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# Max-analogues of N-infinite Divisibility and N-stability

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**Abstract:** Here we discuss the max-analogues of random infinite divisibility and random stability developed by Gnedenko and Korolev [5]. We give a necessary and sufficient condition for the weak convergence to a random max-infinitely divisible law from that to a max-infinitely divisible law. Introducing random max-stable laws we show that they are indeed invariant under random maximum. We then discuss their domain of max-attraction.

**Keywords:** Max-infinite Divisibility, Max-stability, Domain of Max-attraction, Extremal Processes

## 1. Introduction

In the classical summation scheme a characteristic function (CF) $f(t)$ is infinitely divisible (ID) if for every $n \geq 1$ integer there exists a CF $f_n(t)$ such that $f(t) = \{f_n(t)\}^n$. The classical de-Finetti theorem for ID laws states that $f(t)$ is ID *iff* $f(t) = \lim_{n\to\infty} \exp\{-a_n(1 - h_n(t))\}$ where $\{a_n\}$ are some positive constants and $h_n(t)$ are CFs.

Klebanov, *et al.* [1] extended the notion of ID laws to geometrically ID (GID) laws using geometric (with mean $1/p$) sums. According to this, $f(t)$ is GID if for every $p \in (0,1)$ there exists a CF $f_p(t)$ such that $f(t) = \sum_{n=1}^{\infty} \left(f_p(t)\right)^n p(1-p)^{(n-1)}$, the geometric law being independent of the distribution of $f_p(t)$ for every $p \in (0,1)$. They also proved an analogue of the de-Finetti theorem in the context, *viz.* $f(t)$ is GID *iff* $f(t) = \lim_{n\to\infty} 1/\{1 + a_n(1 - h_n(t))\}$, where $\{a_n\}$ and $h_n(t)$ are as above. Consequently, $f(t)$ is GID *iff* $f(t) = 1/\{1 - \log\omega(t)\}$ where $\omega(t)$ is a CF that is ID. Subsequently [2] (also reported in [3]), [4], [5] and [6] have discussed attraction and the first three works that of partial attraction for GID laws.

Later [2] (also reported in [7]), [4], [5], [6] and [8] extended the notion of GID to random ($\mathcal{N}$) ID laws based on $N_\theta$-sums. [2] and [7] defined $\mathcal{N}$-ID laws as: a CF $f(t)$ is $\mathcal{N}$-ID, where $N_\theta$ is a positive integer-valued random variable (*r.v*) having finite mean with probability generating function (*p.g.f*) $P_\theta$ if there exists a CF $f_\theta(t)$ such that $f(t) = P_\theta\{f_\theta(t)\}$ for every $\theta\epsilon\Theta$. We need the distributions of $P_\theta$ and $f_\theta$ to be independent for every $\theta$. She noticed that when $f(t)$ and $f_\theta(t)$ are of the same type, the above is an Abel (Poincare) equation. She also gave two examples of non-geometric laws for $N_\theta$. [5] (section 4.6) and [6] went further by proving the de-Finetti analogue for $\mathcal{N}$-ID laws *viz.* a CF $f(t)$ is $\mathcal{N}$-ID *iff* $f(t) = \lim_{n\to\infty} \varphi\{a_n(1 - h_n(t)\}$ where $\varphi$ is a Laplace transform (LT) that is also a solution to the Poincare (Abel) equation. They then concluded that a CF $f(t)$ is $\mathcal{N}$-ID *iff* $f(t) = \varphi\{-\log\omega(t)\}$ where $\omega(t)$ is CF that is ID. In this description $P_\theta$ and $\varphi$ are related by $P_\theta(s) = \varphi\{\frac{1}{\theta}\varphi^{-1}(s)\}$, $0 < s \leq 1$, $\theta\epsilon\Theta$, where $P_\theta$ is the *p.g.f* of the *r.v* $N_\theta$ that is positive integer-valued having finite mean. [8] also arrived at the same conclusion under the same assumptions but the arguments were based on Levy processes instead of proving the de-Finetti analogue enroute. Poincare equation is given by $\varphi(s) = P\big(\varphi(\theta s)\big), s \geq 0, \theta\epsilon\,\Theta, P$ being a *p.g.f.* [5], [6] and [9] discussed Poincare equation and examples of deriving a *p.g.f* from $\varphi$.

To circumvent the main constraints in the development of $\mathcal{N}$-ID laws *viz.* that $N_\theta$ is a positive integer-valued *r.v* having finite mean, $\varphi$ is a LT that is also a solution to the Poincare equation, [10] introduced $\varphi$-ID laws for any LT $\varphi$ and $N_\theta$ a non-negative integer-valued *r.v* derived from $\varphi$. The important case of compound Poisson distributions was thus brought under random-ID laws. [5], [6] and [10] also discussed attraction and partial attraction for $\mathcal{N}$-ID/ $\varphi$-ID laws. The discrete analogue of this was developed in [9]. The *r.v* $N_\theta$ in $\mathcal{N}$-ID laws has the following property.

*Lemma 1.1* $\theta N_\theta \xrightarrow{d} U$ as $\theta \downarrow 0$, and the LT of $U$ is $\varphi$, see [5], p.138.

Coming to the max-analogue, [11] introduced the notion of max infinitely divisible (MID) laws. A distribution function (*d.f*) $F$ is MID if $F^{1/n}$ is a *d.f* for each integer $n \geq 1$. Since

$F^{1/n}$ is always a *d.f* in the univariate case all *d.f*s in $R$ are MID, see [13]. Hence a discussion of MID laws is relevant for *d.f* s in $R^d, d \geq 2$, integer and the max operations are to be taken component wise. Thus in this paper all *d.f*s are assumed to be in $R^d, d \geq 2$ integer, unless stated otherwise. Later [12] introduced geometric max infinitely divisible (GMID) laws and geometric max stable (GMS) laws, see also [13]. [12] also discussed certain connections between GMID/ GMS laws and extremal processes. From [11] we have the max-analogue of the classical de Finetti's theorem.

*Theorem 1.1* A *d.f* $F(x)$ is MID *iff* for some *d.f*s $\{G_n\}$ and constants $\{a_n > 0\}$

$F(x) = \lim_{n\to\infty} \exp\{-a_n(1 - G_n(x))\}$.

Using the transfer theorem for maximums in [14] we can study the limit distributions of random maximums. [15] briefly discussed the max-analogue of $\mathcal{N}$-ID laws to obtain stationary solutions to a generalized max-AR(1) scheme. However, there was an inadvertent omission, as the discussion did not stress that the LT $\varphi$ should also be a standard solution to the Poincare equation.

Proceeding from [15], we discuss random ($\mathcal{N}$) MID ($\mathcal{N}$-MID) laws that is the max-analogue of $\mathcal{N}$-ID laws, in section 2. In section 3 we discuss random ($\mathcal{N}$) max-stable laws, generalise certain results on GMS laws in [12] to $\mathcal{N}$-max-stable laws and their domain of max-attraction. The convergence discussed here is weak convergence of *d.f*s, unless stated otherwise.

## 2. Random MID Laws

We begin by defining $\mathcal{N}$-MID laws analogous to the $\mathcal{N}$-ID laws in [5] correcting the omission mentioned above.

*Definition 2.1* Let $\varphi$ be a standard solution to the Poincare equation and $N_\theta$, a positive integer-valued *r.v* having finite mean with *p.g.f* $P_\theta(s) = \varphi\left(\frac{1}{\theta}\varphi^{-1}(s)\right), \theta \in \Theta \subset (0,1)$. A *d.f* $F(x)$ in $R^d$ is $\mathcal{N}$-MID if for each $\theta \in \Theta$, there exists a *d.f* $G_\theta(x)$ that is independent of $N_\theta$, such that $F(x) = P_\theta(G_\theta(x))$ for all $x \in R^d$.

*Theorem 2.1* A *d.f* $F$ which is the weak limit of a sequence $F_n$ of $\mathcal{N}$-MID *d.f*s is itself $\mathcal{N}$-MID.

*Proof.* By virtue of the continuity of *p.g.f*s, for every $\theta \in \Theta$, we have

$F(x) = \lim_{n\to\infty} F_n(x) = P_\theta(\lim_{n\to\infty} G_{\theta,n}(x)) = P_\theta(G_\theta(x))$.

We now have an analogue of theorem 1.1, a de Finetti type theorem, for $\mathcal{N}$-MID laws.

*Theorem 2.2* Let $\varphi$ be a standard solution to the Poincare equation. A *d.f* $F(x)$ in $R^d$ is $\mathcal{N}$-MID *iff* for some *d.f*s $\{G_n\}$ and constants $\{a_n > 0\}$, $F(x) = \lim_{n\to\infty} \varphi\{a_n(1 - G_n(x))\}$.

*Proof.* See the proof of theorem 3.5 in [15].

Notice that for a LT $\varphi(s), s > 0, \varphi(\lambda(1-s)), 0 < s \leq 1, \lambda > 0$ is a *p.g.f.* Hence the above representation is essentially the weak limit of random-maximums under the transfer theorem for maximums. The next result facilitates the construction and/ or identification of $\mathcal{N}$-MID *d.f*s.

*Theorem 2.3* A *d.f* $F(x)$ is $\mathcal{N}$-MID *iff* $F(x) = \varphi\{-\log H(x)\}$, where $\varphi$ is a standard solution to the Poincare equation and $H(x)$ is a MID *d.f.*

*Proof.* We have seen that an $\mathcal{N}$-MID *d.f* $F(x)$ admits the representation for some *d.f*s $G_\theta$,

$$F(x) = \lim_{\theta\downarrow 0} \varphi\left\{\frac{1}{\theta}\left(1 - G_\theta(x)\right)\right\}.$$

Since $\varphi$ is continuous we can proceed as

$$F(x) = \lim_{\theta\downarrow 0} \varphi\{-\log\left(\exp\left\{\frac{1}{\theta}(G_\theta(x) - 1)\right\}\right)\} = \varphi(-\log H(x)),$$

Where $H(x) = \lim_{\theta\downarrow 0} \exp\left\{\frac{1}{\theta}(G_\theta(x) - 1)\right\}$ is MID. Note the fact that every Poisson maximum is MID and every MID *d.f* is the weak limit of Poisson maximums [11]. Conversely, consider

$$\varphi(-\log H(x)) = \int_0^\infty \exp\{t \log H(x)\}\, d\Lambda(t), t > 0,$$

where $H(x)$ is MID and $\varphi$ is the LT of the *d.f* $\Lambda$. Now $\varphi(-\log H(x))$ is $\mathcal{N}$-MID since the above is the integral representation of a *d.f* that is the weak limit under the transfer theorem for maximums. This completes the proof.

*Corollary 2.1* A *d.f* is $\mathcal{N}$-MID *iff* it is the limit distribution, as $\theta \downarrow 0$, of a random maximum of *i.i.d r.v*s.

Now we proceed to prove the max-analogue of theorem 4.6.5 in [5]. Let, for every $\theta \in \Theta, \{X_{\theta,i}\}$ with *d.f* $G_\theta$ be *i.i.d* random vectors in $R^d$ and $N_\theta$ a positive integer-valued *r.v* having finite mean with *p.g.f* $P_\theta(s) = \varphi\left(\frac{1}{\theta}\varphi^{-1}(s)\right)$, that is independent of $\{X_{\theta,i}\}$ for every $\theta \in \Theta$ and $i$. Let $\left[\frac{1}{\theta}\right]$ denote the integer part of $\frac{1}{\theta}$.

*Theorem 2.4* Let $F(x) = \varphi(-\log G(x))$ be $\mathcal{N}$-MID. Then

$$\lim_{\theta\downarrow 0} P_\theta(G_\theta(x)) = \varphi(-\log G(x)) \tag{1}$$

*iff* there exists a *d.f* $G(x)$ that is MID and

$$\lim_{\theta\downarrow 0} G_\theta^{\left[\frac{1}{\theta}\right]}(x) = G(x). \tag{2}$$

*Proof.* The sufficiency of the condition (2) follows from the transfer theorem for maximums by invoking the relation $\theta\left[\frac{1}{\theta}\right] \to 1$ and $\theta N_\theta \xrightarrow{d} U$ as $\theta \downarrow 0$. Conversely (1) implies

$$\lim_{\theta\downarrow 0} \varphi\left(\frac{1}{\theta}\varphi^{-1}(G_\theta(x))\right) = \varphi(-\log G(x)). \tag{3}$$

Since $\varphi$ is a LT we have; $\lim_{\theta\downarrow 0}\left(\frac{1}{\theta}\varphi^{-1}(G_\theta(x))\right) = -\log G(x)$.

Again, since $\varphi(0) = 1$, this implies that

$$\lim_{\theta\downarrow 0} G_\theta(x) = 1. \tag{4}$$

Since $\varphi\left(\frac{1-G_\theta(x)}{\theta}\right)$ is a *d.f* that is $\mathcal{N}$-MID for every $\theta \in \Theta$, $\lim_{\theta\downarrow 0} \varphi\left(\frac{1-G_\theta(x)}{\theta}\right)$ is also $\mathcal{N}$-MID by theorem 2.1. Hence there exists a *d.f* $H(x)$ that is MID such that

$$\lim_{\theta\downarrow 0} \varphi\left(\frac{1-G_\theta(x)}{\theta}\right) = -\log H(x). \tag{5}$$

On the other hand for $|\kappa| \leq 1$ we have

$$\log G_\theta^{\left[\frac{1}{\theta}\right]} = \left[\frac{1}{\theta}\right] \log\big(1-(1-G_\theta)\big) = \left[\frac{1}{\theta}\right](G_\theta - 1) + \kappa\left[\frac{1}{\theta}\right]|G_\theta - 1|^2. \quad (6)$$

Hence by (4) and (5) we get from (6)

$$\lim_{\theta\downarrow 0} G_\theta^{\left[\frac{1}{\theta}\right]}(x) = H(x). \quad (7)$$

Now applying the transfer theorem for maximums it follows that

$$\lim_{\theta\downarrow 0} \varphi\left(\frac{1}{\theta}\varphi^{-1}\big(G_\theta(x)\big)\right) = \varphi(-\log H(x)).$$

Hence by (1) $H(x) \equiv G(x)$. That is, by (7), (2) is true with $G(x)$ being MID, completing the proof.

# 3. Random Max-stable Laws

Theorem 2.4 identifies the weak limit of partial $N_\theta$-maximums of certain component *r.v*s as a function of the weak limit of partial maximums of the same component *r.v*s and vice-versa. This description thus enables us to define random max-stable ($\mathcal{N}$-max-stable) laws analogous to the $\mathcal{N}$-stable laws in [5] and their domains of $\mathcal{N}$-max-attraction. This is facilitated by prescribing $\left[\frac{1}{\theta}\right] = n$ in theorem 2.4. Notice also that here the discussion can be for *d.f*s in $\boldsymbol{R}$.

*Definition 3.1* A *d.f* $F(x)$ is $\mathcal{N}$-max-stable *iff* $F(x) = \varphi\{-\log H(x)\}$, where $H(x)$ a max-stable *d.f* and $\varphi$ is a standard solution to the Poincare equation.

*Theorem 3.1* An $\mathcal{N}$-max-stable *d.f* can be represented as $F(x) = P_\theta\big(F_\theta(x)\big)$, for every $\theta \epsilon\, \Theta$, where $F$ and $F_\theta$ are *d.f*s of the same type. Here $F_\theta$ and $P_\theta$ are independent for each $\theta \epsilon\, \Theta$, $P_\theta(s) = \varphi\left(\frac{1}{\theta}\varphi^{-1}(s)\right)$ is the *p.g.f* of $N_\theta$, a positive integer-valued *r.v* having finite mean.

*Proof.* Since $F(x)$ is $\mathcal{N}$-max-stable we have the following representation for every $\theta \epsilon\, \Theta$. $F = \varphi\{-\log H\} = \varphi\{\frac{1}{\theta}\varphi^{-1}(\varphi(-\theta\log H))\} = P_\theta(\varphi(-\theta\log H)) = P_\theta(\varphi(-\log H^\theta)) = P_\theta(F_\theta)$.

Notice that $H$ and $H^\theta$ are *d.f*s of the same type, [16]. Since $H$ is max-stable, $H^\theta$ also is max-stable. Thus the above representation describes an $\mathcal{N}$-max-stable *d.f* as an $N_\theta$-sum of *d.f*s of the same type for every $\theta \epsilon\, \Theta$, proving the result.

We now generalise proposition 3.2 on GMS laws in [12] to $\mathcal{N}$-max-stable laws.

*Theorem 3.2* For a *d.f* $F$ on $R^d$ the following statements are equivalent.

($i$) $F$ is $\mathcal{N}$-max-stable

($ii$) $\exp\{-\varphi^{-1}(F)\}$ is max-stable

($iii$) There exists an $\ell\epsilon[-\infty,\infty)^d$ and an exponent measure $\mu$ concentrated on $[\ell,\infty)$ such that for $x \ge \ell, F(x) = \varphi(\mu[\ell,x]^c)$.

($iv$) There exists a multivariate extremal process $\{Y(t), t > 0\}$ governed by a max-stable law and an independent *r.v* $Z$ with *d.f* $F$ and LT $\varphi$ such that $F(x) = P\{Y(Z) \le x\}$.

*Proof.* $(i) \Rightarrow (ii)$ $F$ is $\mathcal{N}$-max-stable implies $F = \varphi\{-\log H\}$, where $H$ is max-stable. This implies $\exp\{-\varphi^{-1}(F)\} = H$ is max-stable.

$(ii) \Rightarrow (iii)$ From the representation of a max-stable *d.f* by an exponent measure $\mu$ and from ($ii$) we have $H(x) = \exp\{-\varphi^{-1}\big(F(x)\big)\} = \exp\{-\mu[\ell,x]^c\}$. This implies $\varphi^{-1}(F(x)) = \mu[\ell,x]^c$ or $F(x) = \varphi(\mu[\ell,x]^c)$.

$(iii) \Rightarrow (iv)$ By ($iii$) we have the exponent measure $\mu$ corresponding to the max-stable law identified in ($ii$). Let $\{Y(t), t > 0\}$ be the extremal process governed by this max-stable law. That is $P\{Y(t) \le x\} = \exp\{-t\mu[\ell,x]^c\}$.

Hence

$$P\{Y(Z) \le x\} = \int_0^\infty \exp\{-t\mu[\ell,x]^c\}\, dF(t) = \varphi\{\mu[\ell,x]^c\} = F(x).$$

$(iv) \Rightarrow (i)$ is now obvious. Thus the proof is complete.

A notion that is closely associated with max-stable laws is their domain of max-attraction. The notion of geometric max-attraction for GMS laws was discussed in [12] and [13]. We now briefly discuss this for $\mathcal{N}$-max-stable laws.

*Definition 3.2* A *d.f* $G(x)$ belongs to the domain of $\mathcal{N}$-max-attraction (D$\mathcal{N}$MA) of the *d.f* $F(x)$ (with non-degenerate marginals) if there exists constants $a_{i,n} = a_i(\theta_n) > 0$ and $b_{i,n} = b_i(\theta_n)$ such that $\lim_{n\to\infty} P_n(G^n) = F$, meaning that

$\lim_{n\to\infty} P_n(G_i^n) = F_i$, for each $1 \le i \le d$ where $G_i^n(x) = G_i^n(a_{i,n}x + b_{i,n})$ and $\theta_n = \frac{1}{n}$.

Recalling that $\varphi$ is continuous and that max-attraction of $G$ to $H$ is equivalently specified by $n\{1 - G_i\big(a_{i,n}x + b_{i,n}\big)\} \to -\log H_i(x), 1 \le i \le d$, we have the following result as an immediate consequence of theorem 2.2.

*Theorem 3.3* Let $\varphi$ be a standard solution to the Poincare equation. A *d.f* $F(x) = \varphi\{-\log H(x)\}$ is $\mathcal{N}$-max-stable *iff* for some *d.f* $G(x)$ and constants $a_{i,n} = a_i(\theta_n) > 0$ and $b_{i,n} = b_i(\theta_n)$,

$\varphi\big(n\{1 - G_i\big(a_{i,n}x + b_{i,n}\big)\}\big) \to \varphi(-\log H_i(x)) = F_i(x), 1 \le i \le d$.

Again, from theorem 2.4, choosing $G_\theta(x) = \big(G_i\big(a_{i,n}x + b_{i,n}\big), 1 \le i \le d\big)$ and $\theta$ such that $\left[\frac{1}{\theta}\right] = n$, where $a_{i,n} = a_i(\theta_n) > 0$ and $b_{i,n} = b_i(\theta_n)$, from the classical results on max-stable laws and their domains of attraction, we have

*Theorem 3.4* A *d.f* $G(x)$ belongs to the D$\mathcal{N}$MA of the *d.f* $F(x) = \varphi\{-\log H(x)\}$ *iff* it belongs to the DMA of $H(x)$.

# References


[1] Klebanov, L. B; Maniya, G. M. and Melamed, I. A (1984). A problem of Zolotarev and analogues of infinitely divisible and stable distributions in the scheme of summing a random number of random variables, *Theory of Probability and Applications*, 29, 791 – 794.

[2] Sandhya, E. (1991). *Geometric Infinite Divisibility and Applications*, Ph.D. Thesis (unpublished), University of Kerala, January 1991.

[3] Sandhya, E. and Pillai, R. N. (1999). On geometric infinite divisibility, *Journal of Kerala Statistical Association*, 10, 1-7.

[4] Mohan, N. R.; Vasudeva, R. and Hebbar, H. V. (1993). On geometrically infinitely divisible laws and geometric domains of attraction, *Sankhya-A*, 55, 171-179.

[5] Gnedenko, B. V., Korolev, V. Y. (1996). *Random summation: limit theorems and applications*. CRC Press. Section 4.6, pp.137-152.

[6] Klebanov, L. B. and Rachev, S. T. (1996). Sums of a random number of random variables and their approximations with $\nu$ -accompanying infinitely divisible laws, *Serdica Math. Journal*, 22, 471-496.

[7] Sandhya, E. (1996). On a generalization of geometric infinite divisibility, *Proc. 8th Kerala Science Congress*, January-1996, 355-357.

[8] Bunge, J. (1996). Composition semi groups and random stability, *Annals of Probability*, 24, 476-1489.

[9] Satheesh, S; Sandhya, E. and Lovely T Abraham (2010). Limit distributions of random sums of $Z^{+}$-valued random variables, *Communications in Statistics—Theory and Methods*, 39, 1979-1984.

[10] Satheesh, S (2004). Another look at random infinite divisibility, *Statistical Methods*, 6(2), 123-144.

[11] Balkema, A. A., Resnick, S. I. (1977). Max-infinite divisibility. *Journal of Applied Probability*, 309-319.

[12] Rachev, S. T., Resnick, S. (1991). Max-geometric infinite divisibility and stability. *Communications in Statistics. Stochastic Models*, *7*(2), 191-218.

[13] Mohan, N. R. (1998). On geometrically max infinitely divisible laws. *Journal of Indian Statistical Association*, *36*(1), 1-12.

[14] Gnedenko, B. V. (1983). On limit theorems for a random number of random variables. In *Probability Theory and Mathematical Statistics* (pp. 167-176). Springer Berlin Heidelberg.

[15] Satheesh, S., Sandhya, E., Rajasekharan, K. E. (2008). A generalization and extension of an autoregressive model. *Statistics & Probability Letters*, *78*(12), 1369-1374.

[16] Barakat, H. M., Ghitany, M. E., Al-Hussaini, E. K. (2009). Asymptotic distributions of order statistics and record values under the Marshall–Olkin parametrization operation. *Communications in Statistics—Theory and Methods*, *38*(13), 2267-2273.