American Journal of Theoretical and Applied Statistics
2023; 12(4): 92-101
http://www.sciencepublishinggroup.com/j/ajtas
doi: 10.11648/j.ajtas.20231204.15
ISSN: 2326-8999 (Print); ISSN: 2326-9006 (Online)

A Polynomial-Based Mathematical Modelling of Age-Specific Fertility and Marital Fertility Rates of Assam

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To cite this article:

Sourav Jyoti Gogoi, Manab Deka. A Polynomial-Based Mathematical Modelling of Age-Specific Fertility and Marital Fertility Rates of Assam. *American Journal of Theoretical and Applied Statistics*. Vol. 12, No. 4, 2023, pp. 92-101. doi: 10.11648/j.ajtas.20231204.15

Received: July 27, 2023; **Accepted**: August 14, 2023; **Published**: August 28, 2023

Abstract: The emergence of population characteristics has become a significant focus within the field of demographic studies. Among these characteristics, fertility holds a pivotal role. In order to obtain a comprehensive comprehension of the complex dynamics of human fertility behaviour, mathematical models are a commonly used and appropriate tool. However, the scholarly discourse on fertility data modelling in the north-eastern Indian state of Assam has been relatively limited. The objective of this study is to investigate the fertility patterns among women in the reproductive age range by creating a polynomial-based mathematical model for Age-Specific Fertility Rates (ASFRs) and Age-Specific Marital Fertility Rates (ASMFRs) specific to the region of Assam. Moreover, we employed diverse statistical methodologies to ascertain the soundness and reliability of the formulated model. The utilization of a quartic polynomial model facilitates a more precise estimation of both ASFRs and ASMFRs. In addition, the Total Fertility Rate (TFR) and Total Marital Fertility Rate (TMFR) are estimated, as well as the age at which mothers experience elevated fertility rates. Velocity and Elasticity curves are fitted to the ASFRs and ASMFRs, unveiling noteworthy disparities between the velocity curves of 2020 and those of previous years (2015 and 2011). Specifically, the ASFRs in Assam exhibited their highest values among mothers aged approximately 27 years in 2020, while in 2015 and 2011, the peak ASFRs were estimated for mothers around 24 years old. Similarly, the ASMFRs in 2020 reached their zenith among mothers aged approximately 15 years, whereas in 2015 and 2011, the highest ASMFRs were estimated for mothers around 21 and 19 years old, respectively. Furthermore, the study sheds light on the decline in the number of women marrying at an illegal age group in Assam during 2020 compared to 2015 and 2011.

Keywords: Age Specific Fertility Rates, Age Specific Marital Fertility Rates, Mathematical Model, Polynomial Regression, Cross-Validity Prediction Power, Shrinkage

1. Introduction

The role of fertility is of great importance in the field of demographic analysis, as births are a crucial element in understanding and studying population dynamics. The level and trend of fertility serve as indicators of a nation's development standards [9]. The term fertility is commonly employed to signify the actual reproductive ability of a woman or groups of women. It has always been a matter of great concern for all people. Fertility simply means the occurrence of live births [1]. Fertility emerges as a vital demographic indicator alongside other significant factors like migration and mortality. It plays a critical role in the growth and expansion of the human population. Utilising statistical techniques, demographers have placed a significant emphasis on understanding fertility patterns. In order to acquire a deeper understanding of the intricate dynamics of human fertility behaviour, mathematical models are widely employed as highly suitable tools. A mathematical model is essentially an attempt to discover the structural relationships and dynamic behaviour among the various elements of Demography.

In the field of Demography, mathematical models can be broadly categorized into two types: non-deterministic or stochastic models and deterministic models. Stochastic models involve variables in the form of probability distributions. In contrast, deterministic models are employed to elucidate the functional relationships between variables that have specific and definite values. Furthermore, deterministic models can be further divided into two categories: stationary population models and time series population models [16]. This paper, however, focuses exclusively on the discussion and application of stationary population models to build models for Age-Specific Fertility Rates (ASFRs) and Age-Specific Marital Fertility Rates (ASMFRs). The ASFR refers to the number of live births per 1000 women within a specific age category in a given year. In contrast, the ASMFR concentrates on the number of live births per 1000 married women in a specific age group during a specific year.

The discussion on the modelling of fertility data in Assam, a state in India's north-eastern region, has thus far been fleeting. Demographically, the state of Assam can be characterised by its population. According to the 2011 census, the population of Assam was 31 million. Nevertheless, based on the population projection report for India and States from 2011 to 2036 [24], Assam's population has increased to 35 million in 2021 and is projected to reach 39 million by 2036. According to the 2011 census, the population density of Assam was calculated to be 397 persons per square kilometre. Subsequently, in 2021, the population density increased to 447 per square kilometre, and it is anticipated that by 2036, it will reach 502 per square kilometre.

According to the Sample Registration System Statistical Report for the years 2011, 2015, and 2020 [21-23], the Total Fertility Rate (TFR) for Assam was 2.4, 2.3, and 2.1 respectively. In addition, the Total Marital Fertility Rate (TMFR) for Assam was 4.7, 4.6, and 6.1 during these years. Notable is the decline in Assam's TFR, which has reached replacement fertility levels. In contrast, the TMFR increased significantly by 32.6% in 2020 compared to 2015. This increase can be attributed to either an increase in the proportion of women who have never been married or a decline in the number of married women within the illegal age range. To delve deeper, let's examine the 15-19 age range for 2015 and 2020. The ASFR for this age group increased from 0.0187 in 2015 to 0.0200 in 2020. During the same time frame, the ASMFR increased from 0.2897 to 0.5549. These figures indicate that out of 1000 women, approximately 99 were married in 2015, but that number declined to roughly 36 in 2020. This compelling evidence strongly supports the claim that the number of married women of illegal age in Assam is declining. This pattern can also be observed in other age groups, supporting the same conclusion. This paper presents a mathematical model based on polynomials to accurately represent the ASFRs and ASMFRs specific to the state of Assam.

There are numerous fertility models mentioned in the literature. Within this field, demographers have put forward deterministic models that rely on specific parameters, while others have proposed stochastic models that account for probabilistic variations. Visalakshi and Geetha [2] employed non-linear models to examining the prevailing pattern of Age-Specific Fertility Rates and the trends within reproductive age groups across all states of India. Peristera and Kostaki [5] introduced a novel approach incorporating the normal mixture model to address the distortions in both conventional and contemporary Age-Specific Fertility Rates. In a similar vein, Gaire and Aryal [10] employed the Inverse Gaussian distribution model to accurately model the Age-Specific Rates of Nepali females. Asili et al. [11] employed a skew-logistic probability model to fit the Age-Specific Fertility Rates of Italy. Similarly, Mishra et al. [12] utilized the same skew-logistic probability density function to develop a mathematical model for fitting the Age-Specific Fertility Rates of different states of India. Furthermore, Gaire et al. [13] introduced and applied the skew-log-logistic probability distribution model to accurately fit the Age-Specific Fertility Rates of Nepali females. Gilje [14] utilized a generalized Hadwiger function with three parameters to fit the fertility data of both Hungary and Norway. This function proves valuable, particularly when dealing with bi-modal Age-Specific Fertility Rates data. Furthermore, Waseem and Yasmeen [6] utilized the Hadwiger function model to fit the Age-Specific Fertility Rates of Pakistan spanning the years 1985 to 2005. Islam and Ali [15] utilised a quartic polynomial model to match the Age-Specific Fertility Rates of a rural region of Bangladesh between 1980 and 1988. Age-Specific Fertility Rates of Uttar Pradesh were calculated using the cubic polynomial of age and reciprocal of age by Singh et al. [8]. Gaire et al. [17] utilized a quartic polynomial model to fit the Age-Specific Fertility Rates of Nepali mothers. Additionally, they employed a quadratic polynomial model to examine the Forward Cumulative Age-Specific Fertility Rates in their study.

Mitra et al. [3] proposed two models expressing Age-Specific Marital Fertility Rates (ASMFRs) as functions of age. Specifically, the linear function of the logarithm of age and the logarithm of (50-age) have been proposed as estimators for ln (ASMR) and ln[-ln (ASMR)]. Rodriguez and Cleland [4] examined the modelling of marital fertility based on age and distribution since first marriage. Islam [16] utilised multiple polynomial models to assess the compatibility of the age-specific rates of marital fertility in Bangladesh in 1996 and 1999-2000. Using this analysis, he contrasted the various models to determine the best fit. Bongaarts [7] proposed a reproductive model that elucidates the relationship between a group of intermediate fertility variables and the fertility rate of married women.

The objective of this paper is to examine the fertility trend in the fertile age group by developing a polynomial-based mathematical model for Age-Specific Fertility Rates (ASFRs) and Age-Specific Marital Fertility Rates (ASMFRs) of Assam. Various statistical measures, including Cross-Validity Prediction Power (CVPP), Shrinkage, and R^2, F-test statistic are utilised to evaluate the model's validity and stability.

2. Data and Methodology

2.1. Source of Data

For the purposes of this study, secondary data on the Age-Specific Fertility Rate (ASFR) and the Age-Specific Marital Fertility Rate (ASMFR) were extracted from the Sample Registration System (SRS) Statistical Report for the years 2011, 2015, and 2020 [21-23].

2.2. Methodology

2.2.1. Polynomial Regression Model

Spiegel [25] defines the polynomial relationship between the independent variable x and the dependent variable y as a function, expressed in the following form:

$$y = f(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_k x^k \quad (1)$$

Where the constants $\beta_0, \beta_1, \beta_2, \ldots, \beta_k (\beta_0 \neq 0)$ represent coefficients. Each $\beta_i (i = 1, 2, \ldots, k)$ corresponds to the coefficient of x^i. The variable k is a positive integer, denoting the degree of polynomial, while x is known as the indeterminate.

For instance, A polynomial with $k = 0$ is referred to as a constant function. When $k = 1$, it is called a simple linear function. For $k = 2$, it is termed a quadratic function. A polynomial with $k = 3$ is known as a cubic function, and when $k = 4$, it is referred to as a quartic function and so on.

In this context, we define x as the maternal age at the time of giving birth to a child. The Age Specific Fertility Rates (ASFRs) and Age Specific Marital Fertility Rates (ASMFRs) are considered as a function of the age of the mother when the child is born [16, 17].

$$\underset{\cdot}{^{ASFR}}Y = X\dot{\beta} + \dot{u} \quad (2)$$

$$\underset{\cdot}{^{ASMFR}}Y = X\ddot{\beta} + \ddot{u} \quad (3)$$

Where

$$\underset{\cdot}{^{ASFR}}Y = \begin{pmatrix} \underset{\cdot}{^{ASFR}}y_1 \\ \underset{\cdot}{^{ASFR}}y_1 \\ \vdots \\ \underset{\cdot}{^{ASFR}}y_n \end{pmatrix}_{n\times 1} ; X = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^k \\ 1 & x_2 & x_2^2 & \cdots & x_2^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^k \end{pmatrix}_{n\times(k+1)} ; \dot{\beta} = \begin{pmatrix} \dot{\beta}_0 \\ \dot{\beta}_1 \\ \vdots \\ \dot{\beta}_k \end{pmatrix} ; \dot{u} = \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \vdots \\ \dot{u}_n \end{pmatrix}$$

$$\underset{\cdot}{^{ASMFR}}Y = \begin{pmatrix} \underset{\cdot}{^{ASMFR}}y_1 \\ \underset{\cdot}{^{ASMFR}}y_1 \\ \vdots \\ \underset{\cdot}{^{ASMFR}}y_n \end{pmatrix}_{n\times 1} ; X = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^k \\ 1 & x_2 & x_2^2 & \cdots & x_2^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^k \end{pmatrix}_{n\times(k+1)} ; \ddot{\beta} = \begin{pmatrix} \ddot{\beta}_0 \\ \ddot{\beta}_1 \\ \vdots \\ \ddot{\beta}_k \end{pmatrix} ; \ddot{u} = \begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \vdots \\ \ddot{u}_n \end{pmatrix}$$

The variables $\dot{u}$ and $\ddot{u}$ represent the stochastic error terms in the model. To ensure a more accurate fit, an appropriate value of k is selected such that the sum of squared errors is minimized.

2.2.2. Velocity and Elasticity Curves of ASFRs and ASMFRs

The graph showing the rate of change of the dependent variable with respect to the independent variable in a polynomial model is referred to as the velocity curve of a polynomial regression. According to Gasser et al. [26], the velocity curve for a polynomial regression model of ASFRs and ASMFRs is derived by calculating the first derivative of the fitted polynomial regression concerning the age of the mother during the child's birth. This curve represents the rate of change for ASFRs and ASMFRs concerning maternal age. In simpler terms, the velocity curves of ASFRs and ASMFRs indicate the rate at which these rates change in response to variations in maternal age during childbirth. Moreover, the optimum value of these fertility rates corresponds to the age at which the first derivative becomes zero and the second derivative becomes negative. The respective velocity curves are as follows:

$$\frac{\partial}{\partial x} \underset{\cdot}{^{ASFR}}Y = X'\dot{\beta} \quad (4)$$

$$\frac{\partial}{\partial x} \underset{\cdot}{^{ASMFR}}Y = X'\ddot{\beta} \quad (5)$$

Where

$$x = (x_1 \quad x_2 \quad \cdots \quad x_n)'$$

$$X' = \begin{pmatrix} 0 & 1 & 2x_1 & \cdots & kx_1^{k-1} \\ 0 & 1 & 2x_2 & \cdots & kx_2^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2x_n & \cdots & kx_n^{k-1} \end{pmatrix}_{n\times(k+1)}$$

To obtain the estimated Total Fertility Rate (TFR) and estimated Total Marital Fertility Rate (TMFR), we simply integrate the fitted polynomials of the ASFRs and ASMFRs with respect to age within the reproductive age range. The elasticity curve illustrates the correlation between the proportional alteration in one variable and the proportional modification in another variable. In the field of economics, it is frequently employed to analyse the degree of responsiveness or sensitivity of one variable to fluctuations in another variable.

In the context of ASFRs and ASMFRs, the elasticity curve quantifies the sensitivity of these rates in response to variations in the age of women at the time of giving birth. By computing the percentage change in these fertility rates as a result of changes in maternal age, it is easier to evaluate how these rates fluctuate in relation to changes in the mother's age when a child is born [17].

Dewett [18] presents a specific relationship that can be used to estimate the elasticity curve.

$$\text{Elasticity of } {}^{ASFR}_{\cdot}Y = \frac{X}{Y}\frac{\partial}{\partial x}\, {}^{ASFR}_{\cdot}Y \quad (6)$$

$$\text{And Elasticity of } {}^{ASMFR}_{\cdot}Y = \frac{X}{Y}\frac{\partial}{\partial x}\, {}^{ASMFR}_{\cdot}Y \quad (7)$$

2.2.3. Model Performance Assessment

We use a variety of statistical methods to examine the stability and dependability of the suggested model, including cross-validation prediction power (CVPP), model shrinkage, and F test statistics. To quantify the CVPP, as outlined by Stevens [19], we define it according to the following relationship:

$$\rho^2_{CV} = 1 - \frac{(n^2-1)(n-2)}{n(n-k-1)(n-k-2)}(1-R^2) \quad (8)$$

Where n represents the total number of cases, k indicates the number of regressors included in the model ($k \leq n-2$), and cross-validated R represents the correlation between the actual fertility rates and the predicted values of the fertility rates.

To evaluate the stability of the coefficient of determination, R^2 in the model, we assess the expression (1 - shrinkage), where

$$\text{Shrinkage} = |\rho^2_{CV} - R^2| \quad (9)$$

The F-test is employed to assess the overall significance of the model and the significance of R^2 [20]. The formula for the F-test is as follows:

$$F = \frac{R^2/k}{1-R^2/n-k-1} \sim F_{k,n-k-1} \quad (10)$$

The model will be significant with a p-value less than 5% for the computed value of F-test statistics.

3. Results and Discussions

The information regarding the model fitting of Age-Specific Fertility Rates (ASFRs) and Age-Specific Maternal Fertility Rates (ASMRs) for the years 2020, 2015, and 2011 is respectively provided in the Appendix section.

The quartic polynomial regression model demonstrates a strong fit for the ASFRs of Assam in the years 2011, 2015, and 2020, as presented below:

$${}^{ASFR}_{2020}Y = -1.053 + 0.1033x - 0.00268x^2 + 0.00001326x^3 + 0.0000001527x^4$$

$${}^{ASFR}_{2015}Y = -5.156 + 0.6539x - 0.0287x^2 + 0.0005334x^3 - 0.000003597x^4$$

$${}^{ASFR}_{2011}Y = -3.850 + 0.4926x - 0.02163x^2 + 0.0004019x^3 - 0.000002713x^4$$

Table 1. *Actual and Predicted Values of ASFRs for Assam by using quartic polynomial.*

Age Group	Age Specific Fertility Rates					
	2020		2015		2011	
	Actual	Predicted	Actual	Predicted	Actual	Predicted
15-19	0.0200	0.019396	0.0187	0.018741	0.0458	0.045281
20-24	0.1023	0.104788	0.1806	0.180658	0.1632	0.165095
25-29	0.1277	0.124326	0.1578	0.156898	0.1460	0.144315
30-34	0.0987	0.099406	0.0755	0.077710	0.0827	0.080875
35-39	0.0514	0.053716	0.0218	0.019387	0.0276	0.032022
40-44	0.0153	0.013235	0.0030	0.004268	0.0173	0.014310
45-49	0.0057	0.006234	0.0010	0.000738	0.0029	0.003602

Based on the polynomial regression model above, we have found that, in 2020 the ASFRs in Assam are most elevated among mothers aged around 27 years, which aligns with the actual ASFRs observed in 2020, where the highest rates were found among mothers aged 25-29. Similarly, in 2015 and 2011, the highest ASFRs were observed among 24-year-old mothers, consistent with the actual ASFRs recorded in those years, where the highest rates were found among mothers in the 20-24 age group. Table 1 provides a comprehensive overview of the actual and predicted values of ASFRs for the years 2020, 2015, and 2011. Additionally, Figures 1, 2, and 3 visually display the actual and fitted values of ASFRs in Assam for the years 2020, 2015, and 2011, respectively.

The interval between ages 15 to 49 encompasses a specific region, characterized by the definite integrals calculated from polynomial models representing ASFRs in their corresponding years. Within this region, the obtained values are 2.0497, 2.2369, and 2.40606, reflecting the expected Total Fertility Rate (TFR) based on the fitted ASFR curves. The TFR for Assam in 2020 is 2.1, whereas the fitted model yields a value of 2.04. Similarly, the TFR for Assam in 2015 is 2.3, but the fitted model gives a value of 2.24. Lastly, the

TFR for Assam in 2011 is 2.40, with the fitted model producing a value of 2.41.

The values of Cross Validated Prediction Power, coefficient of determination, (1-shrinkage), and p-values for each model are summarized in Table 2. It is evident that all the fitted models exhibit high statistical significance, explaining over 99% of the variance. The cross-validation results also indicate strong performance for the fitted polynomial models of ASFRs, with shrinkage values of 0.0952, 0.0194, and 0.0774 for the years 2020, 2015, and 2011, respectively. It is noteworthy that the fitted ASFRs polynomial model for 2020 displays stability of over 89%, whereas the models for 2015 and 2011 show stability rates of over 97% and 91%, respectively. Additionally, the coefficient of determination, R^2 for these models surpasses 90% in terms of stability.

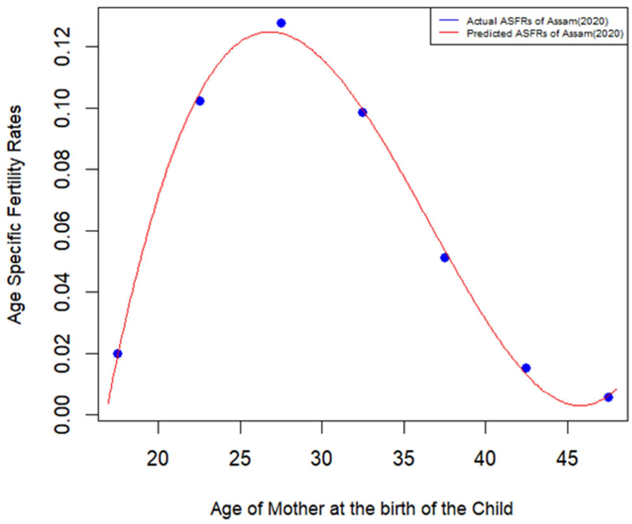

Figure 1. *Actual and Fitted Age Specific Fertility rates of Assam in 2020.*

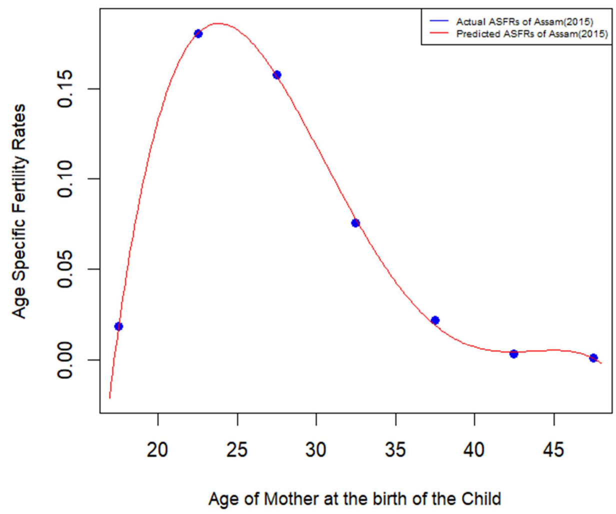

Figure 2. *Actual and Fitted Age Specific Fertility rates of Assam in 2015.*

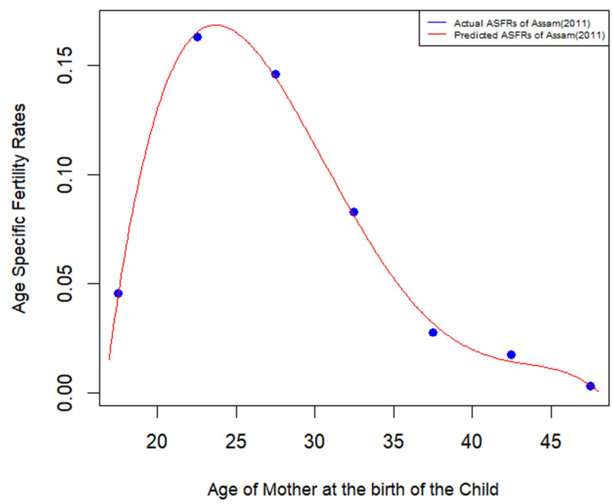

Figure 3. *Actual and Fitted Age Specific Fertility rates of Assam in 2011.*

Table 2. *Estimated Cross Validated Prediction Power of the Fitted Polynomials Models of ASFRs.*

Model	n	k	R^2	CVPP	1-Shrinkage	p-value
ASFR Model (2020)	7	4	0.9941	0.8988	0.9048	0.0117*
ASFR Model (2015)	7	4	0.9988	0.9794	0.9806	0.0024**
ASFR Model (2011)	7	4	0.9952	0.9177	0.9226	0.0096**
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1						

The Age-Specific Marital Fertility Rates (ASMFRs) of Assam for the years 2020, 2015, and 2015 are consistently well fit by the quartic polynomial regression model, illustrated below:

$$_{2020}^{ASMFR}Y = 4.910 - 0.5180x + 0.02166x^2 - 0.0004099x^3 + 0.000002901x^4$$

$$_{2015}^{ASMFR}Y = -4.020 + 0.5933x - 0.02846x^2 + 0.0005633x^3 - 0.00000399x^4$$

$$_{2011}^{ASMFR}Y = -2.07 + 0.3418x - 0.01694x^2 + 0.0003388x^3 - 0.000002409x^4$$

Table 3. *Actual and Predicted Values of ASMFRs for Assam by using quartic Polynomial.*

Age Group	Age Specific Marital Fertility Rates					
	2020		2015		2011	
	Actual	Predicted	Actual	Predicted	Actual	Predicted
15-19	0.5549	0.553695	0.2897	0.291048	0.3145	0.314152
20-24	0.2897	0.294584	0.3195	0.314063	0.2870	0.288118
25-29	0.1854	0.179060	0.1971	0.204068	0.1904	0.190033

Age Group	Age Specific Marital Fertility Rates					
	2020		2015		2011	
	Actual	Predicted	Actual	Predicted	Actual	Predicted
30-34	0.1166	0.117228	0.0847	0.084243	0.0961	0.093351
35-39	0.0573	0.062698	0.0242	0.017918	0.0309	0.035390
40-44	0.0171	0.012593	0.0034	0.008563	0.0201	0.017332
45-49	0.0064	0.007542	0.0011	0.001123	0.0036	0.004223

Based on the aforementioned polynomial regression model, we discovered interesting patterns in Assam's ASMFRs for the years 2020, 2015, and 2011. In 2020, the rates were highest among mothers aged approximately 15 years, while in 2011, the peak rates were observed among mothers aged around 19 years. These findings align with the actual ASMFRs observed in those years, where the highest rates were indeed found among mothers age ranges from 15 to 20. Similarly, in 2015, the highest ASMFRs were observed among 21-year-old mothers, consistent with the recorded ASMFRs for that year, where the highest rate was observed among mothers in the 20-24 age group. Furthermore, as elucidated in the introductory section, the number of unlawfully married women in Assam is diminishing. Nevertheless, it is worth noting that among those who did enter into matrimony, over 55% of couples went on to conceive. Furthermore, among couples who married just prior to attaining the age of 15 years, more than 82% of them experienced childbirth, as per our model's predictions.

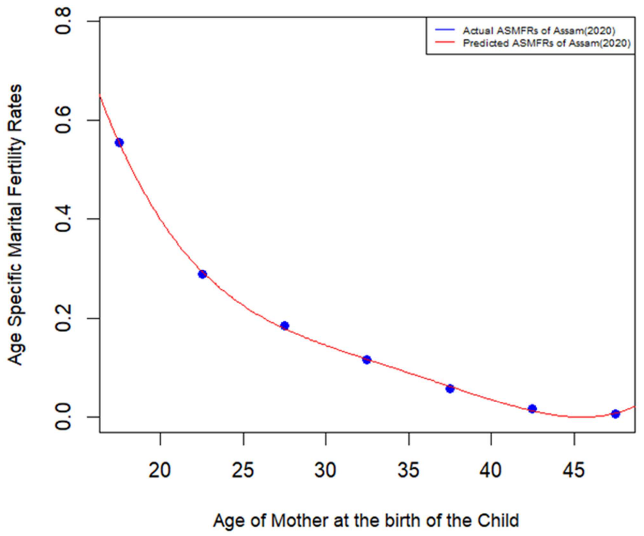

Figure 4. *Actual and Fitted Age Specific Marital Fertility rates of Assam in 2020.*

The actual and predicted values of ASMFRs for the years 2020, 2015, and 2011 are provided in Table 3. Furthermore, Figures 4, 5, and 6 visually depict the ASMFRs in Assam for the corresponding years, showcasing both the actual and fitted values. By calculating the area between the fitted curve for ages 15 to 49, using definite integrals applied to the fitted polynomial models of ASMFRs, we obtained values of 6.30564, 4.64062, and 4.49121 for the years 2020, 2015, and 2011, respectively. These areas represent the expected Total Marital Fertility Rate (TMFR) based on the fitted ASMFRs curves. Comparing the TMFRs for Assam in 2020, it is 6.1, whereas the fitted model produces a value of 6.30. The TMFRs for Assam in 2015 is 4.6, while the fitted model produces a value of 4.64. Lastly, the TMFRs for Assam in 2011 is 4.7, with the fitted model producing a value of 4.49.

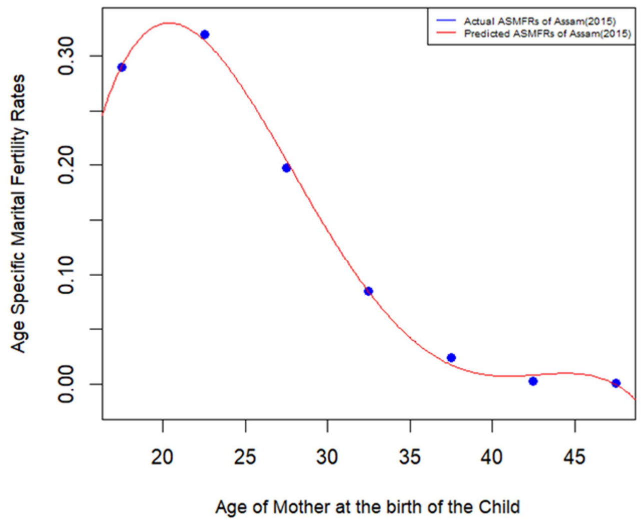

Figure 5. *Actual and Fitted Age Specific Marital Fertility rates of Assam in 2015.*

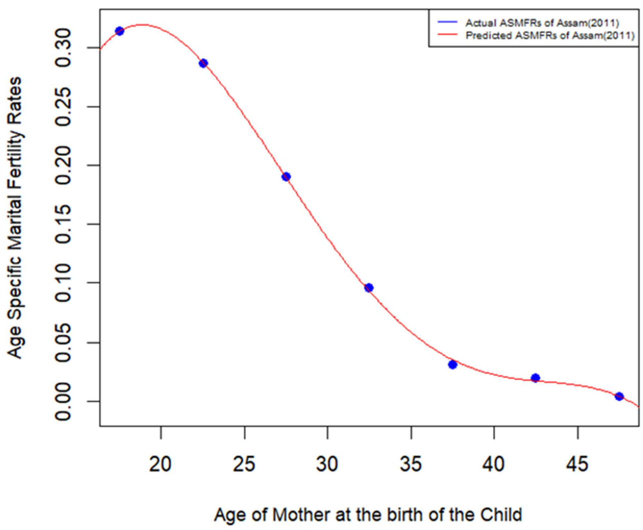

Figure 6. *Actual and Fitted Age Specific Marital Fertility rates of Assam in 2011.*

Table 4 summarizes the values of Cross Validated Prediction Power, coefficient of determination, (1-shrinkage), and p-values for each model. Notably, all of the fitted models have strong statistical significance and account for more than 99% of the variance. With shrinkage values of 0.0242, 0.0646, and 0.0177 for the years 2020, 2015, and 2011, respectively, the cross-validation findings further demonstrate the fitted polynomial models of ASMFRs' high performance. It is noteworthy that all the ASMFRs polynomial models that were fitted have stability rates that are higher than 93%. The coefficient of determination, R^2 for these models exceeds 93% in terms of stability.

Table 4. *Estimated Cross Validated Prediction Power of the Fitted Polynomials Models of ASMFRs.*

Model	n	k	R^2	CVPP	1-Shrinkage	p-value
ASMFR Model (2020)	7	4	0.9985	0.9743	0.9758	0.0029**
ASMFR Model (2015)	7	4	0.9960	0.9314	0.9354	0.0079**
ASMFR Model (2011)	7	4	0.9989	0.9811	0.9823	0.0021**
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1						

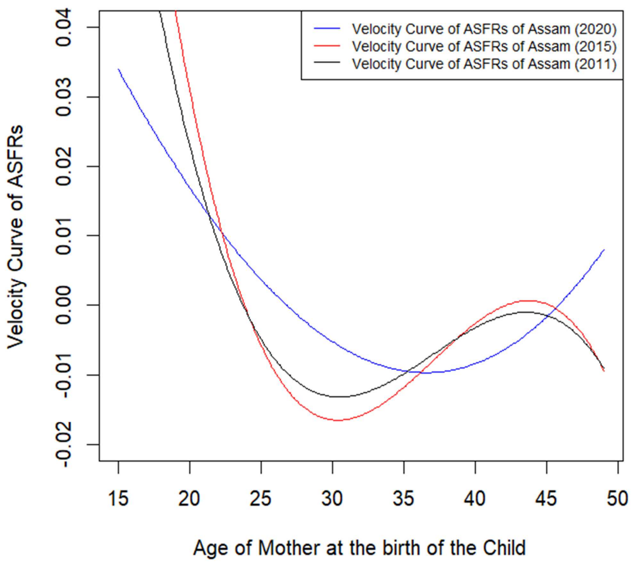

Figure 7. *Velocity Curves of Age-Specific Fertility Rates of Assam.*

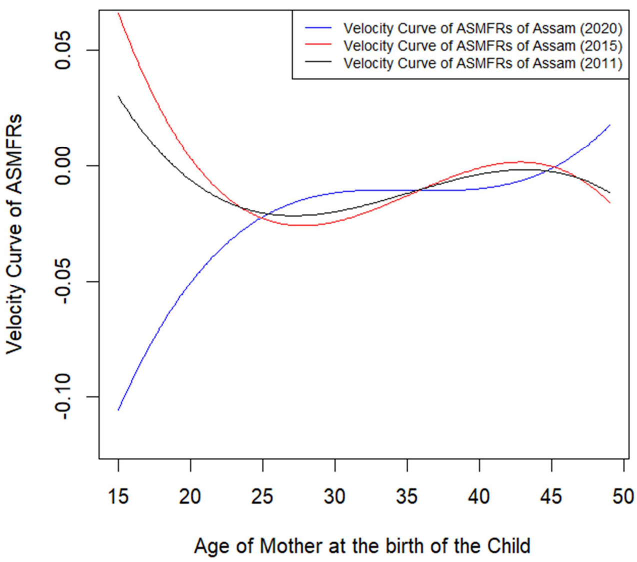

Figure 8. *Velocity Curves of Age-Specific Fertility Rates of Assam.*

Age-Specific Fertility Rates (ASFRs) and Age-Specific Marital Fertility Rates (ASMFRs) velocity curves of Assam for the years 2020, 2015, and 2011 are depicted in Figures 7 and 8, respectively. Notably, the velocity curve of ASFRs in 2020, when Assam attained replacement level fertility, exhibits a distinctive pattern compared to the velocity curves in 2015 and 2011. For the year 2020, the ASFRs velocity declines rapidly until the mothers' age reaches 35, after which it starts to increase. However, the velocity curves for 2015 and 2011 share a similar trend with decreasing velocities until age 30, followed by a slight increase until the age of 45, before declining again. Similarly, the velocity curve of ASMFRs in 2020 differs significantly from those in 2015 and 2011. In 2020, the ASMFRs velocity increases until the age of 30, remains constant between ages 30 to 40, and then rises again. On the contrary, the velocity curves in 2015 and 2011 decrease until the mothers' age reaches 30, experience a slight increase up to age 45, and then decline once more.

Assam's ASFRs and ASMFRs elasticity curves for the years 2020, 2015, and 2011 are depicted in Figures 9 and 10, respectively. These curves depict the percentage change in ASFRs and ASMFRs in relation to the age of the mother at childbirth. Figure 9 reveals that the elasticity of ASFRs remains essentially constant, with some fluctuations at the beginning and end of the reproductive period of mothers. Figure 10 demonstrates that ASMFRs experience fluctuations predominantly at the end of the reproductive period, whereas the elasticity remains constant for all other reproductive age groups.

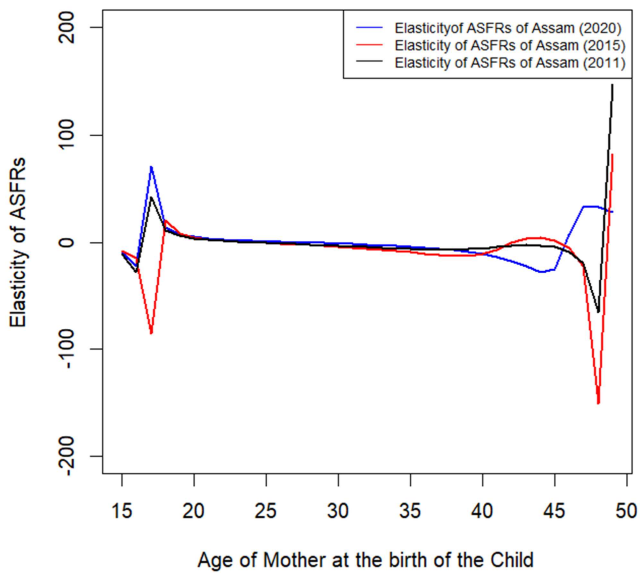

Figure 9. *Elasticity Curves of Age-Specific Fertility Rates of Assam.*

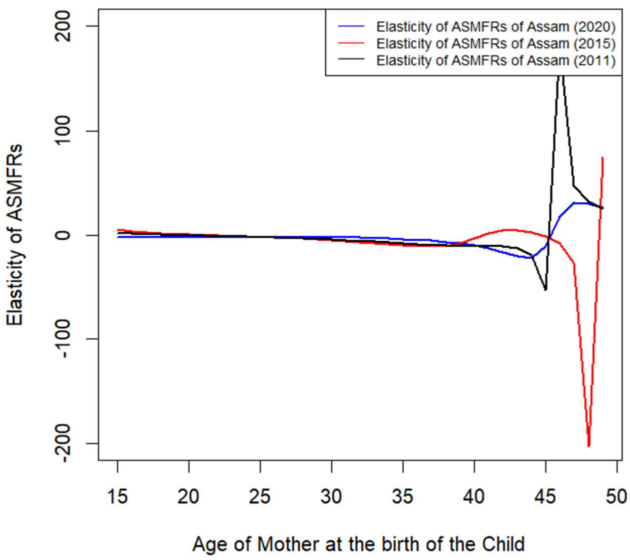

Figure 10. *Elasticity Curves of Age-Specific Marital Fertility Rates of Assam.*

4. Conclusion

Age-Specific Fertility Rate (ASFR) and Age-Specific Marital Fertility Rate (ASMFR) in Assam for the years 2020, 2015, and 2011 were analysed using a polynomial-based mathematical model in this paper. According to the results, the quartic polynomial model provided a more precise estimation of both ASFRs and ASMFRs, effectively explaining the data variance. Furthermore, we have ascertained the specific age at which women exhibit the highest levels of fertility, aligning with the rates that have been reported. The estimates of the Total Fertility Rate (TFR) and the Total Marital Fertility Rate (TMFR) were consistent with the actual TFR and TMFR. In addition, velocity and elasticity curves were fitted to ASFRs and ASMFRs. The velocity curve represented the rate of change in fertility rates as a function of the mother's age at childbirth. Notably, the velocity curves for ASFRs and ASMFRs in 2020 exhibited substantial disparities compared to those seen in 2015 and 2011. This indicates a substantial alteration over time in the rate of change for both fertility rates related to maternal age. On the other hand, the elasticity curve illustrated the sensitivity of fertility rates to the age of the mother at childbirth, and curiously, the elasticity curves for ASFRs and ASMFRs exhibit fluctuations within the same reproductive age group of mothers across all three years. This suggests that both fertility rates fluctuate in a consistent manner with changes in the mother's age at the time of childbirth throughout the years.

Appendix

Table A1. *Information on Model Fitting of ASFRs and ASMFRs of Assam in 2020.*

Models	Age Specific Fertility Rates			Age Specific Marital Fertility Rates		
	R^2	Parameters	p-value	R^2	Parameters	p-value
Model I	0.05425	$\dot{\beta}_0$	0.0208 *	0.8098	$\ddot{\beta}_0$	0.00282 **
		$\dot{\beta}_1$	0.2987		$\ddot{\beta}_1$	0.00361 **
Model II	0.6136	$\dot{\beta}_0$	0.00647 **	0.9655	$\ddot{\beta}_0$	0.000214 ***
		$\dot{\beta}_1$	0.14390		$\ddot{\beta}_1$	0.000267 ***
		$\dot{\beta}_2$	0.04546 *		$\ddot{\beta}_2$	0.008296 **
Model III	0.9955	$\dot{\beta}_0$	1.95e-05 ***	0.9859	$\ddot{\beta}_0$	0.000271 ***
		$\dot{\beta}_1$	0.000456 ***		$\ddot{\beta}_0$	0.000321 ***
		$\dot{\beta}_2$	0.000116 ***		$\ddot{\beta}_0$	0.004716 **
		$\dot{\beta}_3$	0.000345 ***		$\ddot{\beta}_0$	0.079721
Model IV	0.9941	$\dot{\beta}_0$	0.000559 ***	0.9985	$\ddot{\beta}_0$	0.000271 ***
		$\dot{\beta}_1$	0.004585 **		$\ddot{\beta}_1$	0.000304 ***
		$\dot{\beta}_2$	0.001839 **		$\ddot{\beta}_2$	0.001880 **
		$\dot{\beta}_3$	0.003809 **		$\ddot{\beta}_3$	0.015645 *
		$\dot{\beta}_4$	0.643655		$\ddot{\beta}_4$	0.037047 *
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1						

Table A2. *Information on Model Fitting of ASFRs and ASMFRs of Assam in 2015.*

Models	Age Specific Fertility Rates			Age Specific Marital Fertility Rates		
	R^2	Parameters	p-value	R^2	Parameters	p-value
Model I	0.1731	$\dot{\beta}_0$	0.0525	0.8703	$\ddot{\beta}_0$	0.000875 ***
		$\dot{\beta}_1$	0.1934		$\ddot{\beta}_1$	0.001357 **
Model II	0.2554	$\dot{\beta}_0$	0.056	0.8704	$\ddot{\beta}_0$	0.00211 **
		$\dot{\beta}_1$	0.189		$\ddot{\beta}_1$	0.00301 **
		$\dot{\beta}_2$	0.281		$\ddot{\beta}_2$	0.37296
Model III	0.8648	$\dot{\beta}_0$	0.00824 **	0.9470	$\ddot{\beta}_0$	0.00158 **
		$\dot{\beta}_1$	0.03393 *		$\ddot{\beta}_0$	0.00210 **
		$\dot{\beta}_2$	0.06126		$\ddot{\beta}_0$	0.21513
		$\dot{\beta}_3$	0.02227 *		$\ddot{\beta}_0$	0.08014
Model IV	0.9988	$\dot{\beta}_0$	0.000220 ***	0.9960	$\ddot{\beta}_0$	0.000612 ***
		$\dot{\beta}_1$	0.000623 ***		$\ddot{\beta}_1$	0.000741 ***
		$\dot{\beta}_2$	0.001005 **		$\ddot{\beta}_2$	0.029181 *

Models	Age Specific Fertility Rates			Age Specific Marital Fertility Rates		
	R^2	Parameters	p-value	R^2	Parameters	p-value
		$\dot{\beta}_3$	0.000452 ***		$\ddot{\beta}_3$	0.010872 *
		$\dot{\beta}_4$	0.002874 **		$\ddot{\beta}_4$	0.025288 *
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1						

Table A3. *Information on Model Fitting of ASFRs and ASMFRs of Assam in 2011.*

Models	Age Specific Fertility Rates			Age Specific Marital Fertility Rates		
	R^2	Parameters	p-value	R^2	Parameters	p-value
Model I	0.3125	$\dot{\beta}_0$	0.0177 *	0.9195	$\ddot{\beta}_0$	0.000201 ***
		$\dot{\beta}_1$	0.1114		$\ddot{\beta}_1$	0.000405 ***
Model II	0.4126	$\dot{\beta}_0$	0.0197 *	0.9513	$\ddot{\beta}_0$	0.000241 ***
		$\dot{\beta}_1$	0.1050		$\ddot{\beta}_1$	0.000429 ***
		$\dot{\beta}_2$	0.2452		$\ddot{\beta}_2$	0.107983
Model III	0.8896	$\dot{\beta}_0$	0.00321 **	0.9790	$\ddot{\beta}_0$	0.000322 ***
		$\dot{\beta}_1$	0.01703 *		$\ddot{\beta}_0$	0.000500 ***
		$\dot{\beta}_2$	0.05169		$\ddot{\beta}_0$	0.051569
		$\dot{\beta}_3$	0.02348 *		$\ddot{\beta}_0$	0.087350
Model IV	0.9952	$\dot{\beta}_0$	0.000579 ***	0.9989	$\ddot{\beta}_0$	0.000147 ***
		$\dot{\beta}_1$	0.001876 **		$\ddot{\beta}_1$	0.000197 ***
		$\dot{\beta}_2$	0.004403 **		$\ddot{\beta}_2$	0.005283 **
		$\dot{\beta}_3$	0.002380 **		$\ddot{\beta}_3$	0.008278 **
		$\dot{\beta}_4$	0.014668 *		$\ddot{\beta}_4$	0.017683 *
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1						

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