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Constant Specific Heat Approximation in Multifractal Thermodynamics in Multiparticle Production in Relativistic Heavy-Ion Collisions

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Abstract: The study of specific heat is motivated by the fact that a sudden change in the value of specific heat might be interpreted as a signal of a phase transition. It is also an established fact that multifractal analysis has been proved to be highly effective in characterizing fluctuations which is considered to be important tool for understanding the mechanism of quark-gluon plasma (QGP) in high energy nucleus-nucleus collisions. The Present research work provides some fascinating investigations on multifractal specific heat, c using the concept of entropy, f_q which is found as a potential procedure in the study of multifractal specific heat along with the earlier known approaches. The investigations are done for the produced shower particles in nuclear emulsion detector for ^{28}Si-nucleus interactions at 14.5 A GeV/c in the framework of generalized dimension. An attempt is also made to discuss certain universal properties of multifractal specific heat and entropy. We have computed c, by applying the methodology of modified and Takagi moments (T_q). Experimental results are compared with the predictions of LUND model FRITIOF. Moreover, the constant-specific heat method, which is based on the concept of entropy and is commonly accepted in conventional thermodynamics, is demonstrated to be suitable in multifractal thermodynamics also. The values of 'c' calculated from these methods are compared with constant specific heat approximations (CSHs) obtained using multifractal entropy (f_q). It is found that the values of 'c', estimated using Takagi approach are consistent with those of Bershadskii's work as compared to those calculated using ($G_a{}^m$) moments and multifractal entropy, f_q. This is obtained for both experimental and for the FRITIOF generated data for the three types of interactions namely, CNO, emulsion and AgBr. The findings of this paper reveal useful information regarding the choice of method used and our results are consistent with CSH approximation for both experimental and simulated data and also in conjunction with recent studies on multifractal specific heat.

Keywords: Multiparticle Production, Entropy, Multifractal Specific Heat, Relativistic Nuclear Collisions, Quark-Gluon Plasma, Multifractality

1. Introduction

The description of multifractality in thermodynamic terms [1] enables the investigation of fractal features of stochastic systems using ordinary thermodynamic concepts such as phase transition [2]. The methodology of modified G_q^m and Takagi moments, widely documented [3-5] and applied in our earlier works [6-8] assist us in obtaining multifractal spectra for both experimental and simulated data. The methodologies of the generalized dimensions, Dq, obtained from these two methods are given in appendix I and II of this paper. The thermodynamical picture of multi-particle production in high energy relativistic nuclear collisions can be very well understood by extracting a basic quantity, the multifractal specific heat, 'c' in terms of G_q^m and T_q moments [3-5] for both data sets. This parameter 'c', might also be used as a customary attribute of the particle production in high energy nuclear collisions [9]. Bershadskii [10] provided a thermodynamic explanation of the concluded outcome with reference to CSH. It is well known that in regular thermodynamics, constant specific heat is extensively significant in many important circumstances; for example, the specific heat of gases and solids is actually a consistent absolute temperature throughout huge or lower temperature intervals [11].

In this paper an attempt is made to explore the thermal properties and their manifestation due to underlying physics which is considered important aspect for understanding the mechanism of multiparticle production and hunt to discern the hadronization process with regard to fractal measures. As stated earlier we have calculated multifractal specific heat, c using the modified G_q^m and Takagi moments for the interaction of 14.5A GeV/c ^{28}Si nuclei with the three groups of targets, namely CNO, emulsion and AgBr. The experimental results are also analyzed to the results acquired from LUND model, FRITIOF generated data.

It may be worth mentioning that we have also extracted c, using CSH approximation by calculating the multifractal entropy f_q, as we found scarcity of the literature on multifractal entropy [10]. This paper strengthen some significant information as it apply the approach of multifractal entropy f_q for estimating c, for both experimental and FRITIOF simulated events on 14.5 A GeV/c for the three types of collisions namely, CNO, Em and AgBr.

The mathematical formalism of Bershadskii concept of multifractal specific heat, c is widely discussed in the literature [6, 8, 10, 12, 13]. The generalized dimension equation, D_q, may be derived from modified G_q^m and Takagi moments, T_q which helps extracting 'c' is given below for the ready reference for the readers.

$$D_q \cong (a-c)+c\frac{\ln q}{(q-1)} \tag{1}$$

In Eq. (1) 'a' denotes the information dimension, D_1, and 'c' is the constant specific heat. In this case q might be understood as the temperature's inverse q = (T^{-1}) [11, 14]. In the next section the concept of multifractal entropy is briefly discussed along with its mathematical formalism used to obtain constant specific heat approximation, denoted by 'c_a'.

2. Multifractal Entropy and CSH-Approximation[1]

The concept of entropy owes its origin to the work of Rudolf Claudius [15] who popularized its usage in thermodynamics, the quantity of energy in a system that cannot create work is measured. Physicists, like Boltzmann [16, 17], Gibbs [18], etc. are generally credited in the literature to have contributed towards the development and refinement of this concept further. Since then, the role and applicability of the concept of entropy, as a measure of various properties (energy that cannot be converted into work, disorder, uncertainty, randomness, complexity, etc.) has widened to appear in several scenarios, such as thermodynamics, statistical mechanics, information theory, measure-preserving dynamical systems, typological dynamics, and so on [19]. Due to its fascinating features, the science of entropy attracted the attention of researchers, not only from Physics, but also from other disciplines, including, Logic and Statistics, Biology and Economics. However, the notion of entropy has been termed differently depending on what context it is applied in. For example, when entropy concerns energy loss and is applied to explain thermodynamic principles, it is termed as thermodynamic entropy, when it concerns data loss, it is known as information entropy (or Shannon's entropy) [20], similarly, Hartley's entropy (which precedes Shannon's entropy) [21] and R´enyi's entropy (generalization of Shannon's entropy) [22]. High energy physicists [23–32] have intensively used the science of entropy to investigate the possibility of the occurrence of the formation of Quark-Gluon-Plasma (QGP) and to get a better understanding of the underlying mechanism of multi-particle process in high energy nuclear collisions (also often referred to as multifractal entropy). In reality, the study of entropy is important not only in the search for the development of QGP states but also in studying correlations and event-by-event variations. Guided by the purpose of the present paper, the method of entropy (or multifractal entropy for our purpose) is applied to calculate specific heat, 'c', defined by Eq. (1) and the mathematical procedure to do this are outlined below.

Let us consider a thermodynamic interpretation of multifractality [1, 2]. As discussed earlier it is well accepted that in usual thermodynamics, constant specific heat is extensively relevant in various important cases, for instance, the specific heat of gases and solids is truly consistent and independent of temperature, spanning a bigger or smaller temperature gaps [11].

Let us assume that the entire volume of a sample is a d-dimensional cube of size L. We partition this volume into N boxes of linear dimension l (N~ $(L/l)^d$). Each box is labeled with the index i, and the measure function of a field $\mu(x,t)$ is constructed for each box.

$$\mu_{i\,(l)} = \int_{v_i}^{\infty} \mu(x)dv \tag{2}$$

The volume of the i^{th} box is denoted by v_i. The generalised dimension, D_q, can thus be expressed as follows [2]

$$D_q = lim_{(l/L\to 0)} \frac{ln(Z(q))}{(q-1)(ln(l/L))} \tag{3}$$

The partition function may now be described using the following equation:

$$Z(q) = \sum_i \mu_i^q \tag{4}$$

As a result, the following scaling relationship emerges:

$$Z(q) \sim (l/L)^{\tau(q)} \tag{5}$$

[1] This mathematical formalism of CSH approximation using multifractal entropy draws upon [10]

$$\text{Here, } \tau(q) = D_q(q-1) \tag{6}$$

Furthermore, the partition function may be written as:

$$Z(q) \simeq \int \rho(\alpha)(l/L)^{q\alpha - f(\alpha)} d\alpha \tag{7}$$

Here α represents the singularity strength of the measure μ and the f(α)-singularity spectrum characterize the statistical distribution of the singularity exponent α. In the limit (l/L) → 0 , the sum (7) is dominated by the term $(l/L)^{min_\alpha(q\alpha - f(\alpha))}$. Now from $\tau(q)$, one gets the following equation

$$\tau(q) = min_\alpha(q\alpha - f(\alpha)) \tag{8}$$

As a result, $\tau(q)$ is achieved by Legendre converting the f(α). When f (α) and $\tau(q)$ are smooth functions, the equation (8) may be expressed as follows:

$$\tau(q) = q\alpha - f(\alpha), \frac{df}{d\alpha} = \alpha \tag{9}$$

As stated already, according to the thermodynamic interpretation of these equations, q may be described as an inverse of temperature $q=T^{-1}$ and the limit (l/L)→ 0 can be seen as the thermodynamic limit of infinite volume (V= ln (L/l) → ∞). The partition function may then be rewritten under the known form by associating $\alpha_i = {ln\mu_i}/{\ln(L/l)}$ with the energy E_i (per unit of volume) of a microstate i,

$$Z(q) = \sum_i exp(-qE_i) \tag{10}$$

From the definition f(α) = $\ln N_\alpha((l/L))$ / ln(L/l), the singularity spectrum f(α) plays the role of the entropy (per unit of volume). The attribute structure of plots f (α) versus α is reminiscent of plots of the dependence on E of the entropy for the thermodynamic systems.

Let us now propose the concept of constant specific heat (CSH) approximation. In this situation, the entropy is approximated by [11]:

$$f(q) = a - c_a lnq \tag{11}$$

The constant specific heat is denoted by 'c_a.' Now, using Eqs. (6), (9), and (11), one may calculate the CSH-approximation from the generalized dimensions provided by Eq (1).

3. Details of the Data

In the present experiment arandom data sample which consists of 605 interactions with $n_h \geq 7$ have been examined, where n_h represents the number of relativistic charged particles emitted from interaction having energy of 14.5A GeV $^{28}Si_{14}$-nucleus interactions with their relative velocities β≤0.7 are analyzed. Moreover, other significant information concerning the stacks, criteria for selecting the events and the method of measuring emission angles may also be found in Powel's book [33] and elsewhere [34-36]. Furthermore, the experimental results of the present work are compared with the predictions of LUND model, FRITIOF, which are simulated to match the true experimental data.

4. Results and Discussion

Figure 1 depicts the variations of the generalized dimensions, D_q, versus $\frac{lnq}{(q-1)}$ obtained from modified G_q^m moments for order, q = 2-6 for experimental data on 14.5 A GeV/c $^{28}Si_{14}$-nucleus collisions. The best linear fit straight lines are drawn for comparison with CSH-approximation (Eq. (1)). Nearly comparable patterns were found for the FRITIOF generated events. The values of multifractal specific heats, 'c', estimated from this figure along with FRITIOF generated data are presented in Table 1. It is interesting to mention that the value of 'c' is found to increase with increasing target size, while it is consistent around 0.24 for the three classes of interactions for the FRITIOF simulated events. Table 1 seems to suggest that the values of 'c' for the experimental events using modified G_q^m are far less consistent for all the three categories of collisions. Furthermore, several high-energy physicists have obtained remarkably comparable multifractal specific heat values [7, 12, 37-38].

Table 1. *Values of multifractal specific heat, 'c', from modified G_q^m, Takagi moments and multifractal entropy, f_q for the experimental and FRITIOF simulated events on 14.5A GeV/c ^{28}Si-nucleus interactions.*

DATA TYPE	'c_m' (Modified G_q^m moments)	'c_T' (Takagi moments)	'c_a' (Multifractal entropy)
EXPT.			
CNO	0.78±0.08	0.21±0.01	0.47±0.004
EMULSION	0.88±0.07	0.15±0.01	0.49±0.003
AgBr	0.98±0.09	0.28±0.003	0.62±0.005
FRITIOF			
CNO	0.23±0.01	0.16±0.02	0.15±0.03
EMULSION	0.28±0.02	0.21±0.02	0.23±0.04
AgBr	0.22±0.01	0.20±0.006	0.19±0.02

Note: In this table 'c' stands for constant specific heat approximation and the subscripts m, T and a indicate the method of modifiedG_q^m, Takagi moments and multifractal entropy f(q) respectively used to compute 'c'.

For extracting multifractal specific heat, 'c', from Takagi moments (T_q), we have plotted $D_q$against $\frac{lnq}{(q-1)}$ for experimental data for ^{28}Si-nucleus-collisions in Figure 2 and carryout linear best fits with respect to CSH-approximation (Eq. (1)). It is worth noting that a similar pattern was seen for the LUND model, FRITIOF simulated data set, suggesting the importance of Bershadski's interpretation of multifractality [39] in the context of a phase transition. Furthermore, for the FRITIOF data, multifractal specific heat c values diverge somewhat from the comparable experimental values. On the other hand, a finite non-zero value of c appears to be a good indicator of the presence of multifractality in the distribution of relativistic charged particles produced [13, 37].

The values of c for both the data sets are furnished in Table 1. It may be pointed out that the values of c, evaluated from both the data sets for all the three classes of interactions are increases with increasing target size and reveal some universality.

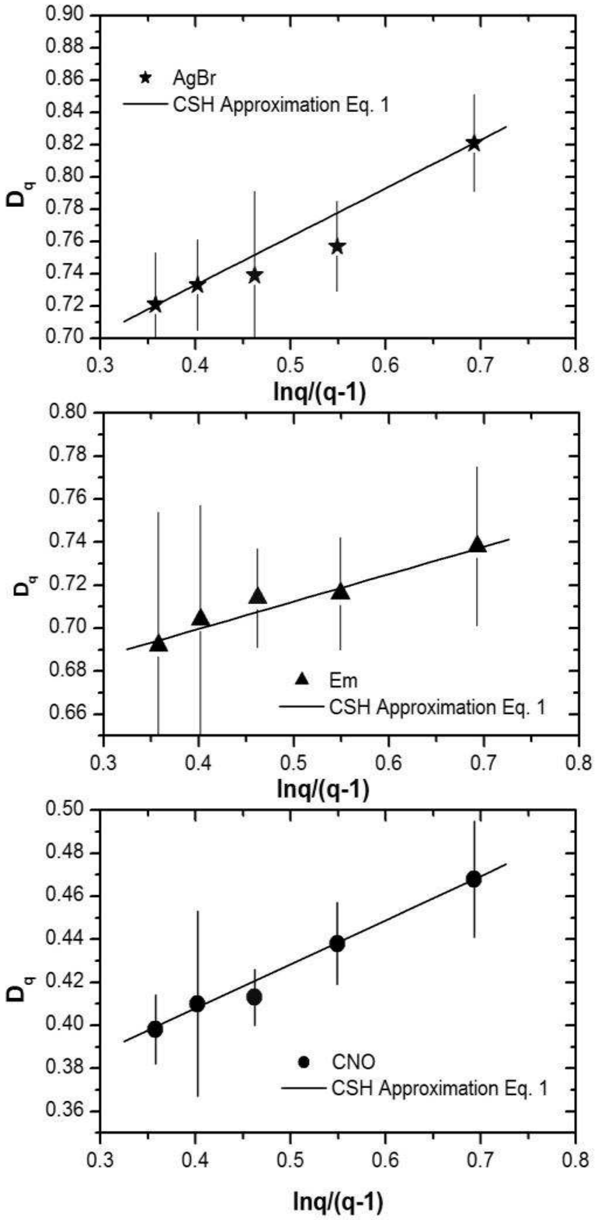

Figure 1. *The generalized dimensions Dq vs lnq / (q-1) plots for the experimental data on 14.5A GeV/c 28Si-nucleus interactions (Modified* G_q^m *moments).*

Figure 3 depicts the variations of multifractal entropy f(q) against lnq which is calculated using Eq. (11). One can also extract the values of specific heat, c_a, from this figure. A linear best fitted straight line shows a good agreement between CSH-approximation and the experimental data points. The estimated values of constant specific heat, 'c_a' ~ 0.52 from this figure is less consistent for the experimental data points. However, for the FRITIOF generated events the consistency in the values of 'c_a' is observed even using multifractal entropy f_q. This may be attributed to the fact that lower statistics of experimental data are considered in the present work. Our findings, however, are consistent with those of Bershadski [10].

A noticeable trend to observe from Table 1 is the increase in the values of 'c' for the three classes of collisions, CNO, emulsion and AgBr for the experimental data points across all the three methods used and consistency in the values of c have been observed for the FRITIOF simulated data.

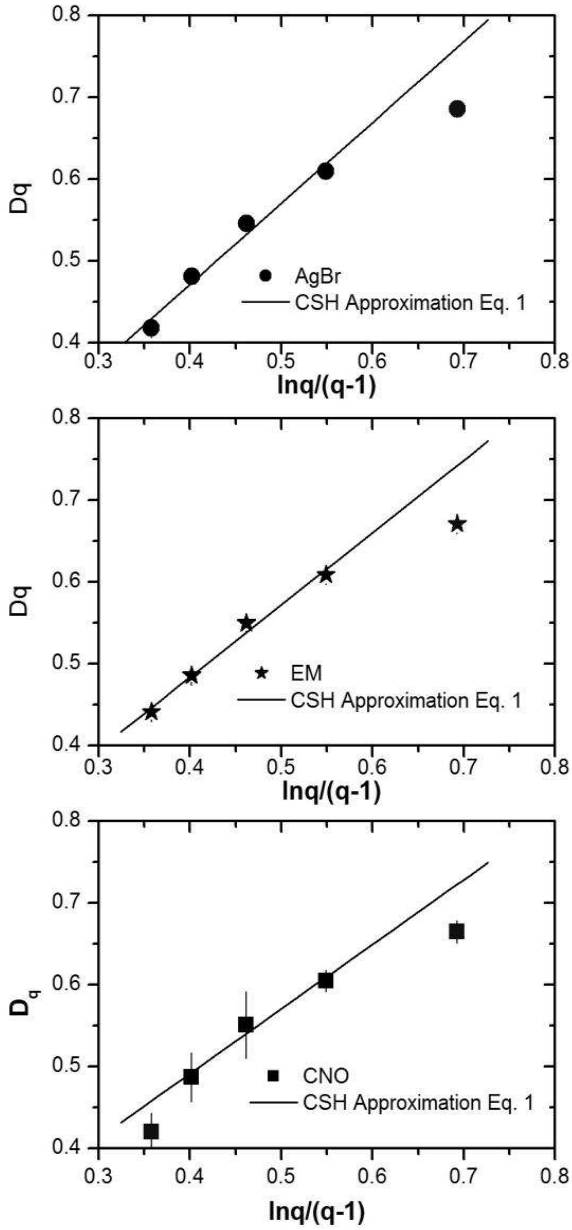

Figure 2. *Dependence of the generalized dimensions,* D_q *on lnq / (q-1) for the experimental data (Takagi moments, Tq).*

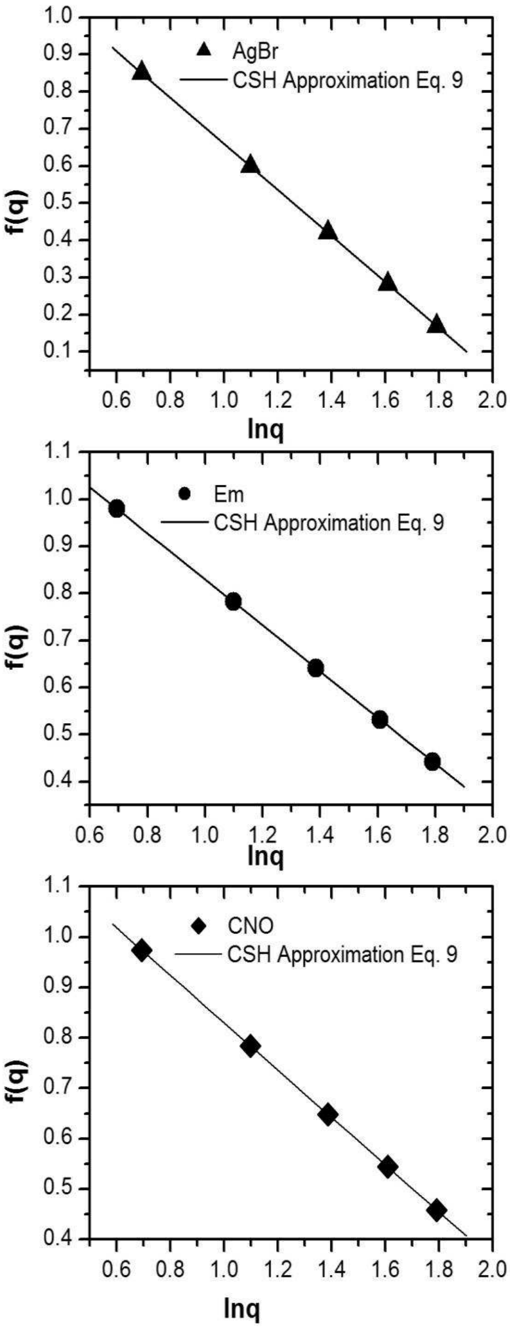

Figure 3. *Variations of multifractal entropy f (q) with lnq for 14.5A GeV/c ^{28}Si-nucleus interactions.*

5. Conclusions

The empirical results of this paper lead to the following observations:

The values of the multifractal specific heat, c obtained using modified G_q^m as well as by multifractal entropy (CSH) are less consistent as compared to those computed using Takagi moments, for the experimental data set for all the three target sizes. However, a close consistency is observed for the values of 'c' estimated using all these three methods and across three categories of interactions for FRITIOF simulated data. It is also to be noted that the values of c, computed using Takagi moments are almost in conformity with CSH approximation obtained by Bershadskii. On the other hand, a finite non-zero value of c appears to be a good indicator of the presence of multifractality in the distribution of relativistic charged particles produced.

These results seem to suggest the superiority of Takagi approach over modified G_q^m and also over multifractal entropy, while extracting the values of constant specific heat as a universal characteristic of thermodynamics process in multi-particle production in relativistic nuclear collisions. They also signify the role of high multiplicity events for the better explanation of the mechanism of multi-particle production process in these collisions. The common feature of multifractal specific heat, c extracted using the three approaches is that the values of 'c' increase with increasing target size (CNO, emulsion and AgBr) for the experimental data. Thus, analysis of multiparticle production data develop quite interesting and effective way for describing fluctuations and describe effectively mechanism and thermodynamic behavior of hadronization process in regard to fractal measures. The new frontier created by the LHC's high multiplicity events provides fertile ground for investigation of fluctuations and phase transitions that would not be possible at lower energies.

Conflict of Interest

The authors state that they do not have any conflicts of interest.

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Appendix

Appendix 1. Determination of Generalized Dimension Using Modified G_q^m Moments

To separate the dynamical and statistical fluctuations and examine the self-similar cascade process, the statistical component must be suppressed. Hwa and Pan's modified G_q^m moments are employed for this purpose.

$$G_q^m = \sum_{j=1}^{M'} P_j^q \,\theta(n_j - q) \tag{12}$$

Where $\theta(n_j - q)$ represents the step function, which is unity for $n_j > 1$ and disappears in all other situations. Summation is performed exclusively on non-empty bins M' in this case. The symbols P_j and n_j have the same meaning as in the case of normal G_q moments. When the particle multiplicity is great enough $n/_M \gg q$, G_q and

G_q^m moments are almost similar. According to the idea of multifractality, a self-similar particle emission process should display power-law behavior of the form:

$$G_q^m \propto M^{-t_q^m} \quad (13)$$

where t_q^m denotes the modified mass exponent which can be retrieved from the following equation.

$$t_q^m = \frac{\Delta \ln\langle G_q^m \rangle}{\Delta \ln M} \quad (14)$$

The dynamical component of $\langle G_q^m \rangle$ can be estimated from:

$$\langle G_q^{dyn} \rangle = \left[\frac{\langle G_q^m \rangle}{\langle G_q^{stat} \rangle}\right] M^{1-q} \quad (15)$$

MonteCarlo produced events are utilized in the calculation of $\langle G_q^{stat} \rangle$. It is also worth noting that $\langle G_q^{dyn} \rangle$ has the following power-law dependency on M:

$$< G_q^{dyn} > \propto M^{-t_q^{dyn}} \quad (16)$$

where

$$t_q^{dyn} = t_q^m - t_q^{stat} + q - 1 \quad (17)$$

Where t_q^{stat} is the statistical part's slope. If $\langle G_q^m \rangle = \langle G_q^{stat} \rangle$, then $\langle G_q^{dyn} \rangle$ is M^{1-q} is the outcome for trivial dynamics, according to Eq. (15). The result of trivial dynamics is $t_q^{dyn} = q - 1$. Any departure from this would indicate the presence of dynamical fluctuations.

The generalized dimensions, D_q^{tyn}, are considered to be the most and important property of the fractals and are envisaged to represent scaling behavior. Generalized dimensions are defined as:

$$D_q^{tyn} \cong \frac{t_q^{dyn}}{q-1} \quad (18)$$

We calculated the values of D_q^{tyn} using Eq. (18) for various orders of the moments.

Appendix 2. Determination of Generalized Dimension Using Takagi Moments

The multifractal moments, G_q, and modified G_q^m moments all have one or more limitations. In an event, the mathematical limit of phase space partition number ($M \rightarrow \infty$) cannot be realized in actuality due to the finite number of relativistic charged particle multiplicity, n_s. Even the step function, θ which was incorporated in the formulation of the modified G_q^m moments, cannot totally eliminate the saturation effects, especially at higher $|q|$ levels. Non-statistical fluctuations of the high multiplicity' of hadrons produced in hadronic and nuclear collisions at extremely high energy may therefore be investigated. Takagi and Miyamura conducted substantial study on this topic.

Takagi believes that the particle density, P_{ij}, is linked to $T_q(\delta\eta)$ in the following fashion:

$$T_q(\delta\eta) = \ln \sum_{j=1}^{M} \sum_{i=1}^{m} P_j^q \text{ for } q > 0 \quad (19)$$

where q denotes the moment's order It should be noted that the above formula is analogous to a linear logarithmic function of resolution, $R(\delta\eta)$, of the form:

$$T_q(\delta\eta) = A_q + B_q \ln R(\delta\eta) \quad (20)$$

where the constants A_q and B_q are independent of $\delta\eta$ and for a given range of $R(\delta\eta)$, generalized dimension, D_q, can be computed from:

$$D_q = \frac{B_q}{(q-1)} \quad (21)$$

Takagi has modified Eq. (20) to estimate the fluctuations as follows:

$$ln\langle n^q \rangle = A_q + \left(B_q + 1\right) \ln R(\delta\eta) \quad (22)$$

It's worth noting that the parameters B_q and D_q are unaffected by the bin width,$\delta\eta$. The generalized dimensions, D_q, for q = 2,3,4,5, and 6, may be calculated using the slope values found by Eq (22). However, the value of the information dimension D_1 may be computed using the following relationship:

$$\frac{\langle nlnn \rangle}{\langle n \rangle} = C_1 + D_1 ln\langle n \rangle \quad (23)$$

Thus, D_1 can be obtained from the slope of the variation of $\frac{\langle nlnn \rangle}{\langle n \rangle}$ with$ln\langle n \rangle$.

A declining trend in the values of D_q with the order of moment, q would suggest the existence of multifractality in the multiplicity distribution. However, if the D_q remains constant with increasing order of moments, q the conclusion is monofractality.

Takagi also proposed that the information on entropy may be calculated from a simple equation:

$$S(\delta\eta) = -\frac{\langle nlnn \rangle}{\langle n \rangle} + \ln N \quad (24)$$

This is not the same as Simak's study on entropy [14].

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