

Calculation of Perfect Bayesian Equilibrium in a Signaling Game

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To cite this article:

Yuming Xiao. Calculation of Perfect Bayesian Equilibrium in a Signaling Game. *Higher Education Research*. Vol. 8, No. 4, 2023, pp. 193-200. doi: 10.11648/j.her.20230804.20

Received: July 24, 2023; **Accepted:** August 21, 2023; **Published:** August 31, 2023

Abstract: In the teaching of dynamic games with incomplete information, the solution of perfect Bayesian equilibrium is a difficult point. Although there is no general solution to the general dynamic games with incomplete information, there is a general solution method to the signaling games. Most of the existing game theory textbooks do not give clear solving methods and specific calculation steps. This paper analyses how to calculate the perfect Bayesian equilibrium of an extensive-form signaling game through an example. In this paper, Gibbons' definition of perfect Bayesian equilibrium is taken as the definition of perfect Bayesian equilibrium. The definition attaches four requirements to Nash equilibrium. That is, in a dynamic game with incomplete information, the Nash equilibrium satisfying these four requirements is a perfect Bayesian equilibrium. The calculation process is divided into 4 steps. First, the belief hypothesis. Second, analysis of the signal receiver's strategy and the requirements for the posterior probability. Third, analysis of the signal sender's behavior according to different belief combinations. Finally, the posterior probabilities are analyzed using the requirements 3 and 4 in the perfect Bayesian equilibrium definition. On the basis of the previous analyses, the perfect Bayesian equilibria of the signaling game can be calculated step by step.

Keywords: Signaling Game, Perfect Bayesian Equilibrium (PBE), Requirement, Strategy, Belief

1. Introduction

In the learning or teaching of game theory, the concept of perfect Bayesian equilibrium (PBE) usually can be understood, but the solution of perfect Bayesian equilibrium is a difficult problem. The definition of perfect Bayesian equilibrium is given in many game theory textbooks, but it seems that few textbooks or papers give detailed solutions to this problem. For example, Weiyang Zhang [1], Guangming Hou and Cunjin Li [2] both give the same example and give the perfect Bayesian equilibria of the game. But they don't give the specific solution method. Other game theory textbooks (Gibbons [3]; Fudenberg and Tirole [4]; Osborne and Rubinstein [5]; Dutta [6]; Shiyu Xie [7]; Changde Zheng [8]) and papers (Fudenberg and Tirole [9]; Giacomo Bonanno [10]; Julio and Miguel [11]) don't either give the specific solution method. Of course, there are a few game theory textbooks, for example, Steven [12], Peters [13] and Rasmusen [14], which have analysed this problem. When many students study this section, they usually can understand

the concept of perfect Bayesian equilibrium, but they don't know how to solve the equilibrium.

This brings some confusion to the study and application of the dynamic game theory of incomplete information. Tirole thinks that there exists no general method. However, a few systematic tricks are of use in solving these games [15]. This paper considers that the role of this systematic technique may be more embodied in solving the perfect Bayesian equilibrium of the signaling games.

This paper attempts to illustrate how to solve the perfect Bayesian equilibrium of a dynamic game of incomplete information by an example.

A perfect Bayesian equilibrium is a set of strategies and beliefs such that, at any stage of the game, strategies are optimal given the beliefs, and the beliefs are obtained from equilibrium strategies and observed actions using Bayes' rule [9].

In order to make the calculation more operable in this paper, the definition of Gibbons is used as the definition of perfect Bayesian equilibrium.

2. Definition of Perfect Bayesian Equilibrium

The definition involves four requirements [3].

Requirement 1 At each information set, the player with the move must have a belief about which node in the information set has been reached by the play of the game.

For a non-singleton information set, a belief is a probability distribution over the nodes in the information set; for a singleton information set, the player's belief puts probability one on the single decision node.

Requirement 2 Given their beliefs, the players' strategies must be sequentially rational.

Requirement 3 At information sets on the equilibrium path, beliefs are determined by Bayes' rule and the players' equilibrium strategies.

Requirement 4 At information sets off the equilibrium path, beliefs are determined by Bayes' rule and the players' equilibrium strategies where possible.

Requirement 1 implies that if the play of the game reaches certain player's non-singleton information set, then the player must have a belief about which node has been reached. Requirement 2 implies that given the player's belief, how the player should choose his own strategies. Requirements 3 and 4 stipulate how the belief should be inferred.

Definition A perfect Bayesian equilibrium consists of strategies and beliefs satisfying Requirements 1 through 4.

3. Calculation of Perfect Bayesian Equilibrium

This section uses the example mentioned in the introduction to illustrate how to calculate the perfect Bayesian equilibrium that satisfies the above 4 requirements (only pure strategies are discussed in this paper).

Consider the following signaling game (market entry

game).

Suppose that there are two stages and two firms in the signaling game. Firm 1, the incumbent, is a monopoly at the first stage and chooses a price. Firm 2, the entrant, then decides whether to enter the market at the second stage after observing the price of firm 1. If firm 2 enters, there is duopolistic competition between the two firms at stage 2. Otherwise, firm 1 remains a monopoly.

Information is asymmetric: firm 1 knows its cost type from the start. Firm 2 does not know firm 1's specific cost type, but knows that firm 1 has two possible types: high cost (with probability μ) or low cost (with probability $1-\mu$). These two types are denoted by t_1 and t_2 , respectively. Suppose that the entrant has only one type: in the case of entry, its cost is the same as the cost of the high cost incumbent.

At the first stage, firm 1, as a monopolistic incumbent, decides its price. Suppose that it has three prices to choose: $p=4$, $p=5$ or $p=6$. If the incumbent's cost is high, the profits corresponding to the three prices are 2, 6 or 7; if the incumbent's cost is low, the profits corresponding to the three prices are 6, 9 or 8. Therefore, the single stage optimal monopoly price of the high cost incumbent is $p=6$, and the single stage optimal monopoly price of the low cost incumbent is $p=5$.

At the second stage, if the entrant enters, the cost function of the incumbent becomes common knowledge. If the incumbent is a high cost type, the cost functions of the two firms are the same. The price of symmetrical Cournot model is $p=5$, the profit of each firm is 3. Deducting the entry cost 2, the net profit of the entrant is 1. If the incumbent is low cost type, the cost functions of the two firms are different. The price of asymmetrical Cournot model is $p=4$, the profit of firm 1 is 5, deducting the entry cost 2, the net profit of the entrant is -1. If firm 2 does not enter, firm 1 remains a monopoly, the profits under different prices are the same as the first stage's. So, under such hypotheses, in the complete information case, the incumbent would enter if and only if firm 1's cost is high.

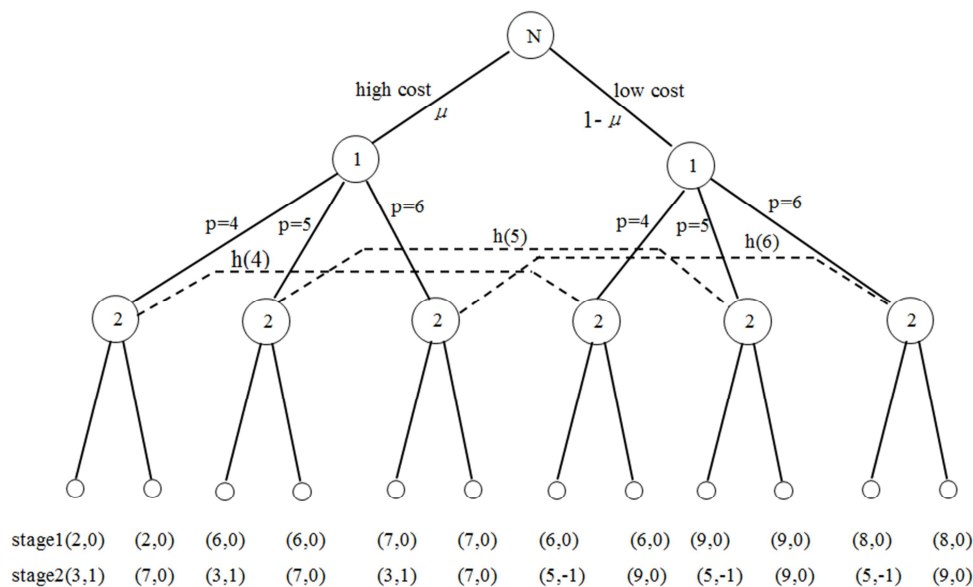


Figure 1. Market entry game.

The above game can be depicted in Figure 1.

Obviously, this is a signaling game.

Let $T=\{t_1, t_2\}$ denotes the type set of the incumbent, where t_1 and t_2 denote high cost and low cost types, respectively. $M=\{4, 5, 6\}$ denotes the signal set of the incumbent, where 4, 5 and 6 denote prices. $A=\{u, d\}$ denotes the action set of the entrant, where u and d represent entry and exit, respectively.

We can calculate the perfect Bayesian equilibrium of the signaling game according to the following steps.

Step 1 belief hypothesis

Suppose that firm 2 has beliefs p_1, p_2 and p_3 in its three information sets (denoted by $h(4), h(5)$ and $h(6)$)

$$\begin{aligned} \max_{a_k \in A} Eu_R[t_i, 4, a_k(4)] &= \max\{p_1 \times 1 + (1 - p_1) \times (-1), p_1 \times 0 + (1 - p_1) \times 0\} \\ &= \max\{2p_1 - 1, 0\} = \begin{cases} 2p_1 - 1 & p_1 \geq 1/2 \\ 0 & p_1 \leq 1/2 \end{cases} \end{aligned} \quad (2)$$

The receiver's optimal actions can be deduced.

$$a^*(4) = \begin{cases} u & p_1 \geq 1/2 \\ d & p_1 \leq 1/2 \end{cases} \quad (3)$$

In the same way, when $p=5$ and $p=6$ are observed, his decisions are the same as $a^*(4)$.

$$a^*(5) = a^*(6) = \begin{cases} u & p_i \geq 1/2 \\ d & p_i \leq 1/2 \end{cases} \quad i=2, 3 \quad (4)$$

respectively), these beliefs are depicted in Figure 1.

Step 2 Analysis of the behavior choice of the receiver (entrant)

Now, we analyse the actions of the receiver and the requirements for the posterior probability.

For the receiver (firm 2), when he receives the signal $p=4$, his action, $a^*(4)$, should maximize his expected returns. That is, $a^*(4)$ is the solution of the following equation.

$$a^*(4) \in \arg \max_{a_k \in A} Eu_R[t_i, 4, a_k(4)] \quad (1)$$

From equations (3) and (4), If the incumbent considers the posterior probability $p_i \geq 1/2$ ($i=1, 2, 3$), no matter what price the incumbent chooses in the first stage, the entrant will choose to enter. Otherwise, he will choose not to enter.

Step 3 Analysis of the behavior choice of the sender (incumbent)

The following different eight cases are discussed according to step 2. In table 1, eight belief combinations are listed.

Table 1. 8 combinations of belief.

	p_1	p_2	p_3
1	$\geq 1/2$	$\geq 1/2$	$\geq 1/2$
2	$\geq 1/2$	$\geq 1/2$	$\leq 1/2$
3	$\geq 1/2$	$\leq 1/2$	$\geq 1/2$
4	$\leq 1/2$	$\geq 1/2$	$\geq 1/2$
5	$\leq 1/2$	$\leq 1/2$	$\geq 1/2$
6	$\leq 1/2$	$\geq 1/2$	$\leq 1/2$
7	$\geq 1/2$	$\leq 1/2$	$\leq 1/2$
8	$\leq 1/2$	$\leq 1/2$	$\leq 1/2$

When nature endows the incumbent with type t_i ($i=1, 2$), under the condition of inferring the receiver's optimal action $a^*(t_i)$, The sender selects the signal $m^*(t_i)$ to maximize its own revenue. that is, $m^*(t_i)$ is the solution of the following equation.

$$m^*(t_i) \in \arg \max_{m_j \in M} u_s[t_i, m_j, a^*(m_j)] \quad (5)$$

$$\begin{aligned} &\max_{m_j \in \{4,5,6\}} u_s[t_1, m_j, a^*(m_j)] \\ &= \begin{cases} \max\{u_s(t_1, 4, u), u_s(t_1, 5, u), u_s(t_1, 6, u)\} & p_1 \geq 1/2, p_2 \geq 1/2, p_3 \geq 1/2 \\ \max\{u_s(t_1, 4, d), u_s(t_1, 5, d), u_s(t_1, 6, d)\} & p_1 \leq 1/2, p_2 \leq 1/2, p_3 \leq 1/2 \end{cases} \\ &= \begin{cases} \max\{5, 9, 10\} & p_1 \geq 1/2, p_2 \geq 1/2, p_3 \geq 1/2 \\ \max\{9, 13, 14\} & p_1 \leq 1/2, p_2 \leq 1/2, p_3 \leq 1/2 \end{cases} \end{aligned}$$

Where $m_j \in M = \{4, 5, 6\}, j=1, 2, 3$.

For the type of t_1 , according to different combinations of beliefs, this problem is discussed in eight different cases.

(1) $p_1 \geq 1/2, p_2 \geq 1/2, p_3 \geq 1/2$ or $p_1 \leq 1/2, p_2 \leq 1/2, p_3 \leq 1/2$

Since

The sender's optimal signals can be deduced.

$$m^*(t_1) = \begin{cases} 6 & p_1 \geq 1/2, p_2 \geq 1/2, p_3 \geq 1/2 \\ 6 & p_1 \leq 1/2, p_2 \leq 1/2, p_3 \leq 1/2 \end{cases} \quad (6)$$

From equation (6), If $p_1 \geq 1/2, p_2 \geq 1/2, p_3 \geq 1/2$ or $p_1 \leq 1/2, p_2 \leq 1/2, p_3 \leq 1/2$, the sender will choose high price, i. e. $p=6$, in the first stage.

(2) $p_1 \geq 1/2, p_2 \geq 1/2, p_3 \leq 1/2$ or $p_1 \leq 1/2, p_2 \leq 1/2, p_3 \geq 1/2$

Since

$$\begin{aligned} & \max_{m_j \in \{4,5,6\}} u_s[t_1, m_j, a^*(m_j)] \\ &= \begin{cases} \max\{u_s(t_1, 4, u), u_s(t_1, 5, u), u_s(t_1, 6, d)\} & p_1 \geq 1/2, p_2 \geq 1/2, p_3 \leq 1/2 \\ \max\{u_s(t_1, 4, d), u_s(t_1, 5, d), u_s(t_1, 6, u)\} & p_1 \leq 1/2, p_2 \leq 1/2, p_3 \geq 1/2 \end{cases} \\ &= \begin{cases} \max\{5, 9, 14\} & p_1 \geq 1/2, p_2 \geq 1/2, p_3 \leq 1/2 \\ \max\{9, 13, 10\} & p_1 \leq 1/2, p_2 \leq 1/2, p_3 \geq 1/2 \end{cases} \end{aligned}$$

The sender's optimal signals can be deduced.

$$m^*(t_1) = \begin{cases} 6 & p_1 \geq 1/2, p_2 \geq 1/2, p_3 \leq 1/2 \\ 5 & p_1 \leq 1/2, p_2 \leq 1/2, p_3 \geq 1/2 \end{cases} \quad (7)$$

From equation (7), If $p_1 \geq 1/2, p_2 \geq 1/2, p_3 \leq 1/2$, the sender will choose $p=6$. if $p_1 \leq 1/2, p_2 \leq 1/2, p_3 \geq 1/2$, the sender will choose $p=5$ in the first stage.

(3) $p_1 \geq 1/2, p_2 \leq 1/2, p_3 \geq 1/2$ or $p_1 \leq 1/2, p_2 \geq 1/2, p_3 \leq 1/2$
Since

$$\begin{aligned} & \max_{m_j \in \{4,5,6\}} u_s[t_1, m_j, a^*(m_j)] \\ &= \begin{cases} \max\{u_s(t_1, 4, u), u_s(t_1, 5, d), u_s(t_1, 6, u)\} & p_1 \geq 1/2, p_2 \leq 1/2, p_3 \geq 1/2 \\ \max\{u_s(t_1, 4, d), u_s(t_1, 5, u), u_s(t_1, 6, d)\} & p_1 \leq 1/2, p_2 \geq 1/2, p_3 \leq 1/2 \end{cases} \\ &= \begin{cases} \max\{5, 13, 10\} & p_1 \geq 1/2, p_2 \leq 1/2, p_3 \geq 1/2 \\ \max\{9, 9, 14\} & p_1 \leq 1/2, p_2 \geq 1/2, p_3 \leq 1/2 \end{cases} \end{aligned}$$

The sender's optimal signals can be deduced.

$$m^*(t_1) = \begin{cases} 5 & p_1 \geq 1/2, p_2 \leq 1/2, p_3 \geq 1/2 \\ 6 & p_1 \leq 1/2, p_2 \geq 1/2, p_3 \leq 1/2 \end{cases} \quad (8)$$

From equation (8), If $p_1 \geq 1/2, p_2 \leq 1/2, p_3 \geq 1/2$, the sender will choose $p=5$. if $p_1 \leq 1/2, p_2 \geq 1/2, p_3 \leq 1/2$, the sender will choose $p=6$ in the first stage.

(4) $p_1 \leq 1/2, p_2 \geq 1/2, p_3 \geq 1/2$ or $p_1 \geq 1/2, p_2 \leq 1/2, p_3 \leq 1/2$

Since

$$\begin{aligned} & \max_{m_j \in \{4,5,6\}} u_s[t_1, m_j, a^*(m_j)] \\ &= \begin{cases} \max\{u_s(t_1, 4, d), u_s(t_1, 5, u), u_s(t_1, 6, u)\} & p_1 \leq 1/2, p_2 \geq 1/2, p_3 \geq 1/2 \\ \max\{u_s(t_1, 4, u), u_s(t_1, 5, d), u_s(t_1, 6, d)\} & p_1 \geq 1/2, p_2 \leq 1/2, p_3 \leq 1/2 \end{cases} \\ &= \begin{cases} \max\{9, 9, 10\} & p_1 \leq 1/2, p_2 \geq 1/2, p_3 \geq 1/2 \\ \max\{5, 13, 14\} & p_1 \geq 1/2, p_2 \leq 1/2, p_3 \leq 1/2 \end{cases} \end{aligned}$$

The sender's optimal signals can be deduced.

$$m^*(t_1) = \begin{cases} 6 & p_1 \leq 1/2, p_2 \geq 1/2, p_3 \geq 1/2 \\ 6 & p_1 \geq 1/2, p_2 \leq 1/2, p_3 \leq 1/2 \end{cases} \quad (9)$$

For the type of t_2 , according to different combinations of beliefs, this problem is discussed in eight different cases.
(1) $p_1 \geq 1/2, p_2 \geq 1/2, p_3 \geq 1/2$ or $p_1 \leq 1/2, p_2 \leq 1/2, p_3 \leq 1/2$
Since

From equation (9), If $p_1 \leq 1/2, p_2 \geq 1/2, p_3 \geq 1/2$ or $p_1 \geq 1/2, p_2 \leq 1/2, p_3 \leq 1/2$, the sender will choose high price, i.e. $p=6$, in the first stage.

$$\begin{aligned} & \max_{m_j \in \{4,5,6\}} u_s[t_2, m_j, a^*(m_j)] \\ &= \begin{cases} \max\{u_s(t_2, 4, u), u_s(t_2, 5, u), u_s(t_2, 6, u)\} & p_1 \geq 1/2, p_2 \geq 1/2, p_3 \geq 1/2 \\ \max\{u_s(t_2, 4, d), u_s(t_2, 5, d), u_s(t_2, 6, d)\} & p_1 \leq 1/2, p_2 \leq 1/2, p_3 \leq 1/2 \end{cases} \\ &= \begin{cases} \max\{11, 14, 13\} & p_1 \geq 1/2, p_2 \geq 1/2, p_3 \geq 1/2 \\ \max\{15, 18, 17\} & p_1 \leq 1/2, p_2 \leq 1/2, p_3 \leq 1/2 \end{cases} \end{aligned}$$

The sender's optimal signals can be deduced.

$$m^*(t_2) = \begin{cases} 5 & p_1 \geq 1/2, p_2 \geq 1/2, p_3 \geq 1/2 \\ 5 & p_1 \leq 1/2, p_2 \leq 1/2, p_3 \leq 1/2 \end{cases} \quad (10)$$

From equation (10), If $p_1 \geq 1/2, p_2 \geq 1/2, p_3 \geq 1/2$ or $p_1 \leq 1/2, p_2 \leq 1/2, p_3 \leq 1/2$, the sender will choose $p=5$ in the first stage.

(2) $p_1 \geq 1/2, p_2 \geq 1/2, p_3 \leq 1/2$ or $p_1 \leq 1/2, p_2 \leq 1/2, p_3 \geq 1/2$
Since

$$\begin{aligned} & \max_{m_j \in \{4,5,6\}} u_s[t_2, m_j, a^*(m_j)] \\ &= \begin{cases} \max\{u_s(t_2, 4, u), u_s(t_2, 5, u), u_s(t_2, 6, d)\} & p_1 \geq 1/2, p_2 \geq 1/2, p_3 \leq 1/2 \\ \max\{u_s(t_2, 4, d), u_s(t_2, 5, d), u_s(t_2, 6, u)\} & p_1 \leq 1/2, p_2 \leq 1/2, p_3 \geq 1/2 \end{cases} \\ &= \begin{cases} \max\{11, 14, 17\} & p_1 \geq 1/2, p_2 \geq 1/2, p_3 \leq 1/2 \\ \max\{15, 18, 13\} & p_1 \leq 1/2, p_2 \leq 1/2, p_3 \geq 1/2 \end{cases} \end{aligned}$$

The sender's optimal signals can be deduced.

$$m^*(t_2) = \begin{cases} 6 & p_1 \geq 1/2, p_2 \geq 1/2, p_3 \leq 1/2 \\ 5 & p_1 \leq 1/2, p_2 \leq 1/2, p_3 \geq 1/2 \end{cases} \quad (11)$$

From equation (11), If $p_1 \geq 1/2, p_2 \geq 1/2, p_3 \leq 1/2$, the sender will choose $p=6$. if $p_1 \leq 1/2, p_2 \leq 1/2, p_3 \geq 1/2$, the sender will choose $p=5$ in the first stage.

(3) $p_1 \geq 1/2, p_2 \leq 1/2, p_3 \geq 1/2$ or $p_1 \leq 1/2, p_2 \geq 1/2, p_3 \leq 1/2$
Since

$$\begin{aligned} & \max_{m_j \in \{4,5,6\}} u_s[t_2, m_j, a^*(m_j)] \\ &= \begin{cases} \max\{u_s(t_2, 4, u), u_s(t_2, 5, d), u_s(t_2, 6, u)\} & p_1 \geq 1/2, p_2 \leq 1/2, p_3 \geq 1/2 \\ \max\{u_s(t_2, 4, d), u_s(t_2, 5, u), u_s(t_2, 6, d)\} & p_1 \leq 1/2, p_2 \geq 1/2, p_3 \leq 1/2 \end{cases} \\ &= \begin{cases} \max\{11, 18, 13\} & p_1 \geq 1/2, p_2 \leq 1/2, p_3 \geq 1/2 \\ \max\{15, 14, 17\} & p_1 \leq 1/2, p_2 \geq 1/2, p_3 \leq 1/2 \end{cases} \end{aligned}$$

The sender's optimal signals can be deduced.

$$m^*(t_2) = \begin{cases} 5 & p_1 \geq 1/2, p_2 \leq 1/2, p_3 \geq 1/2 \\ 6 & p_1 \leq 1/2, p_2 \geq 1/2, p_3 \leq 1/2 \end{cases} \quad (12)$$

From equation (12), If $p_1 \geq 1/2, p_2 \leq 1/2, p_3 \geq 1/2$, the sender will choose $p=5$. if $p_1 \leq 1/2, p_2 \geq 1/2, p_3 \leq 1/2$, the sender will

choose $p=6$ in the first stage.

(4) $p_1 \leq 1/2, p_2 \geq 1/2, p_3 \geq 1/2$ or $p_1 \geq 1/2, p_2 \leq 1/2, p_3 \leq 1/2$

Since

$$\begin{aligned} & \max_{m_j \in \{4,5,6\}} u_s[t_2, m_j, a^*(m_j)] \\ &= \begin{cases} \max\{u_s(t_2, 4, d), u_s(t_2, 5, u), u_s(t_2, 6, u)\} & p_1 \leq 1/2, p_2 \geq 1/2, p_3 \geq 1/2 \\ \max\{u_s(t_2, 4, u), u_s(t_2, 5, d), u_s(t_2, 6, d)\} & p_1 \geq 1/2, p_2 \leq 1/2, p_3 \leq 1/2 \end{cases} \\ &= \begin{cases} \max\{15, 14, 13\} & p_1 \leq 1/2, p_2 \geq 1/2, p_3 \geq 1/2 \\ \max\{11, 18, 17\} & p_1 \geq 1/2, p_2 \leq 1/2, p_3 \leq 1/2 \end{cases} \end{aligned}$$

The sender's optimal signals can be deduced.

$$m^*(t_2) = \begin{cases} 4 & p_1 \leq 1/2, p_2 \geq 1/2, p_3 \geq 1/2 \\ 5 & p_1 \geq 1/2, p_2 \leq 1/2, p_3 \leq 1/2 \end{cases} \quad (13)$$

From equation (13), If $p_1 \leq 1/2, p_2 \geq 1/2, p_3 \geq 1/2$ the sender will choose $p=4$, if $p_1 \geq 1/2, p_2 \leq 1/2, p_3 \leq 1/2$, the sender will choose $p=5$ in the first stage.

The above analyses are summarized as following:

Receiver's strategies

$$a^*(4) = \begin{cases} u & p_1 \geq 1/2 \\ d & p_1 \leq 1/2 \end{cases}$$

$$a^*(5) = \begin{cases} u & p_2 \geq 1/2 \\ d & p_2 \leq 1/2 \end{cases}$$

$$a^*(6) = \begin{cases} u & p_3 \geq 1/2 \\ d & p_3 \leq 1/2 \end{cases}$$

Sender's strategies

$$(1) m^*(t_1) = \begin{cases} 6 & p_1 \geq 1/2, p_2 \geq 1/2, p_3 \geq 1/2 \\ 6 & p_1 \leq 1/2, p_2 \leq 1/2, p_3 \leq 1/2 \end{cases}$$

$$m^*(t_2) = \begin{cases} 5 & p_1 \geq 1/2, p_2 \geq 1/2, p_3 \geq 1/2 \\ 5 & p_1 \leq 1/2, p_2 \leq 1/2, p_3 \leq 1/2 \end{cases}$$

$$(2) m^*(t_1) = \begin{cases} 6 & p_1 \geq 1/2, p_2 \geq 1/2, p_3 \leq 1/2 \\ 5 & p_1 \leq 1/2, p_2 \leq 1/2, p_3 \geq 1/2 \end{cases}$$

$$m^*(t_2) = \begin{cases} 6 & p_1 \geq 1/2, p_2 \geq 1/2, p_3 \leq 1/2 \\ 5 & p_1 \leq 1/2, p_2 \leq 1/2, p_3 \geq 1/2 \end{cases}$$

$$(3) m^*(t_1) = \begin{cases} 5 & p_1 \geq 1/2, p_2 \leq 1/2, p_3 \geq 1/2 \\ 6 & p_1 \leq 1/2, p_2 \geq 1/2, p_3 \leq 1/2 \end{cases}$$

$$m^*(t_2) = \begin{cases} 5 & p_1 \geq 1/2, p_2 \leq 1/2, p_3 \geq 1/2 \\ 6 & p_1 \leq 1/2, p_2 \geq 1/2, p_3 \leq 1/2 \end{cases}$$

$$(4) m^*(t_1) = \begin{cases} 6 & p_1 \leq 1/2, p_2 \geq 1/2, p_3 \geq 1/2 \\ 6 & p_1 \geq 1/2, p_2 \leq 1/2, p_3 \leq 1/2 \end{cases}$$

$$m^*(t_2) = \begin{cases} 4 & p_1 \leq 1/2, p_2 \geq 1/2, p_3 \geq 1/2 \\ 5 & p_1 \geq 1/2, p_2 \leq 1/2, p_3 \leq 1/2 \end{cases}$$

Therefore, the eight preliminary results are obtained.

Step 4 Analysis of requirements for the posterior probability

It can be seen from the above analysis that we obtain the eight preliminary results without considering the requirements 3 and 4. On this basis, the requirements 3 and 4 will be considered. We analyse these eight strategy combinations and beliefs one by one, so as to obtain the final analysis results.

(1) $m^*(t_1)=6, m^*(t_2)=5, a^*(4)=a^*(5)=a^*(6)=u; p_1 \geq 1/2, p_2 \geq 1/2, p_3 \geq 1/2$

i.e. strategy combination $((6, 5), (u, u, u))$ and belief $p_1 \geq 1/2, p_2 \geq 1/2, p_3 \geq 1/2$.

The strategy combination means

$$p(4 | t_1) = 0, p(5 | t_1) = 0, p(6 | t_1) = 1, p(4 | t_2) = 0, p(5 | t_2) = 1, p(6 | t_2) = 0$$

According to the given equilibrium strategy, the information sets $h(5)$ and $h(6)$ are on the equilibrium paths, but $h(4)$ is not on the equilibrium path.

According to requirement 3

$$p_2 = p(t_1 | 5) = \frac{p(5 | t_1) \cdot p(t_1)}{p(5 | t_1) \cdot p(t_1) + p(5 | t_2) \cdot p(t_2)} = \frac{0 \times \mu}{0 \times \mu + 1 \times (1 - \mu)} = 0$$

$$p_3 = p(t_1 | 6) = \frac{p(6 | t_1) \cdot p(t_1)}{p(6 | t_1) \cdot p(t_1) + p(6 | t_2) \cdot p(t_2)} = \frac{1 \times \mu}{1 \times \mu + 0 \times (1 - \mu)} = 1$$

This result, $p_2=0$, is not in conformity with $p_2 \geq 1/2$, so this strategy combination and belief is not a perfect Bayesian equilibrium.

(2) strategy combination $((6, 6), (u, u, d))$ and belief $p_1 \geq 1/2, p_2 \geq 1/2, p_3 \leq 1/2$.

The strategy combination means

$$p(4 | t_1) = 0, p(5 | t_1) = 0, p(6 | t_1) = 1, p(4 | t_2) = 0, p(5 | t_2) = 0, p(6 | t_2) = 1$$

According to the given equilibrium strategy, the information set $h(6)$ is on the equilibrium path, but $h(4)$ and $h(5)$ are not on the equilibrium paths.

(5) are not on the equilibrium paths.

According to requirement 3

$$p_3 = p(t_1 | 6) = \frac{p(6 | t_1) \cdot p(t_1)}{p(6 | t_1) \cdot p(t_1) + p(6 | t_2) \cdot p(t_2)} = \frac{1 \times \mu}{1 \times \mu + 1 \times (1 - \mu)} = \mu$$

Here, because the sender's strategy is pooling, the signal does not bring the receiver new useful information, so the posterior probability $p(t_1|6)$ is equal to the prior probability μ .

Since $p(4|t_1)=0, p(4|t_2)=0, p(5|t_1)=0$ and $p(5|t_2)=0, p(t_1|4), p(t_2|4), p(t_1|5), p(t_2|5)$ cannot be calculated by the Bayes formula.

According to Requirement 4, the inference of the information set, which is off the equilibrium path, must conform the player's equilibrium strategy.

We now analyse how the receiver should infer $p(t_1|4), p(t_2|4)$ when the game reaches information set $h(4)$.

If the receiver infers $p_1=0$, i.e. $1-p_1=1$, then entrance is obviously not the optimal choice. And this inference is incompatible with the preliminary analysis result ($p_1 \geq 1/2$) either.

If the receiver infers $p_1=1$, the inference is both consistent with his equilibrium strategy and compatible with the preliminary analysis result ($p_1 \geq 1/2$). So in the given equilibrium strategy combination, the receiver will infer $p_1=1$ in information set $h(4)$.

In the same way, in the given equilibrium strategy combination, the receiver will infer $p_2=1$ in information set $h(5)$.

Therefore, if μ is not greater than $1/2$, strategy combination $((6, 6), (u, u, d))$ and belief $p_1=1, p_2=1, p_3=\mu$ is a perfect Bayesian equilibrium.

(3) strategy combination $((5, 5), (u, d, u))$ and belief $p_1 \geq 1/2, p_2 \leq 1/2, p_3 \geq 1/2$.

The strategy combination means

$$p(4 | t_1) = 0, p(5 | t_1) = 1, p(6 | t_1) = 0, p(4 | t_2) = 0, p(5 | t_2) = 1, p(6 | t_2) = 0$$

According to the given equilibrium strategy, the information set $h(5)$ is on the equilibrium path, but $h(4)$ and $h(6)$ are not on the equilibrium path.

According to requirement 3

$$p_2 = p(t_1 | 5) = \frac{p(5 | t_1) \cdot p(t_1)}{p(5 | t_1) \cdot p(t_1) + p(5 | t_2) \cdot p(t_2)} = \frac{1 \times \mu}{1 \times \mu + 1 \times (1 - \mu)} = \mu$$

Since $p(4 | t_1) = 0, p(4 | t_2) = 0, p(6 | t_1) = 0, p(6 | t_2) = 0, p(t_1 | 4), p(t_2 | 4), p(t_1 | 6), p(t_2 | 6)$ can not be calculated by the Bayes formula.

According to Requirement 4, the inference of the information set, which is off the equilibrium path, must conform the player's equilibrium strategy.

We now analyse how the receiver should infer $p(t_1 | 4), p(t_2 | 4)$ when the game reaches information set $h(4)$.

If the receiver infers $p_1=0$, i.e. $1-p_1=1$, then entrance is obviously not the optimal choice. And this inference is incompatible with the preliminary analysis result ($p_1 \geq 1/2$) either.

If the receiver infers $p_1=1$, the inference is both consistent with his equilibrium strategy and compatible with the preliminary analysis result ($p_1 \geq 1/2$). So in the given equilibrium strategy combination, the receiver will infer $p_1=1$ in information set $h(4)$.

In the same way, in the given equilibrium strategy combination, the receiver will infer $p_3=1$ in information set $h(6)$.

Therefore, if μ is not greater than $1/2$, strategy combination

$((5, 5), (u, d, u))$ and belief $p_1=1, p_2=\mu, p_3=1$ is a perfect Bayesian equilibrium.

Similar to cases (2) and (3), for cases (4), (5) and (6), we can obtain the following results.

(4) Strategy combination $((6, 4), (d, u, u))$ and belief $p_1=0, p_2=1, p_3=1$ is a perfect Bayesian equilibrium.

(5) if μ is not greater than $1/2$, strategy combination $((5, 5), (d, d, u))$ and belief $p_1=0, p_2=\mu, p_3=1$ is a perfect Bayesian equilibrium.

(6) if μ is not greater than $1/2$, strategy combination $((6, 6), (d, u, d))$ and belief $p_1=0, p_2=1, p_3=\mu$ is a perfect Bayesian equilibrium.

(7) strategy combination $((6, 5), (u, d, d))$ and belief $p_1 \geq 1/2, p_2 \leq 1/2, p_3 \leq 1/2$.

The strategy combination means

$$p(4 | t_1) = 0, p(5 | t_1) = 0, p(6 | t_1) = 1, p(4 | t_2) = 0, p(5 | t_2) = 1, p(6 | t_2) = 0$$

According to the given equilibrium strategy, the information sets $h(5)$ and $h(6)$ are on the equilibrium path,

but $h(4)$ is not on the equilibrium paths.

According to requirement 3

$$p_2 = p(t_1 | 5) = \frac{p(5 | t_1) \cdot p(t_1)}{p(5 | t_1) \cdot p(t_1) + p(5 | t_2) \cdot p(t_2)} = \frac{0 \times \mu}{0 \times \mu + 1 \times (1 - \mu)} = 0$$

$$p_3 = p(t_1 | 6) = \frac{p(6 | t_1) \cdot p(t_1)}{p(6 | t_1) \cdot p(t_1) + p(6 | t_2) \cdot p(t_2)} = \frac{1 \times \mu}{1 \times \mu + 0 \times (1 - \mu)} = 1$$

This result, $p_3=1$, is not in conformity with $p_3 \leq 1/2$, so this strategy combination and belief is not a perfect Bayesian equilibrium.

(8) strategy combination $((6, 5), (d, d, d))$ and belief $p_1 \leq 1/2$, $p_2 \leq 1/2$, $p_3 \leq 1/2$.

The strategy combination means

$$p(4 | t_1) = 0, p(5 | t_1) = 0, p(6 | t_1) = 1, p(4 | t_2) = 0, p(5 | t_2) = 1, p(6 | t_2) = 0$$

According to the given equilibrium strategy, the information sets $h(5)$ and $h(6)$ are on the equilibrium paths, but $h(4)$ is not on the equilibrium path.

According to requirement 3

$$p_2 = p(t_1 | 5) = \frac{p(5 | t_1) \cdot p(t_1)}{p(5 | t_1) \cdot p(t_1) + p(5 | t_2) \cdot p(t_2)} = \frac{0 \times \mu}{0 \times \mu + 1 \times (1 - \mu)} = 0$$

$$p_3 = p(t_1 | 6) = \frac{p(6 | t_1) \cdot p(t_1)}{p(6 | t_1) \cdot p(t_1) + p(6 | t_2) \cdot p(t_2)} = \frac{1 \times \mu}{1 \times \mu + 0 \times (1 - \mu)} = 1$$

This result, $p_3=1$, is not in conformity with $p_3 \leq 1/2$, so this strategy combination and belief is not a perfect Bayesian equilibrium.

Synthesizing the above analyses, the signaling game has five perfect Bayesian equilibria.

(2) strategy combination $((6, 6), (u, u, d))$ and belief $p_1=1$, $p_2=1$, $p_3=\mu$, $\mu \leq 1/2$.

(3) strategy combination $((5, 5), (u, d, u))$ and belief $p_1=1$, $p_2=\mu$, $p_3=1$, $\mu \leq 1/2$.

(4) strategic combination $((6, 4), (d, u, u))$ and belief $p_1=0$, $p_2=1$, $p_3=1$.

(5) strategy combination $((5, 5), (d, d, u))$ and belief $p_1=0$, $p_2=\mu$, $p_3=1$, $\mu \leq 1/2$.

(6) strategy combination $((6, 6), (d, u, d))$ and belief $p_1=0$, $p_2=1$, $p_3=\mu$, $\mu \leq 1/2$.

It can be seen from (3), (5) that although the strategies of the incumbent are the same, the strategies of the entrant are different. This is because of different beliefs. The similar scenarios occur in the cases of (2), (6).

If the incumbent can only use the separation equilibrium, the game has only one perfect Bayesian equilibrium. If $\mu > 1/2$, there also exists only one perfect Bayesian equilibrium.

4. Conclusion

The purpose of this paper is to illustrate a PBE solution method of a signaling game in detail. It can be seen from the analysis process of this paper that PBE of a signaling game

can be calculated by the method of this paper.

The PBE of a signaling game may be multiple. If we want to reduce the number, this involves refinements of PBE.

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