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A New Discrete Family of Reduced Modified Weibull Distribution

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Abstract: Discretization of continuous lifetime distribution is an interesting and intuitively appealing approach to derive a discrete lifetime model. This study derived a discretized form of Reduced Modified Weibull distribution known as the Marshall-Olkin Discrete Reduced Modified Weibull (MDRMW) distribution. The mathematical and statistical properties of MDRMW distribution were derived and compared with existing distributions of Discrete Reduced Modified Weibull distribution (DRMW), Exponentiated Discrete Weibull distribution (EDW) and Two Parameters Discrete Lindley distribution (TDL). Maximum likelihood method was used to derive the statistics of MDRMW parameters. The Aarset Reliability dataset was fitted for the existing and derived distribution and AIC and Kolmogorov Smirrnoff (KS) were compared. The shape of MDRMW distribution was unimodal and monotonic decreasing. The plot of hazard rate function could be decreasing or bath-tub. The AIC and KS values of Aarset reliability data analysis were 483.9 and 0.17579; 507.8 and 0.24435; 485.2 and 0.17897 for MDRMW, DRMW and TDL respectively. The AIC and KS values of Leukemia survival data analysis were 668.2 and 0.11053; 751.9 and 0.39285 respectively. The Aarset reliability data analysis showed that MDRMW compared favorably with existing distributions. The MDRMW and DRMW handled Leukemia survival data set as against EDW and TDL. The values of AIC and KS for MDRMW were lower than DRMW, EDW and TDL. This showed that MDRMW was better than the existing distributions.

Keywords: Weibull, TDL, DRMW, EDW, MDRMW

1. Introduction

The Weibull distribution is a lifetime distribution, this make it to be important and desirable. Weibull distribution can be used in different fields with many applications. Its survival and hazard rate functions have simple expression and its flexibility make it useful to fit different lifetime data sets in different fields.

The cumulative distribution function (CDF) of the two-parameter Weibull distribution is given by the

$$F(x) = 1 - \exp\left(-\alpha x^{\theta}\right), \qquad x > 0, \tag{1}$$

Where $\alpha > 0$, and $\theta > 0$ are the scale and shape parameters respectively.

The probability density function (PDF) is given by

$$f(x) = \alpha\theta x^{\theta-1}\exp\left(-\alpha x^{\theta}\right), \qquad x > 0. \tag{2}$$

And the hazard rate function is

$$h(x) = \alpha\theta x^{\theta-1}, \quad x > 0, \tag{3}$$

This can either increase; decrease or constant depending on $\theta > 1, \theta < 1$ or $\theta = 1$. For many years, using different techniques, many researchers have developed various modified forms of the Weibull distribution to achieve non-monotonic shapes. Bebbington *et al* (2007) proposed that the hazard rate function of the two-parameter flexible Weibull extension can be increasing, decreasing or bathtub shaped. Also Mudholkar and Srivastave (1993) proposed a two-parameter model, called the exponentiated Weibull distribution.

The authors are interested in using the Marshal-Olkin discretized family to discretize the modified Weibull distribution (MW) due to the wide ability to discretize any continuous distribution unlike the general method being used. There is no limit at which the method can discretize continuous distributions in as much as the continuous distribution has a survival function. Also, a shape parameter from the Marshal-Olkin family is an added advantage in the analysis of data.

The proposed distribution is referred to as Marshall-Olkin Discrete Reduced Modified Weibull distribution (MDRMW). The statistical properties of the proposed distribution were derived. The proposed distribution is compared with existing competing distributions and applied to medical and reliability. The new distribution has a desired characteristic, it has a closed form and it can be highly monotonically decreasing in shape, its hazard function can be bathtub and it is useful in modeling medical and reliability data.

Marshall-Olkin Discrete Family

A methodology to add a parameter to obtain a new family of distributions was introduced by Marshall and Olkin (1997). The family of Marshall – Olkin distributions with survival function given as

$$S(x;\beta) = \frac{\beta\bar{F}(x)}{1-\bar{\beta}\bar{F}(x)}, \quad (4)$$

where $\beta > 0, \bar{\beta} = 1-\beta$ and $\bar{F}(x)$ is a survival function of the continuous distribution.

Here, the authors consider the class of continuous scale family of distributions with scale parameter $\sigma > 0$ and denote probability density function and the cumulative distribution function of the same by $f(.|\sigma)$ and $F(.|\sigma)$ respectively. The survival function of the new scale family of continuous distributions on the positive real line using equation (4) is given by

$$S(x;\beta,\sigma) = \frac{\beta\bar{F}(x|\sigma)}{1-\bar{\beta}\bar{F}(x|\sigma)}, \quad (5)$$

where $\bar{F}(x|\sigma) = 1 - F(x|\sigma)$.

The survival function (5) can be considered as a generalization of the scale family of distributions and the corresponding family of distributions.

Let X be a discrete random variable associated to a continuous random variable belonging to RMW.

The probability mass function is given by

$$P(X = x;\beta) = p_x$$

$$p_x = S(x;\beta) - S(x+1;\beta)$$

$$p_x = \frac{\beta[\bar{F}(x)-\bar{F}(x+1)]}{\left(1-\bar{\beta}\bar{F}(x)\right)\left(1-\bar{\beta}\bar{F}(x+1)\right)}, x = 0,1,2,\ldots \quad (6)$$

where $\beta > 0, \bar{\beta} = 1-\beta$

2. Methodology

A two-parameter discrete distribution will be introduced. It will be a new discrete distribution allowing for bathtub shaped hazard rate functions. Its mathematical properties will be discussed, and its applications to real data sets will be studied. The proposed distribution will be shown to outperform the existing models including the ones allowing for bathtub shaped hazard rate functions.

The proposed discrete distribution is based on a two-parameter modification of the Reduced Modified Weibull (RMW) distribution proposed by Almaki, (2014). The two-parameter distribution would be shown to be flexible, have a bathtub shaped hazard rate function. The authors refer to it as the Marshall-Olkin Discrete Reduced Modified Weibull distribution and denoted by MDRMW.

3. Proposed Distribution: Marshall-Olkin Discrete Reduced Modified Weibull Distribution (MDRMW)

The cumulative distribution function and survival function of RMW distribution are respectively as follows

$$F(x) = 1 - e^{-\alpha x^{1/2} - \delta x^{1/2} e^{\lambda x}}, \quad (7)$$

and

$$\bar{F}(x) = e^{-\alpha x^{1/2} - \delta x^{1/2} e^{\lambda x}}, \quad (8)$$

respectively.

$$p_x = \frac{\beta\left(q^{x^{1/2}(1+bc^x)} - q^{(x+1)^{1/2}(1+bc^{x+1})}\right)}{\left(1-\bar{\beta}q^{x^{1/2}(1+bc^x)}\right)\left(1-\bar{\beta}q^{(x+1)^{1/2}(1+bc^{x+1})}\right)} \quad (9)$$

Equation (9) is referred to as Marshall-Olkin Discrete Reduced Modified Weibull distribution (MDRMW) which is the proposed model.

3.1. Properties of Proposed Distribution (MDRMW)

The cumulative distribution function and survival function of the discrete random variable having the probability mass function (9) is given by

$$F(x;\beta,q,b,c) = \frac{1-q^{(x+1)^{1/2}(1+bc^{x+1})}}{1-\bar{\beta}q^{(x+1)^{1/2}(1+bc^{x+1})}}, x = 0,1,2,\ldots \quad (10)$$

and

$$S(x;\beta,q,b,c) = \frac{\beta q^{x^{1/2}(1+bc^x)}}{1-\bar{\beta}q^{x^{1/2}(1+bc^x)}}, x = 0,1,2,\ldots \quad (11)$$

3.2. Shape of the Proposed Distribution (MDRMW)

In Figure 1, it can be seen that the MDRMW distribution is flexible. Its PMF can take one of the following shapes: i) a unimodal shape; ii) a monotonic decreasing shape; iii) a decreasing shape followed by an increasing shape followed by a decreasing.

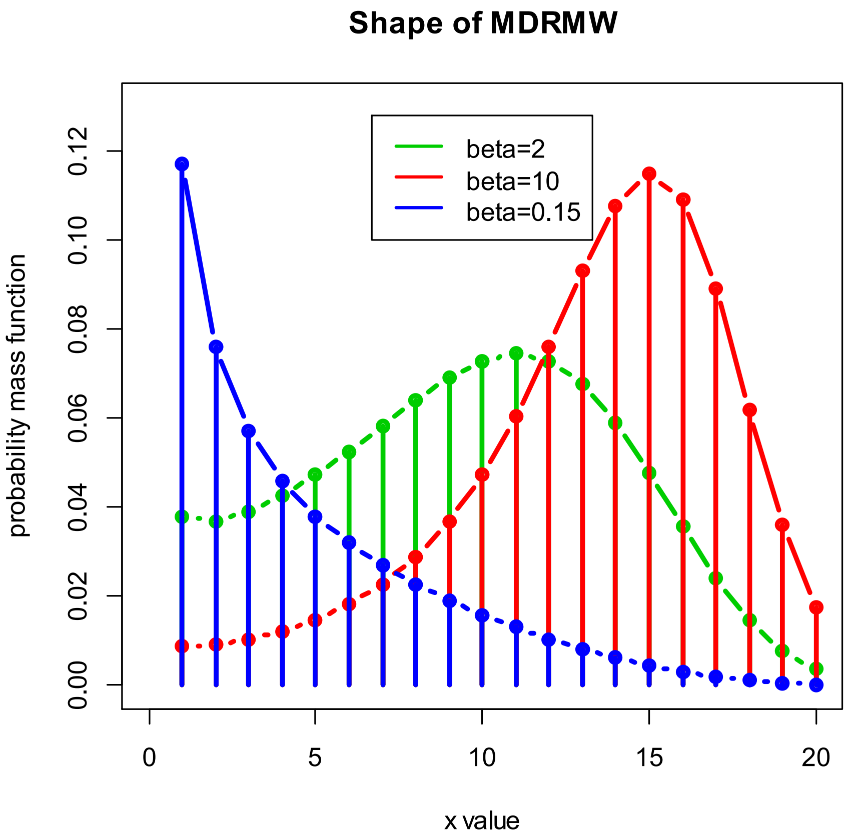

Figure 1. The shape of pmf of the proposed distribution.

3.3. Series Expansion of the Proposed Distribution

Here, a series expansion for the survival function is derived, and hence for the probability mass function. Using the series expansion for the exponential, it can be written as

$$q^{x^{1/2}(1+bc^x)}$$

$$\sum_{j=0}^{\infty}\frac{ln^j\,(q)x^{j/2}}{j!}\sum_{k=0}^{j}\binom{j}{k}b^k\left(\sum_{i=0}^{\infty}\frac{x^i ln^i\,c}{i!}\right)^k \tag{12}$$

Now using the partial exponential Bell polynomial, $B_{rk}(x)$ (Comtet, 1974), defined by

$$\left(\sum_{r=1}^{\infty}x_r t^r/r!\right)^k\Big/k! = \sum_{r=k}^{\infty}B_{rk}(x)t^r/r!$$

$$S(x)=\sum_{j=0}^{\infty}\frac{ln^j\,(q)x^{j/2}}{j!}\sum_{k=0}^{j}\binom{j}{k}b^k k!\sum_{l=k}^{\infty}B_{rk}(\ln c\,,\ln c\,,\ldots)\frac{x^l}{l!}$$

$$=\sum_j^{\infty}\sum_k^{j}\sum_{l=k}^{\infty}\frac{ln^j\,(q)x^{j/2}B_{rk}(\ln c,\ln c,\ldots)}{(j-k)l!}x^{l+\frac{j}{2}}. \tag{13}$$

The representation in (13) can be used to derive similar expansions for the probability mass function as the following

$$\beta\left(\sum_j^{\infty}\sum_k^{j}\sum_{l=k}^{\infty}\frac{ln^j\,(q)x^{j/2}B_{rk}(\ln c\,,\ln c\,,\ldots)}{(j-k)l!}x^{l+\frac{j}{2}}-\sum_j^{\infty}\sum_k^{j}\sum_{l=k}^{\infty}\frac{ln^j\,(q)x^{j/2}B_{rk}(\ln c\,,\ln c\,,\ldots)}{(j-k)l!}(x+1)^{l+\frac{j}{2}}\right)$$

$$\left(1-\bar{\beta}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=k}^{\infty}\frac{ln^{j}\,(q)x^{j/2}B_{rk}(\ln c\,,\ln c\,,\dots)}{(j-k)l!}x^{l+\frac{j}{2}}\right)\left(1-\bar{\beta}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=k}^{\infty}\frac{ln^{j}\,(q)x^{j/2}B_{rk}(\ln c\,,\ln c\,,\dots)}{(j-k)l!}(x+1)^{l+\frac{j}{2}}\right)$$

using the binomial expansion for non-integer powers:

$$(A+B)^{P}=\sum_{d=0}^{\infty}c(p,d)A^{p-d}B^{p},$$

where $c(p,d)=\frac{p(p-1)(p-2)\dots(p-d+1)}{d!!}$,

$$\varphi=\beta\left(\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=k}^{\infty}\frac{ln^{j}\,(q)x^{j/2}B_{rk}(\ln c\,,\ln c\,,\dots)}{(j-k)l!}x^{l+\frac{j}{2}}-\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=k}^{\infty}\sum_{p=0}^{\infty}\frac{c\left(\frac{j}{2}+l,p\right)ln^{j}\,(q)x^{j/2}B_{rk}(\ln c\,,\ln c\,,\dots)}{(j-k)l!}x^{\frac{j}{2}+l-p}\right)$$

$$\omega=\left(1-\bar{\beta}\sum_{j}^{\infty}\sum_{k}^{j}\sum_{l=k}^{\infty}\frac{ln^{j}\,(q)x^{j/2}B_{rk}(\ln c\,,\ln c\,,\dots)}{(j-k)l!}x^{l+\frac{j}{2}}\right)\left(1-\bar{\beta}\sum_{j}^{\infty}\sum_{k}^{j}\sum_{l=k}^{\infty}\sum_{p=0}^{\infty}\frac{c\left(\frac{j}{2}+l,p\right)ln^{j}\,(q)x^{j/2}B_{rk}(\ln c\,,\ln c\,,\dots)}{(j-k)l!}x^{\frac{j}{2}+l-p}\right)$$

Then

$$\frac{\varphi}{\omega}=\frac{\beta\left(\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=k}^{\infty}\frac{ln^{j}\,(q)x^{j/2}B_{rk}(\ln c,\ln c,\dots)}{(j-k)l!}x^{l+\frac{j}{2}}-\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=k}^{\infty}\sum_{p=0}^{\infty}\frac{c\left(\frac{j}{2}+l,p\right)ln^{j}\,(q)x^{j/2}B_{rk}(\ln c,\ln c,\dots)}{(j-k)l!}x^{\frac{j}{2}+l-p}\right)}{\left(1-\bar{\beta}\sum_{j}^{\infty}\sum_{k}^{j}\sum_{l=k}^{\infty}\frac{ln^{j}\,(q)x^{j/2}B_{rk}(\ln c,\ln c,\dots)}{(j-k)l!}x^{l+\frac{j}{2}}\right)\left(1-\bar{\beta}\sum_{j}^{\infty}\sum_{k}^{j}\sum_{l=k}^{\infty}\sum_{p=0}^{\infty}\frac{c\left(\frac{j}{2}+l,p\right)ln^{j}\,(q)x^{j/2}B_{rk}(\ln c,\ln c,\dots)}{(j-k)l!}x^{\frac{j}{2}+l-p}\right)} \tag{14}$$

3.4. Hazard Rate Function of the Proposed Distribution

The Hazard rate function is

$$\frac{p(x)}{S(x)},$$

$$h(x)=\frac{q^{x^{1/2}(1+bc^{x})}-q^{(x+1)^{1/2}(1+bc^{x+1})}}{q^{x^{1/2}(1+bc^{x})}\left(1-\bar{\beta}q^{(x+1)^{1/2}(1+bc^{x+1})}\right)}$$

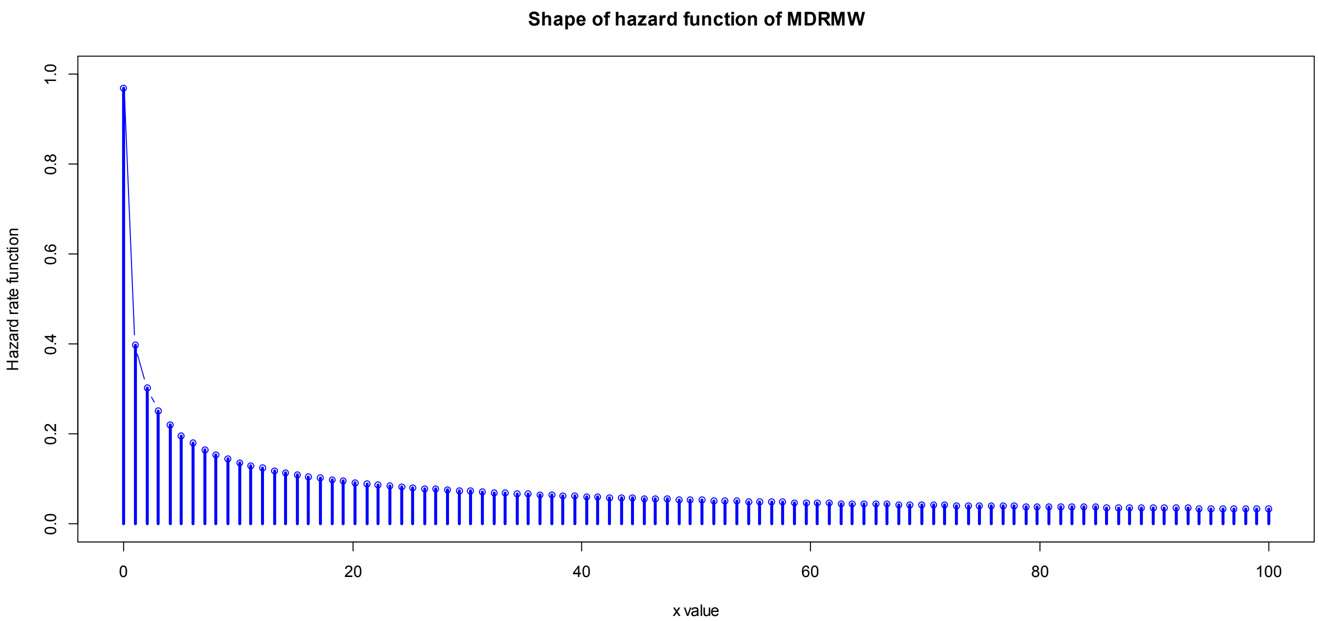

Figure 2. *The decreasing hazard shape of the proposed distribution.*

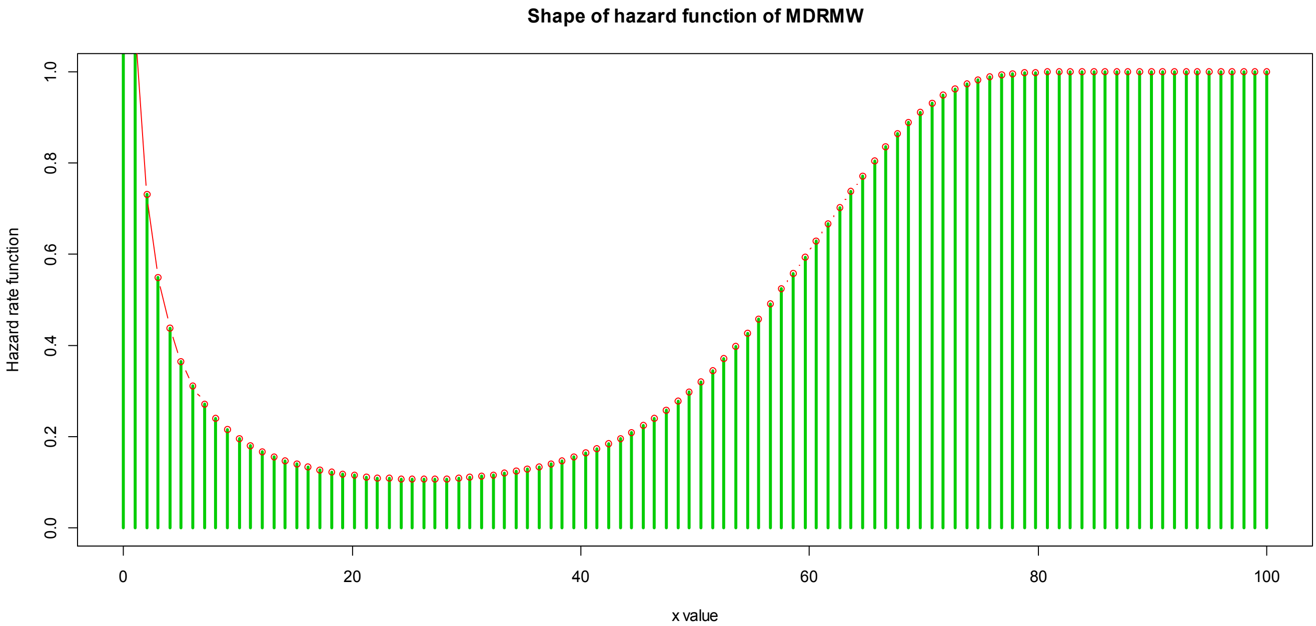

Figure 3. *The decreasing then increasing hazard shape.*

3.5. The Probability Generating Function of the Proposed Distribution

The probability generating function (pgf) of discrete random variable having the MDRMW (q, b, c, β) is given by

$$p_x(t) = 1 + \beta(t-1)\sum_{x=1}^{\infty} t^{x-1}\frac{\bar{G}(x)}{1-\bar{\beta}\bar{G}(x)},$$

$$= 1 + \beta(t-1)\sum_{x=1}^{\infty} t^{x-1}\frac{q^{x^{1/2}(1+bc^x)}}{1-\bar{\beta}q^{x^{1/2}(1+bc^x)}}$$

mean and variance of the random variable having MDRMW (q, b, c, β)

$$\mu(q,b,c,\beta) = E(X) = \beta\sum_{x=1}^{\infty}\frac{q^{x^{1/2}(1+bc^x)}}{1-\bar{\beta}q^{x^{1/2}(1+bc^x)}},$$

$$V(X) = \beta\sum_{x=1}^{\infty}(2x-1)\frac{q^{x^{1/2}(1+bc^x)}}{1-\bar{\beta}q^{x^{1/2}(1+bc^x)}} - \left(\beta\sum_{x=1}^{\infty}\frac{q^{x^{1/2}(1+bc^x)}}{1-\bar{\beta}q^{x^{1/2}(1+bc^x)}}\right)^2 \quad (15)$$

3.6. Recurrence Relation of Probability Generating Function of the Proposed Distribution

The recurrence relation for generating probabilities given by

$$p_{x+1} = \frac{\bar{G}(x+1)-\bar{G}(x+2)}{\bar{G}(x)-\bar{G}(x+1)}\cdot\frac{1-\bar{G}(x)}{1-\bar{G}(x+2)}p_x, x = 0,1,2\ldots$$

where $p_0 = \frac{1-\bar{G}(1)}{1-\bar{\beta}\bar{G}(1)}$.

then,

$$p_x = \frac{q^{(x+1)^{1/2}(1+bc^{x+1})}-q^{(x+2)^{1/2}(1+bc^{x+2})}}{q^{x^{1/2}(1+bc^x)}-q^{(x+1)^{1/2}(1+bc^{x+1})}}\cdot\frac{1-\bar{\beta}q^{x^{1/2}(1+bc^x)}}{1-\bar{\beta}q^{(x+2)^{1/2}(1+bc^{x+2})}}p_x, x = 0,1,2\ldots \quad (16)$$

where $p_0 = \frac{1-q^{(1+bc)}}{1-\bar{\beta}q^{(1+bc)}}$.

3.7. Maximum Likelihood Estimation of the Proposed Distribution (MDRMW)

The maximum likelihood estimation (MLE) is used to estimate unknown parameters. Consider a random sample $x_1, x_2, \ldots, x_n$ from the Marshall-Olkin Discrete Reduced Modified Weibull (MDRMW) distribution.

The likelihood function is given by

$$\log L = n\log\beta + \sum_{i=1}^{n}\log\left(q^{x_i^{1/2}(1+bc^{x_i})} - q^{(x_i+1)^{1/2}(1+bc^{x_i+1})}\right) - \sum_{i=1}^{n}\log\left(1-\bar{\beta}q^{x_i^{1/2}(1+bc^{x_i})}\right) - \sum_{i=1}^{n}\log\left(1-\bar{\beta}q^{(x_i+1)^{1/2}(1+bc^{x_i+1})}\right) \quad (17)$$

Two applications using well-known data sets were used to demonstrate the robustness and applicability of the proposed model. These data present different degrees of skewness and kurtosis. The new distribution was compared with the existing distributions which are Discrete Reduced Modified Weibull distribution (DRMW) (Amalki, 2014), Two parameter Discrete Lindley distribution (TDL) (Tassaddaq *et al.,* 2016).

4. Numerical Illustration

The goodness of fit statistic and the goodness of fit plot were provided in order to check the model that best fit the data among the models for each dataset used for this research.

Two real data sets were considered in this section. One of them has bathtub shaped hazard rate functions while the second one has an increasing hazard rate function.

4.1. Discrete Aarset Data

The data are integer parts of the lifetimes of fifty devices. The data are listed in Table 1.

Nooghabi et al. (2011) had shown that the DMW distribution provides a good fit for this data.

Table 1. Aarset data (in weeks).

Time of failure	0	1	2	3	6	7	11	12	18	21
No of failure	2	5	1	1	1	1	1	1	5	1
Time of failure	32	36	40	45	46	47	50	55	60	63
No of failure	1	1	1	1	1	1	1	1	1	2
Time of failure	67	72	75	79	82	83	84	85	86	
No of failure	4	1	1	1	2	1	3	5	2	

Table 2 shows the Maximum Likelihood Estimation (MLEs) of the parameters and their standard errors. Table 3 shows the AIC and BIC values for the fitted MDRMW, DRMW, TDL and EDW distributions.

Table 2. The Maximum Likelihood Estimation of the MDRMW distribution for the Aarset Data and Standard Error.

MODEL	q	β	b	c
MDRMW	0.6183 (0.0443)	18.682 (10.7933)	2.5905 (4.1552)	−0.1776 (0.2084)
DRMW	0.8475 (0.0198)	−	7.4787 (22.5427)	−0.0817 (0.2335)
TDL	0.9687 (0.0051)	0.0275 (0.0234)	−	−

Table 3. The AIC and BIC for the fitted distributions.

MODEL	-2Log-Likelihood	AIC	BIC
MDRMW	475.9	483.9	491.5
DRMW	501.8	507.8	513.5
TDL	481.2	485.2	489.0

4.2. Leukemia Data

The data set for this example was collected from the Ministry of Health Hospital in Saudi Arabia. The data are lifetimes in days of forty three blood patients who had leukemia. The data set exhibits an increasing hazard rate.

Table 4. The Leukemia data.

115	181	255	418	441	461	516	739	743	789	807	865
1062	924	983	1025	1063	1165	1191	1222	1222	1251	1277	1290
1578	1357	1369	1408	1478	1549	1455	1578	1599	1603	1605	1696
1735	1799	1815	1852	1899	1925	1965					

Table 5. The Maximum Likelihood Estimation of the MDRMW distribution for the Leukemia Data and Standard Errors.

MODEL	q	β	b	c
MDRM	$0.8091(2.37\times10^{-2})$	$1.494.8(1.615\times10^{3})$	$-2214.9(6.78\times10^{3})$	$0.9274(2.676\times10^{-2})$
DRMW	0.9706 (0.0044)	−	1.0197 (−)	0.8921 (0.0548)

Table 6. The AIC and BIC for the fitted distributions.

MODEL	-2Log-Likelihood	AIC	BIC
MDRMW	660.2	668.2	675.3
DRMW	745.9	751.9	757.2

5. Conclusion

A new distribution called Marshall-Olkin Discrete Reduced Modified Weibull distribution (MDRMW) was proposed and its properties studied. In this work, the new discrete RMW was used to analyze all the datasets used by Amalki, (2014) in order to properly compare the new distribution with DRMW of Amalki. In addition, two most recent distributions which are Exponentiated Discrete Weibull distribution and Two Parameter discrete Lindley distribution (TDL) proposed by Tassaddaq et al., (2016) were also used to analyze the datasets.

The MDRMW distribution is flexible to model discrete data such as survival data and over-dispersed data. Its hazard function is monotonically decreasing, followed by an increasing shape and upside down bathtub. The closed form expressions for the moments, distribution of order statistics were obtained. Maximum likelihood estimation technique was used to estimate the model parameters.

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