4y^4z^4)t^{n\alpha}}{\Gamma(n\alpha+1)} \quad (48)$$

Now when $N \to \infty$ we obtained the follow solution

$$u_n(x,y,z,t) = \sum_{n=0}^{\infty} \frac{(x^4y^4z^4)t^{n\alpha}}{\Gamma(n\alpha+1)} - (x^4y^4z^4) = (x^4y^4z^4)[E_\alpha(t^\alpha) - 1] \quad (49)$$

Where $E_\alpha(t^\alpha)$ is the generalized Mittag-Leffler function. Note that in the case $\alpha = 1$

$$u(x,y,z,t) = (xyz)^4[e^t - 1] \quad (50)$$

This is the exact solution for this case.

6. Conclusion

The main concern of this paper was to combine Aboodh transform and homotopy perturbation method (ATHPM). This method has been successfully employed to obtain an analytical solution for Nonlinear Fractional Heat -Like Equations. The results showed the efficiency and accuracy of the combined Aboodh transform and homotopy perturbation method.

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