Vibration Analysis of Non-uniform Beams Resting on Two Layer Elastic Foundations Under Axial and Transverse Load Using (GDQM)

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Abstract: The natural frequencies of non-uniform beams resting on two layer elastic foundations are numerically obtained using the Generalized Differential Quadrature (GDQ) method. The Differential Quadrature (DQ) method is a numerical approach effective for solving partial differential equations. A new combination of GDQM and Newton's method is introduced to obtain the approximate solution of the governing differential equation. The GDQ procedure was used to convert the partial differential equations of non-uniform beam vibration problems into a discrete eigenvalues problem. We consider a homogeneous isotropic beam with various end conditions. The beam density, the beam inertia, the beam length, the linear (k_1) and nonlinear (k_2) Winkler (normal) parameters and the linear (k_3) Pasternak (shear) foundation parameter are considered as parameters. The results for various types of boundary conditions were compared with the results obtained by exact solution in case of uniform beam supported on elastic support.

1. Introduction

Beams resting on linear and non-linear elastic foundations have many practical engineering applications as railroad tracks, highway pavement, buried pipelines and foundation beams. Due to the difficulty of mathematical nature of the problem, a few analytical solutions limited to special cases for vibrations of non-uniform beams resting on non-linear elastic foundations are found. Many methods are used to obtain the vibration behavior of different types of linear or nonlinear beams resting on linear or nonlinear foundations such as finite element method [1-3], transfer matrix method [4], Rayleigh-Ritz method [5], differential quadrature element method (DQEM) [6-10], Galerkin procedure [11, 12]

and [13-15]. There are various types of foundation models such as Winkler, Pasternak, Vlasov, etc. that have been used in the analysis of structures on linear and non-linear elastic foundations. Also, There are different beam types such as the Euler-Bernoulli which for slender beams and Timoshenko beam model for moderately short and thick beams. Balkaya et al. [16] studied vibration of a uniform Euler beam on elastic foundation using Differential Transform Method. Also, Ozturk and Coskun [17] studied the same problem using HPM. Avramidis and Morfidis [18] analyzed bending of beams on three-parameter elastic foundation.

Abrate et al. [19, 21] studied the vibrations of non-uniform rods and beams using the Rayleigh-Ritz scheme. Hodges et al. [22] used a discrete transfer matrix scheme to compute the fundamental frequencies and the corresponding modal

shapes. Sharma and DasGupta [23] used the Green's functions to study the bending of axially constrained beams resting on nonlinear Winkler type elastic foundations. Beaufait and Hoadley [24] used the midpoint difference technique to solve the problem of elastic beams resting on a linear foundation. Kuo and Lee [25] used the perturbation method to investigate the deflection of non-uniform beams resting on a nonlinear elastic foundation. Chen [26] used the differential quadrature element approach to obtain the numerical solutions for beams resting on elastic foundations.

Bagheri *et al.* [27] studied the nonlinear responses of clamped–clamped buckled beam. They used two efficient mathematical techniques called variational approach and Laplace iteration method in order to obtain the responses of the beam vibrations. Nikkar *et al.* [28] studied the nonlinear vibration of Euler-Bernoulli using analytical approximate techniques. Li and Zhang [29] used the B-spline function to derive a dynamic model of a tapered beam. Ramzy et al. [30] presented a new technique of GDQM for determining the deflection of a non-uniform beam resting on a non-linear elastic foundation, subjected to axial and transverse distributed force. Ramzy et al. [31] presented some problems in structural analysis resting on fluid layer using GDQM. Ramzy et al. [32] studied free vibration of uniform and non-uniform beams resting on fluid layer under axial force using the GDQM. The details of the DQM and its applications can be found in [33-34].

From previous studies, there are no any attempts to study vibration of non-uniform beam resting on two non-linear elastic foundations with the linear and nonlinear Winkler (normal) parameters and the linear Pasternak (shear) foundation parameter. The main goal of this study, to present a new combination of a GDQM and Newton's method to obtain the fundamental frequencies and the corresponding modal shapes of non-uniform beams resting on two layer elastic foundations under appropriate boundary conditions.

2. Formation

The kinetic energy of a beam with a non-uniform cross-section resting on an elastic foundation is as follows:

$$T = \frac{\rho A}{2} \int_0^L \left[\frac{\partial v(x,t)}{\partial t} \right]^2 dx, \ 0 \le x \le L. \tag{1}$$

Where the length of the non-uniform beam L, the vertical displacement v, the cross-sectional area of the beam A and the density of the beam material ρ.

The strain energy of a non-uniform beam resting on an elastic foundation can be derived as follows:

$$U = \frac{1}{2} \int_0^L EI \left[\frac{\partial^2 v(x,t)}{\partial x^2} \right]^2 dx + \frac{1}{2} k \int_0^L \left[v(x,t) \right]^2 dx. \tag{2}$$

Where the inertia of the beam I, the constant of the foundation k and Young's modulus of the beam material E.

Due to the axial loading, the work done can be written as follows,

$$\text{Work} = \frac{1}{2} \int_0^L p \left[\frac{\partial v(x,t)}{\partial x} \right]^2 dx. \tag{3}$$

Where, p is the axial force.

The Hamilton's principle is given by:

$$\int_{t_1}^{t_2} \left(\delta T - \delta U + \delta W \right) . dt = 0, \tag{4}$$

where δW is the virtual work.

Substituting from Equations (1), (2) and (3) into Equation (4) yields,

$$\frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 v}{\partial x^2} \right) + \rho A \frac{\partial^2 v}{\partial t^2} + p \frac{\partial^2 v}{\partial x^2} + k v = 0 \tag{5}$$

Equation (5) represents the equation of motion of non-uniform beam resting on elastic foundations under axial force.

The corresponding boundary conditions are as follows:
For Clamped-Clamped supported (C-C);

$$W(0) = \frac{dW(0)}{dX} = 0 \tag{6}$$

$$W(L) = \frac{dW(L)}{dX} = 0 \tag{7}$$

For Simply–Simply supported (S–S);

$$W(0) = \frac{d^2 W(0)}{dX^2} = 0 \tag{8}$$

$$W(L) = \frac{dW^2(L)}{dX^2} = 0 \tag{9}$$

For Clamped–Simply supported (C–S);

$$W(0) = \frac{dW(0)}{dX} = 0 \tag{10}$$

$$W(L) = \frac{dW^2(L)}{dX^2} = 0 \tag{11}$$

The vibration equation of a flexural non-uniform beam resting on two-layer elastic foundation is given as:
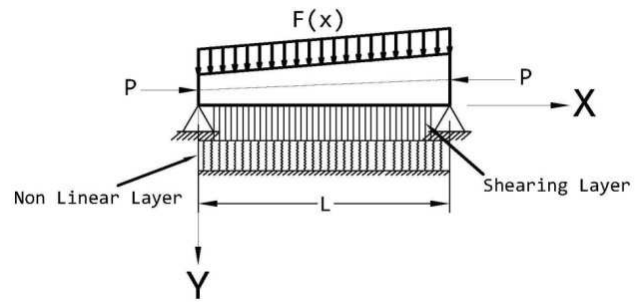
$$\frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 v}{\partial x^2} \right) + \rho A \frac{\partial^2 v}{\partial t^2} + p \frac{\partial^2 v}{\partial x^2} + k_1 v + k_2 v^3 - k_3 \frac{\partial^2 v}{\partial x^2} = f(x,t) , \ 0 \le x \le L \tag{12}$$

To obtain the natural frequencies and mode shapes, one can assume:

$$v(x, t) = V(x)\, e^{i\omega t}, \qquad (13)$$

$$f(x, t) = q(x)\, e^{i\omega t}. \qquad (14)$$

Where the amplitude of free vibration V(x), the natural frequency of the beam ω and the external dynamic distributed load applied q(x, t).

Figure 1. *Non uniform beam on two-layer elastic foundation under axial and transverse load.*

Substituting form Equations (13) and (14) into Equation (12) yields

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{d^2V}{dx^2}\right)\exp(i\omega t) + \rho A V\omega^2 \exp(i\omega t) + p\frac{\partial^2 V}{\partial x^2}\exp(i\omega t) + k_1\left(V\exp(i\omega t)\right) +$$

$$k_2\left(V^3\exp(3i\omega t)\right) - k_3\frac{d^2V}{dx^2}\exp(i\omega t) = q(x)\,\exp(i\omega t),\ \ 0 < x < L \qquad (15)$$

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{d^2V}{dx^2}\right) + p\frac{\partial^2 V}{\partial x^2} + k_1 V + k_2 V^3 - k_3\frac{d^2V}{dx^2} - q(x) = \rho A V\omega^2,\ \ \ \ 0 < x < L \qquad (16)$$

Through the normalization process we can transform Equation (16) into non-dimensional form as follows,

$$S(X).\frac{d^4W}{dX^4} + 2\frac{dS(X)}{dX}.\frac{d^3W}{dX^3} + \frac{dS^2(X)}{dX^2}.\frac{d^2W}{dX^2} + P\frac{d^2W(X)}{dX^2} + K_1.W + K_2.W^3 - K_3\frac{d^2W(X)}{dX^2} - F = \Omega^2.W \qquad (17)$$

The non-dimensional coefficients are:

$$W = \frac{V}{L}, \qquad X = \frac{x}{L}, \qquad P = \frac{pL^2}{EI_0}, \qquad K_1 = \frac{k_1 L^4}{EI_0}, \qquad K_2 = \frac{k_2 L^6}{EI_0},$$

$$K_3 = \frac{k_3 L^2}{EI_0}, \qquad \Omega^2 = \frac{\rho A L^4\omega}{EI_0}, \qquad S(X) = \frac{EI}{EI_0}, \qquad and\ \ F(X) = \frac{q(x)L^3}{EI_0}.$$

Where the non-dimensional deflection of the beam W, the non-dimensional axial loading P, the non-dimensional foundation linear stiffness K, the non-dimensional frequency of the beam Ω, the beam's flexural rigidity EI, the mass per unit length ρA and the inertia ratio S(X).

Equation (17) is a 4ᵗʰ order ordinary differential equation with inertia ratio $S(X) = (1 + \alpha_1 X)^{\alpha_2}$. In this section, we will study two cases of inertia ratio S(X); the first case $\alpha_1 = 0.5$, $\alpha_2 = 1$ and the second case $\alpha_1 = -1$, $\alpha_2 = 1$.

3. Solution of the Problem

The method of GDQ is employed to solve the problem. This method requires to descretize the domain of the problem into *N* pointes. Then the derivatives at any points are approximated by a weighted linear summation of all the functional values along the descretized domain, as follows [33-34]:

$$f_x(x_i) = \frac{df}{dx}\bigg|_{x_i} = \sum_{j=1}^{N} A_{ij}.f(x_j),\ \text{for i=1, 2, 3,..., N.} \qquad (18)$$

Where, A_{ij} represented the weighting coefficient, and *N* is the number of grid points in the whole domain. Equation (18) is called Differential Quadrature (DQ) technique. It should be noted that the weighting coefficients A_{ij} are different at different location of x_i. The key to DQ is to determine the weighting coefficients for the discretization of a derivative of any order.

The weighting coefficient can be determined by making use of Lagrange interpolation formula as follows:

$$g_k(x) = \frac{M(x)}{(x - x_k)M^{(1)}(x_k)},\ \text{where k = 1, 2,3,..., N} \qquad (19)$$

$$M(x) = (x - x_1)(x - x_2).......(x - x_N) \qquad (20)$$

$$M^{(1)}(x_i) = \prod_{k=1, k\neq i}^{N} (x_i - x_N) \qquad (21)$$

By applying Equation (19) at N grid points, they obtained the following algebraic formulations to compute the weighting coefficients A_{ij}.

$$A_{ij} = \frac{1}{x_i - x_j} \prod_{k=1, k \neq i, j}^{N} \frac{(x_i - x_k)}{(x_j - x_k)}, \quad j \neq i \tag{22}$$

$$A_{ii} = \sum_{k=1, k \neq i}^{N} \frac{1}{x_i - x_k} \tag{23}$$

For calculating the weighting coefficients of mth order

$$\left[A^{(m)} \right] = \left[A^{(1)} \right].\left[A^{(m-1)} \right] = \left[A^{(m-1)} \right].\left[A^{(1)} \right] \; m = 2, 3, 4,\ldots, N\text{-}1 \tag{24}$$

The accuracy of the results obtained by DQM, is affected by choosing of the number of grid points, N, and the type of sampling points, x_i. It is found that the optimal selection of the sampling grid points in the vibration problems, are chosen according to Gauss-Chebyshev-lobatto points [33-34],

$$X(i) = \frac{1}{2}\left[1 - \cos\left(\frac{i-1}{N-1}\pi \right) \right], \; i=1, 2, 3,\ldots, N \tag{25}$$

Applying the GDQ discretization scheme to the non-dimensional governing Equation (17) yields;

$$S(X)\left(\sum_{j=1}^{N} D_{ij}.W_j \right) + 2\, S^{(1)}(X_i)\left(\sum_{j=1}^{N} C_{ij}.W_j \right) + S^{(2)}(X_i)\left(\sum_{j=1}^{N} B_{ij}.W_j \right) + P\left(\sum_{j=1}^{N} B_{ij}.W_j \right) + K_1\, W_i +$$

$$K_2\, W_i^3 - K_3 \left(\sum_{j=1}^{N} B_{ij}\, W_j \right) - F = \Omega^2 W_i, \; i=1, 2, 3,\ldots, N. \tag{26}$$

Where W_i is the functional value at the grid points X_i. B_{ij}, C_{ij} and D_{ij} is the weighting coefficient matrix of the second, third and forth order derivatives.

Applying the GDQ discretization scheme to the boundary conditions are given by Equation (6) through (11) we obtain,

For Clamped–Clamped (C–C) yields;

$$W_1 = \sum_{j=1}^{N} A_{1j} W_j = 0 \tag{27}$$

$$W_N = \sum_{j=1}^{N} A_{Nj} W_j = 0 \tag{28}$$

For Simply–Simply (S–S) yields;

$$W_1 = \sum_{j=1}^{N} B_{1j} W_j = 0 \tag{29}$$

$$W_N = \sum_{j=1}^{N} B_{Nj} W_j = 0 \tag{30}$$

For Clamped–Simply (C–S) yields;

$$W_1 = \sum_{j=1}^{N} A_{1j} W_j = 0 \tag{31}$$

$$W_N = \sum_{j=1}^{N} B_{Nj} W_j = 0 \tag{32}$$

Using the method of direct Substitutions of the Boundary Conditions into the Governing Equations (SBCGE). The essence of the approach is that the Dirichlet condition is

implemented at the boundary points, while the derivative condition is discretized by the DQM. The discretized derivatives conditions at two ends are then combined to give the solutions W_2 and W_{N-1}. The expression for W_2 and W_{N-1} are then substituted into the discrete governing equation which is applied to the interior points $3 \leq i \geq N-2$. The dimension of the equation system using this approach is $(N-4)\times(N-4)$.

For clamped and simply supported end conditions, the discrete boundary conditions using the DQM can be written as:

$$W_1 = 0 \tag{33}$$

$$\sum_{k=1}^{N} C_{1,k}^{(n_0)}.W_k = 0 \tag{34}$$

$$W_N = 0 \tag{35}$$

$$\sum_{k=1}^{N} C_{N,k}^{(n_1)}.W_k = 0. \tag{36}$$

Where n_0 and n_1 may be taken as either 1 or 2. By choosing the value of n_0 and n_1, Equation (33) through (36) can give the following four sets of boundary conditions,

$n_0 = 1, n_1 = 1$ - clamped–clamped supported.
$n_0 = 1, n_1 = 2$ - clamped–simply supported.
$n_0 = 2, n_1 = 1$ - simply–clamped supported.
$n_0 = 2, n_1 = 2$ - simply supported–simply supported.

Equations (33) and (35) can be easily substituted into the governing equation. We can couple Equation (34) and (36) together to give two solutions, W_2 and W_{N-1}, as

$$W_2 = \frac{1}{AXN}.\sum_{k=3}^{N-2} AXK1.W_k, \tag{37}$$

$$W_{N-1} = \frac{1}{AXN} \cdot \sum_{k=3}^{N-2} AXKN \cdot W_k. \qquad (38)$$

Where;

$$AXK1 = C_{1,k}^{(n_0)} \cdot C_{N,N-1}^{(n_1)} - C_{1,N-1}^{(n_0)} \cdot C_{N,k}^{(n_1)}$$

$$AXKN = C_{1,2}^{(n_0)} \cdot C_{N,k}^{(n_1)} - C_{1,k}^{(n_0)} \cdot C_{N,2}^{(n_1)}$$

$$AXN = C_{N,2}^{(n_1)} \cdot C_{1,N-1}^{(n_0)} - C_{1,2}^{(n_0)} \cdot C_{N,N-1}^{(n_1)}$$

According to Equations (37) and (38), W_2 and W_{N-1} are expressed in terms of W_3, W_4,…, W_{N-2}, and can be easily substituted into the governing Equation (26). It should be noted that Equations (33) through (36) provides four boundary equations. In total, we have N unknowns W_2, W_3,…, W_N. In order to close the system, the discretized governing equation (26) has to be applied at $(N-4)$ mesh points. This can be done by applying Equation (26), at grid points X_3, X_4,…, X_{N-2}. Substituting Equations (33), (35), (37) and (38) into Equation (26) gives the system of equations (39).

$$S(X)\left(\sum_{j=3}^{N-2} D_{ij} . W_j \right) + 2.S^{(1)}(X_i).\left(\sum_{j=3}^{N-2} C_{ij} . W_j \right) + S^{(2)}(X_i).\left(\sum_{j=3}^{N-2} B_{ij} . W_j \right) +$$

$$P.\left(\sum_{j=3}^{N-2} B_{ij} . W_j \right) + K_1 . W_i + K_2 . W_i^3 - K_3 . \left(\sum_{j=3}^{N-2} B_{ij} . W_j \right) - F = \Omega^2 . W_i \quad (39)$$

Assuming that the external dynamic distributed load changes as the deflection amplitude change, then the governing equations system (39) can be written as

$$S(X)\left(\sum_{j=3}^{N-2} D_{ij} . W_j \right) + 2.S^{(1)}(X_i).\left(\sum_{j=3}^{N-2} C_{ij} . W_j \right) + S^{(2)}(X_i).\left(\sum_{j=3}^{N-2} B_{ij} . W_j \right) +$$

$$P.\left(\sum_{j=3}^{N-2} B_{ij} . W_j \right) + K_1 . W_i + K_2 . W_i^3 - K_3 . \left(\sum_{j=3}^{N-2} B_{ij} . W_j \right) - F . W_i = \Omega^2 . W_i \quad (40)$$

It is noted that Equations (40) has (N−4) equations with (N−4) unknowns, which can be written in matrix eigen-value form as

$$[A].\{W\} + K_2 .\{W\}^3 = \Omega^2 .\{W\} \qquad (41)$$

$$\{W\} = \{W_3, W_4, \ldots, W_{N-2}\}^T$$

4. Results

In this section forced vibration of Euler–Bernoulli of non-uniform beams resting on two layer elastic foundations under axial and transverse load is analyzed. The GDQM is used to compute the first three natural frequencies and the corresponding mode shapes for the forced vibration of non-uniform under axial and transverse force with two cases of

inertia ratio $S(X) = (1 + \alpha_1 X)^{\alpha_2}$, the first case: 1 $\alpha_1 = 0.5$, $\alpha_2 = 1.0$ and the second case: $\alpha_1 = -1.0$, $\alpha_2 = 1.0$, with three different types of end conditions. Fifteen non-uniformly spaced grid points were chosen by the previous relation.

4.1. Accuracy and Stability

In order to discuss the stability and accuracy of the GDQM, uniform beams are solved using the present approach for implementing the boundary conditions and the results are compared with the exact results available in the literature. The exact solutions are introduced as found in Qiang [34] and Blevins [35] for uniform beam $S(X) = 1.0$. The results are presented in Tables (1) through (3).

Tables (1) through (3) show the first three non-dimensional natural frequencies of uniform beam Clamped–Clamped Beam (C–C) Supported, Simply–Simply (S–S) Beam Supported and Clamped–Simply Beam (C–S) supported, respectively. Fifteen non-uniformly spaced grid points were chosen by the previous relation.

The absolute relative error typed in Tables (1) through (3) represents the accuracy of the GDQM. This absolute relative error can be defined by the formula, $\left| \frac{\text{Present-Exact}}{\text{Exact}} \times 100 \right|$. Examining the three tables.

Table 1. *The first three non-dimensional frequencies of uniform Clamped–Clamped beam.*

Natural Frequency	Ω1	Ω2	Ω3
Exact (Qiang [34], Blevins [35])	22.3733	61.6728	120.9034
Present (SBCGM)	22.3733	61. 6728	120.9021
Absolute relative error %	0.0000	0.0000	0.0011

Table 2. *The first three non-dimensional frequencies of uniform Simply–Simply beam.*

Natural Frequency	Ω1	Ω2	Ω3
Exact (Qiang [34], Blevins [35])	9.8696	39.4784	88.8264
Present (SBCGM)	9.8696	39.4784	88.8249
Absolute relative error %	0.0000	0.0000	0.0016

Table 3. *The first three non-dimensional frequencies of uniform Clamped–Simply beam.*

Natural Frequency	Ω1	Ω2	Ω3
Exact (Qiang [34], Blevins [35])	15.4182	49.9648	104.2477
Present (SBCGM)	15.4182	49.9648	104.2471
Absolute relative error %	0.0000	0.0000	0.0006

4.2. Results Using a Proposed Technique of GDQM

Tables (4) and (5) show the first three non-dimensional natural frequencies of non-uniform beam resting on two layer elastic foundations with two cases of inertia ratio $S(X) = (1 + \alpha_1 X)^{\alpha_2}$, the first case: 1, $\alpha_1 = 0.5$, $\alpha_2 = 1.0$ and the second case: 2, $\alpha_1 = -1.0$, $\alpha_2 = 1.0$, under the three sets of boundary conditions. Fifteen non-uniformly spaced grid points were chosen by the previous relation.

It can be observed from Table (4) and (5) that, the natural frequencies increase when the beam resting on two layer

elastic foundations.

The corresponding mode shapes are presented in Figures (2-7). Figures (2-4) for the first case of inertia ratio and Figures (5-7) for the second case of inertia ratio.

Case (1):

Table 4. *The first three non-dimensional frequencies of non-uniform supported beam (P = 1, F = 1, S(X) = (1 + 0.5X)).*
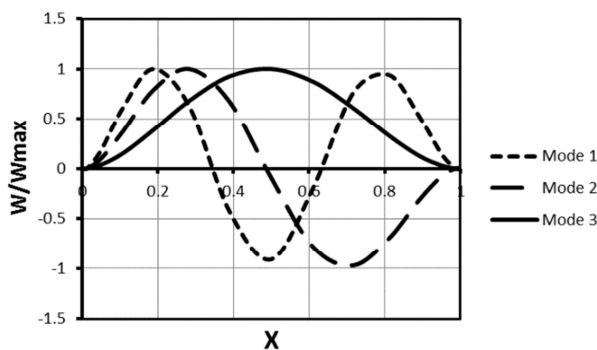
Foundation parameters	Natural Frequency (C-C)	Natural Frequency (S-S)	Natural Frequency (C-S)
K_1 = 1.0,	Ω_1 = 24.8883	Ω_1 = 11.0035	Ω_1 = 16.9279
K_2= 1.0,	Ω_2 = 68.6339	Ω_2 = 43.9763	Ω_2 = 55.3994
K_3= 1.0	Ω_3 =134.5731	Ω_3 = 98.9180	Ω_3 =115.8263

Case (2):

Table 5. *The first three non-dimensional frequencies of non-uniform supported beam (P = 1, F = 1, S(X) = (1 − X)).*

| Foundation parameters	Natural Frequency (C-C)	Natural Frequency (S-S)	Natural Frequency (C-S)
K_1 = 1.0,	Ω_1 = 11.9742	Ω_1 = 6.3249	Ω_1 = 10.3991
K_2= 1.0,	Ω_2 = 34.3839	Ω_2 = 24.6150	Ω_2 = 31.1571
K_3= 1.0	Ω_3 = 68.3635	Ω_3 = 53.5371	Ω_3 = 63.4720

Case (1):

Figure 2. *The first three mode shapes of non-uniform Clamped–Clamped supported beam (K_1= 1.0, K_2= 1.0, K_3= 1.0, P = 1, F = 1, S(X) = (1 + 0.5X)).*
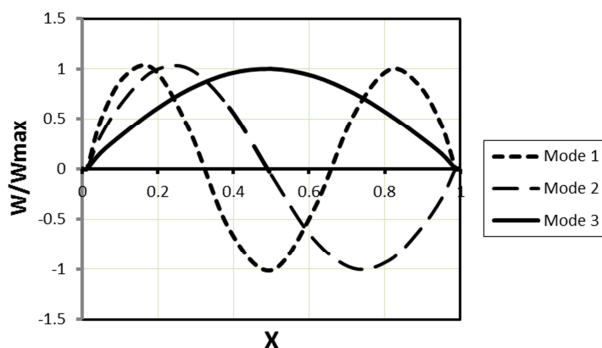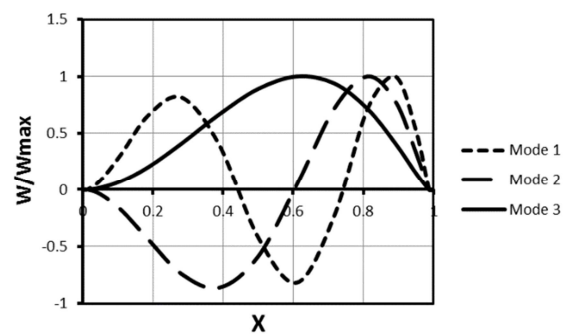
Figure 3. *The first three mode shapes of non-uniform Simply–Simply supported beam (K_1= 1.0, K_2= 1.0, K_3= 1.0, P=1, F=1, S(X) = (1 + 0.5X)).*
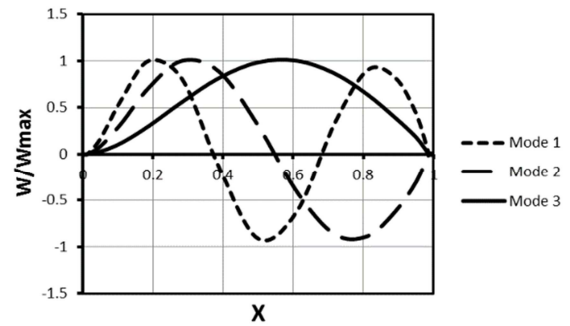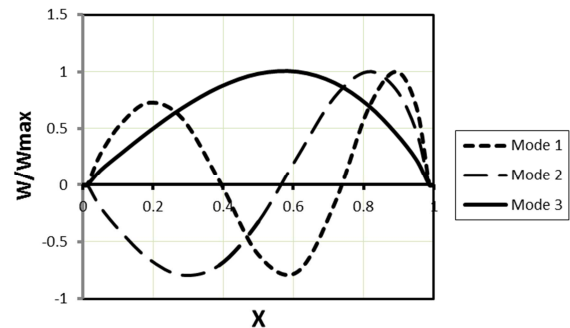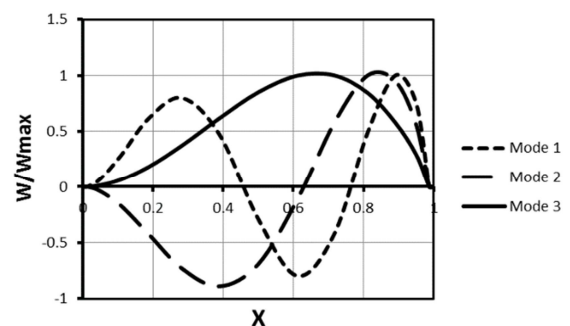
Figure 4. *The first three mode shapes of non-uniform Clamped–Simply supported beam (K_1= 1.0, K_2= 1.0, K_3= 1.0, P=1, F=1, S(X) = (1 + 0.5X)).*

Case (2):

Figure 5. *The first three mode shapes of non-uniform Clamped–Clamped supported beam (K_1= 1.0, K_2= 1.0, K_3= 1.0, P=1, F=1, S(X) = (1 − X)).*

Figure 6. *The first three mode shapes of non-uniform Simply – Simply supported beam (K_1= 1.0, K_2= 1.0, K_3= 1.0, P=1, F=1, S(X) = (1 − X)).*

Figure 7. *The first three mode shapes of non-uniform Clamped–Simply supported beam (K_1= 1.0, K_2= 1.0, K_3= 1.0, P=1, F=1, S(X) = (1 − X)).*

To examine the effect of the non-linear elastic foundation "K_2", we fix the other values of elastic foundations "K_1" and "K_3", the value of axial load "P" and the value of distributed dynamic force "F". Then draw "K_2" versus the natural frequency. It is clear that increasing the non-linear elastic foundation "K_2" increases the natural frequency of the beam, Figures (8-10) for the first case of inertia ratio and Figures (11-13) for the second case of inertia ratio.
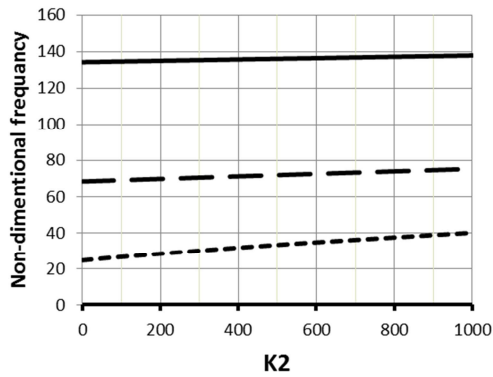
Case (1):

Figure 8. *The first three non-dimensional frequencies of non-uniform Clamped–Clamped beam with various K_2 (P=1.0, F=1.0, $S(X)=(1+0.5X)$).*
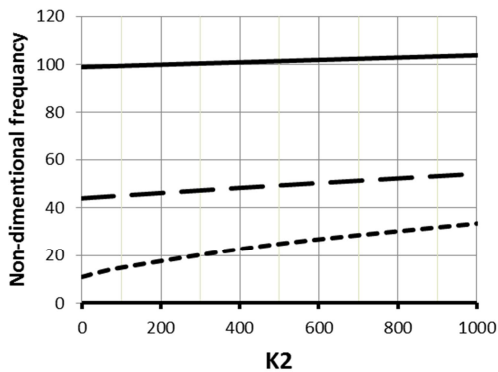
Figure 9. *The first three non-dimensional frequencies of non-uniform Simply –Simply beam with various K_2 (P=1.0, F=1.0, $S(X)=(1+0.5X)$).*
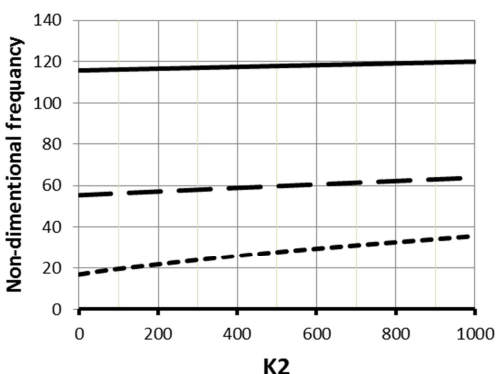
Figure 10. *The first three non-dimensional frequencies of non-uniform Clamped–Simply beam with various K_2 (P=1.0, F=1.0, $S(X)=(1+0.5X)$).*
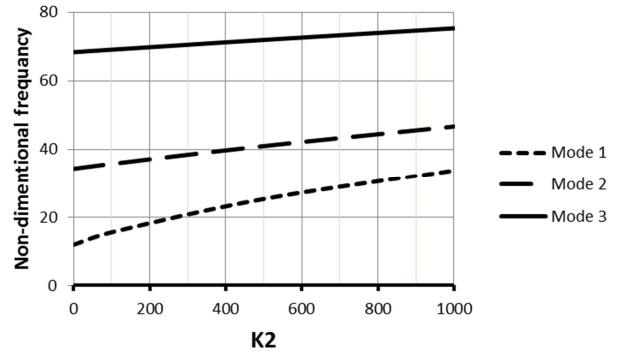
Case (2):

Figure 11. *The first three non-dimensional frequencies of non-uniform Clamped–Clamped beam with various K_2 (P=1.0, F=1.0, $S(X)=(1-X)$).*
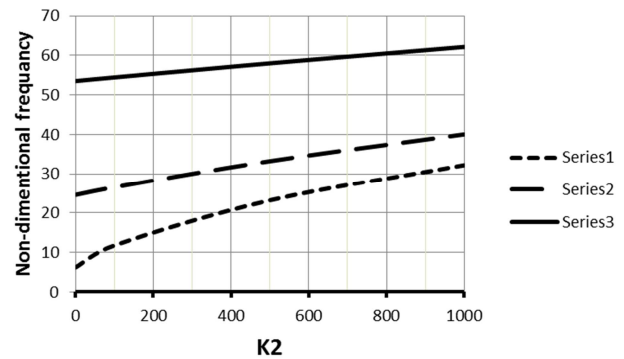
Figure 12. *The first three non-dimensional frequencies of non-uniform Simply –Simply beam with various K_2 (P=1.0, F=1.0, $S(X)=(1-X)$).*
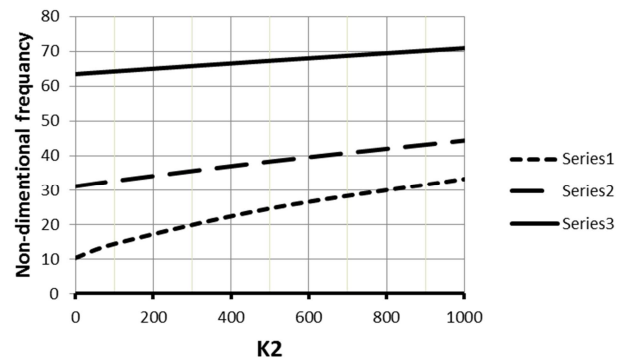
Figure 13. *The first three non-dimensional frequencies of non-uniform Clamped–Simply beam with various K_2 (P=1.0, F=1.0, $S(X)=(1-X)$).*

5. Conclusion

In this paper, an efficient algorithm based on a new combination of a GDQM and Newton's method presented for solving eigenvalue problems of non-uniform beams resting on two layer elastic foundations. Appropriate boundary conditions and the GDQM are applied to transform the partial differential equations of non-uniform beams resting on two layer elastic foundations into discrete eigenvalue problems. The results for various types of boundary conditions were compared with the results obtained by exact solution in case of uniform beam supported on elastic support.

From the parametric study of nonlinear elastic foundation of vibration analysis for various types of boundary conditions, that,

the natural frequency of the beam increases with increasing the nonlinear Winkler (normal) foundation parameter (K_2)

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