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On the Analysis of a Pre – Statically Loaded Nonlinear Cubic Structure Pressurized by an Explicitly Time Dependent Slowly Varying Load

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Abstract: This investigation is concerned with the determination of the dynamic buckling load of a Pre – Statically loaded imperfect elastic cubic model structure that is later struck by a dynamically slowly varying explicitly time - dependent load which is infinitely differentiable and has right hand derivatives of all orders at the initial time. Our initial pre–occupation is the determination of a uniformly valid asymptotic expression of the maximum displacement by means of multi–timing regular perturbation procedures. This is finally followed by a determination of the dynamic buckling load of the structure. The result shows, among other things, that the dynamic buckling load depends on the first derivative of the load function evaluated at the initial time. Besides, the dynamic buckling load is related to the static buckling load and this relationship is independent of the imperfection parameter. The result is, in the final analysis, particularized to cases of a step load with or without a pre–load. All results are asymptotic in nature and so, are valid as the small parameters approach zero.

Keywords: Nonlinear, Slowly Varying, Infinitely Differentiable, Explicitly Time Dependent, Pre – Statically Loaded

1. Introduction

Mathematical investigations into slowly varying time–dependent conditions appear to have originated from the study of Kudiak [1] when he investigated asymptotic solutions of nonlinear second order differential equations with variable coefficients. Later studies on the subject matter include [2-5], among others. Characteristically, none of these studies was made in the landscape of dynamic buckling. Mention is here made of other similar studies like [6-8].

For long in the immediate past, most dynamic buckling investigations were considered devoid of any pre–static load. However, in 1983, Simitses [9] introduced the concept of static pre–loading and further expatiated on it in [10]. Other investigations on the subject matter include [11, 12]. However while most of the earlier investigations on the subject used numerical approach as in [8-10], Ozoigbo and Ette [13], on the other hand, adopted a purely analytical approach by way of perturbation procedures in asymptotic expansions of the variables. Dynamic buckling of structures has been investigated by many researchers including [14-17], among others but the analytical approach adopted here is similar to that in the study of Ozoigbo and Ette [13].

The loading system suggested here is such that the structure investigated is first subjected to a nondimensional per–static load of magnitude λ_0 , $0<\lambda_0<1$, but before the structure could buckle statically, it is next trapped by a dynamically slowly varying load $\lambda f\left(\delta\hat{t}\right)$ where $0<\lambda<1$, $\left|f\left(\delta\hat{t}\right)\right|<1$, $\hat{t}>0$ and $f\left(0\right)=1$, $0<\delta<1$.

We demand that $f\left(\delta\hat{t}\right)$ be explicitly time - dependent, continuous, monotone–decreasing and infinitely differentiable with right hand derivatives of all orders at $\hat{t}=0$

2. Formulation of the Problem

The elastic cubic model structure $(Fig.1)$ considered in the work was first studied by [17, 18] and has continued to serve as a generalized mathematical model of most physical elastic structures encountered in actual engineering practice. It consists of, $(Fig.1)$ two rigid rods each of length L, arranged as in the Figure. The arrangement is struck by a time dependent horizontal load $P(T)$ applied shortly after initial time $T=0$. A mass M is attached at the meeting point of the rods while the ensuing vertical motion is restrained by a string whose rigidity follows a cubic law. From the side of the spring, the mass is affected by the reaction force given by $F_S = KL\left(\frac{X}{L}-b\left(\frac{X}{L}\right)^3\right)$, $b>0$, $K>0$.

were, $\frac{X}{L}$ is the additional displacement from equilibrium position. As in the Figure, $\frac{\bar{X}}{L}$ is the initial displacement serving as the initial imperfection. We let the angle θ be small and characterized by $Cos\theta \approx 1$, $Sin\theta \approx \theta$. By letting Q be the tension on each arm of the rod, it becomes clear that $QCos\theta = P(T)$.

Eventually, the equation of motion is

$$M\frac{d^2}{dT^2}\left(\frac{X}{L}\right)+KL\left(1-\frac{2}{KL}P(T)\right)\frac{X}{L}-bKL\left(\frac{X}{L}\right)^3 = 2P(T)\left(\frac{\bar{X}}{L}\right) \quad (1)$$

The following nondimensional parameters are now adopted:

$$\xi=\frac{X}{L}\ ,\ \bar{\xi}=\frac{\bar{X}}{L}\ ,\ \hat{t}=T\sqrt{\frac{KL}{M}}\ ,\ \left(0<\bar{\xi}<1\right)\ ,\ \lambda=\frac{2P(0)}{KL}\ ,\ \frac{P(T)}{P(0)}=f\left(\delta\hat{t}\right)\ ,\ \left(P(0)\neq 0\right),$$

where ξ_a is the maximum displacement. Our initial pre–occupation is thus to first determine a uniformly valid asymptotic expansion for ξ_a subsequent upon which the condition Eq. (3) will be evoked to determine the dynamic buckling load λ_D.

The nondimensional equation (without pre–static load) is

$$\frac{d^2\xi(\hat{t})}{d\hat{t}^2}+\left(1-\lambda f\left(\delta\hat{t}\right)\right)\xi-b\xi^3=\lambda\bar{\xi}f\left(\delta\hat{t}\right)\ ,\ \hat{t}>0 \quad (2)$$

$$\xi(0) = \frac{d\xi(0)}{d\hat{t}} = 0$$

Here, $b>0$ serves as the imperfection sensitivity parameter while λ is the amplitude of the load $f\left(\delta\hat{t}\right)$. In our case, we are to determine the value of λ, namely λ_D, called the dynamic buckling load which is needed to dynamically buckle the structure under the explicitly time dependent but slowly varying load $f\left(\delta\hat{t}\right)$ assuming that the structure had earlier been struck by the pre–load λ_0. As in Budiansky [18], we define the dynamic buckling load λ_D as the largest load parameter for the response of the system to remain bounded and is determined [13, 18-20] by the condition.

$$\frac{d\lambda}{d\xi_a}=0 \quad (3)$$

3. Pre–Loading Stage [Static Analysis]

At this stage, we let ξ_0 be the displacement and $f\left(\delta\hat{t}\right)=1$. As there is no time dependence, we get the relevant equation at this stage as

$$\left(1-\lambda_0\right)\xi_0-b\xi_0^3=\lambda_0\bar{\xi} \quad (4)$$

Using asymptotics, we get

$$\xi_0=\sum_{i=1}^{\infty}\xi_0^{(i)}\bar{\xi}^i \quad (5)$$

Then, substituting Eq. (5) in Eq. (4) and equating the coefficients of orders of $\bar{\xi}$, we get

$$O\left(\bar{\xi}\right):\ \left(1-\lambda_0\right)\xi_0^{(1)}=\lambda_0\ ;\ O\left(\bar{\xi}^2\right):\ \left(1-\lambda_0\right)\xi_0^{(2)}=0\ ;\ O\left(\bar{\xi}^3\right):\ \left(1-\lambda_0\right)\xi_0^{(3)}=\frac{b\xi_0^{(1)3}}{\left(1-\lambda_0\right)}\ . \quad (6)$$

etc.

From, Eq. (6) we get

$$\xi_0^{(1)}=\frac{\lambda_0}{\left(1-\lambda_0\right)}\ :\ \xi_0^{(2)}=0\ :\ \xi_0^{(3)}=\frac{b\xi_0^{(1)3}}{\left(1-\lambda_0\right)}\ :\ .\ .\ .\ . \quad (7)$$

etc.

While we do not expect static buckling at this stage, we can nevertheless still determine the static buckling load λ_S at this stage by letting λ_0 in Eqs. (4) - (7) to be tentatively written as λ.

Therefore (4) becomes

$$(1-\lambda)\xi_0 - b\xi_0^3 = \lambda\bar{\xi} \tag{8}$$

As in [18, 19], the condition for static buckling is

$$\frac{d\lambda}{d\xi_0} = 0 \tag{9}$$

This gives

$$(1-\lambda_S) - 3b\xi_{0S}^2 = 0$$

where, λ_S and ξ_{0S} are the values of λ and ξ_O respectively at static buckling.

This gives

$$\xi_{OS}^2 = \frac{(1-\lambda_S)}{3b} \quad ; \quad \xi_{OS} = \sqrt{\frac{(1-\lambda_S)}{3b}} \tag{10}$$

By evaluating Eq. (8) at static buckling using Eq. (9), we get

$$(1-\lambda_S)^{\frac{3}{2}} = \frac{3}{2}\sqrt{3}(b)^{\frac{1}{2}}\bar{\xi}\lambda_S \tag{11}$$

We can however still use asymptotic and perturbation procedures to obtain the same result Eq. (11) in the following alternative way.

Here, we let

$$\xi_0 = c_1\bar{\xi} + c_3\bar{\xi}^3 + \quad . \quad . \quad . \quad ; \quad c_1 = \frac{\lambda}{(1-\lambda)} \quad , \quad c_3 = \frac{b\bar{\xi}_0^{(1)^3}}{(1-\lambda)} \tag{12}$$

The process, as in Amazigo and Ette [20], Ozoigbo et. al [16] is to first reverse the series Eq. (12) such that

$$\bar{\xi} = d_1\xi_0 + d_3\xi_0^3 + \quad . \quad . \quad . \tag{13}$$

By substituting in Eq. (12) for ξ_0 from Eq. (10), and equating the coefficients of powers of $\bar{\xi}$, we quickly find that

$$d_1 = \frac{1}{c_1} \quad ; \quad d_3 = -\frac{c_3}{c_1^4} \tag{14}$$

The process in Eq. (9) is now initiated using Eq. (13) to yield.

$$d_1 + 3d_3\xi_{0S}^2 = 0$$

This yield

$$\xi_{OS}^2 = -\frac{d_1}{3d_3} \quad ; \quad \xi_{OS} = \sqrt{\frac{c_1^3}{3c_3}} \tag{15}$$

On determining Eq. (13) at static buckling, we get

$$\bar{\xi} = \xi_{OS}\left(d_1 + d_3\xi_{OS}^2\right) = \frac{2}{2\sqrt{3}}\sqrt{\frac{c_1}{c_3}} \tag{16}$$

On simplification, we obtain

$$(1-\lambda_S)^{\frac{3}{2}} = \frac{3}{2}\sqrt{3}(b)^{\frac{1}{2}}\bar{\xi}\lambda_S \tag{17}$$

and this authenticates Eq. (11) that was earlier derived. This second alternative method will be used in the remaining part of this investigation.

4. Imposition of Slowly Varying Dynamic Load on Pre - Static Load (Dynamic Analysis)

We still let $\xi(\hat{t})$ be the displacement strictly due to $f(\delta\hat{t})$ and equally let $\varsigma(\hat{t})$ be the net displacement at imposition of the two loads. Thus, at imposition of the two loads, we have

$$\varsigma(\hat{t}) = \xi_0 + \xi(\hat{t}) \qquad (18)$$

At imposition of the two loads, we then get

$$\frac{d^2\varsigma(\hat{t})}{d\hat{t}^2} + (1-\lambda_0)\xi_0 + \left(1-\lambda f(\delta\hat{t})\right)\xi - b(\xi_0+\xi)^3 = \lambda_0\bar{\xi} + \lambda f(\delta\hat{t})\bar{\xi} \qquad (19)$$

The equation to be solved is the difference Eq. (19) - Eq. (3) and this gives

$$\frac{d^2\xi(\hat{t})}{d\hat{t}^2} + \left(1-\lambda f(\delta\hat{t})\right)\xi - b\left[\xi^3 + 3\xi_0\xi(\xi+\xi_0)\right] = \lambda\bar{\xi} f(\delta\hat{t}), \ \hat{t} > 0 \qquad (20)$$

$$\xi(0) = \frac{d\xi(0)}{d\hat{t}} = 0, \ ; \ f(0) = 1 \ ; \ \left|f(\delta\hat{t})\right| < 1 \qquad (21)$$

5. Perturbation and Asymptotic Analysis

Let,

$$\tau = \delta\hat{t} \qquad ; \qquad \frac{d\bar{t}}{d\hat{t}} = \left(1-\lambda f(\delta\hat{t})\right)^{\frac{1}{2}} \qquad (22)$$

Further let

$$t = \bar{t} + \frac{1}{\delta}\left\{\omega_2(\tau)\bar{\xi}^2 + \omega_3(\tau)\bar{\xi}^3 + \ . \ . \ . \right\} ; \ \xi(\hat{t}) = \eta(t,\tau) \ ; \ \omega_i(0) = 0 \ , i = 2, 3, \ . \ . \ . \qquad (23)$$

Then, we have

$$\frac{d\xi(\hat{t})}{d\hat{t}} = \frac{\partial\eta}{\partial t}\frac{\partial t}{\partial\bar{t}}\frac{d\bar{t}}{d\hat{t}} + \frac{\partial\eta}{\partial t}\frac{\partial t}{\partial\tau}\frac{d\tau}{d\hat{t}} + \frac{\partial\eta}{\partial\tau}\frac{d\tau}{d\hat{t}}$$

$$= (1-\lambda f)^{\frac{1}{2}}\eta_{,t} + \left\{\omega_2'(\tau)\bar{\xi}^2 + \omega_3'(\tau)\bar{\xi}^3 + \ . \ . \ . \ \right\}\eta_{,t} + \delta\eta_{,\tau} \qquad (24)$$

where,

$$(\ )' = \frac{d}{d\tau}(\ldots) \qquad ; \qquad (\ )_{,t} = \frac{\partial}{\partial t}(\ldots)$$

$$\frac{d^2\xi(\hat{t})}{d\hat{t}^2} = (1-\lambda f)\eta_{,tt} + \left\{\omega_2'(\tau)\bar{\xi}^2 + \omega_3'(\tau)\bar{\xi}^3 + \ . \ . \ . \right\}^2 \xi_{,tt} + \delta^2\eta_{,\tau\tau} + 2(1-\lambda f)^{\frac{1}{2}}\left\{\omega_2'(\tau)\bar{\xi}^2 + \omega_3'(\tau)\bar{\xi}^3 + \ . \ . \ . \right)\eta_{,tt} + 2\delta(1-\lambda f)^{\frac{1}{2}}\eta_{,t\tau} + 2\delta\left\{\omega_2''(\tau)\bar{\xi}^2 + \omega_3''(\tau)\bar{\xi}^3 + \ . \ . \ . \right\}\eta_{,t\tau} - \delta\lambda f'(1-\lambda f)^{-\frac{1}{2}}\xi_{,t} + \delta\left\{\omega_2''(\tau)\bar{\xi}^2 + \omega_3''(\tau)\bar{\xi}^3 + \ . \ . \ . \right)\}\eta_{,t} \qquad (25)$$

On substituting Eqs. (23) and (24) into Eq. (20) and simplifying, we get

$$\eta_{,tt} + \frac{1}{(1-\lambda f)}\left\{\omega_2'(\tau)\bar{\xi}^2 + \omega_3'(\tau)\bar{\xi}^3 + \ . \ . \ . \right\}^2\eta_{,tt} + \frac{\delta^2}{(1-\lambda f)}\eta_{,\tau\tau} + \frac{2}{(1-\lambda f)^{\frac{1}{2}}}\left\{\omega_2'(\tau)\bar{\xi}^2 + \omega_3'(\tau)\bar{\xi}^3 + \ . \ . \ . \right)\eta_{,tt} + \frac{2\delta}{(1-\lambda f)^{\frac{1}{2}}}\eta_{,t\tau} + \frac{2\delta}{(1-\lambda f)}\left\{\omega_2'(\tau)\bar{\xi}^2 + \omega_3'(\tau)\bar{\xi}^3 + \ . \ . \ . \right\}\eta_{,t\tau} - \frac{\delta\lambda f'}{(1-\lambda f)^{\frac{3}{2}}}\eta_{,t\tau} + \frac{\delta}{(1-\lambda f)}\left\{\omega_2''(\tau)\bar{\xi}^2 + \omega_3''(\tau)\bar{\xi}^3 + \ . \ . \ . \right\}\xi_{,t} + \eta - \frac{b}{(1-\lambda f)}\eta^3 + \frac{3}{(1-\lambda f)}b\xi_0\eta(\eta+\xi_0) = \frac{1}{(1-\lambda f)}\lambda f(\tau)\bar{\xi} \qquad (26)$$

Now let,

$$\xi(\hat{t}) = \eta(t,\tau) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\eta^{ij}(t,\tau)\bar{\xi}^{i}\delta^{j} \tag{27}$$

where ij on η^{ij} in Eq. (27) represents indices and not powers.

Then, substituting Eq. (27) into Eq. (26) and equating the coefficients of powers of $\bar{\xi}\,\delta,$ we get

$$O(\bar{\xi}) \ : \ \eta^{10}_{,tt} + \eta^{10} = \frac{1}{\left(1-\lambda f(\tau)\right)}\lambda f(\tau) \cong B(\tau) \tag{28}$$

$$O(\bar{\xi}\delta) \ : \ \eta^{11}_{,tt} + \eta^{11} = -\frac{2}{(1-\lambda f)^{\frac{1}{2}}}\eta^{10}_{,t\tau} + \frac{\lambda f'}{2(1-\lambda f)^{\frac{3}{2}}}\eta^{10}_{,t} \tag{29}$$

$$O(\bar{\xi}\delta^{2}) \ : \ \eta^{12}_{,tt} + \eta^{12} = -\frac{2}{(1-\lambda f)^{\frac{1}{2}}}\eta^{10}_{,t\tau} + \frac{\lambda f'}{2(1-\lambda f)^{\frac{3}{2}}}\eta^{11}_{,t} - \frac{1}{(1-\lambda f)}\eta^{10}_{,\tau\tau} \tag{30}$$

$$O(\bar{\xi}^{3}) \ : \ \eta^{30}_{,tt} + \eta^{30} = -\frac{2}{(1-\lambda f)^{\frac{1}{2}}}\omega_2'(\tau)\,\eta^{10}_{,tt} + \frac{b}{(1-\lambda f)}\left[\left(\eta^{(10)}\right)^{3} - \frac{3b}{(1-\lambda f)}\left\{\xi_0^{(1)}\left(\eta^{(10)}\right)^{2} + \left(\xi_0^{(1)}\right)^{2}\xi_0^{(1)}\right\}\right] \tag{31}$$

$$\begin{aligned} O(\bar{\xi}^{3}\delta) \ : \ \eta^{31}_{,tt} + \eta^{31} = & -\frac{2}{(1-\lambda f)^{\frac{1}{2}}}\omega_2'(\tau)\eta^{11}_{,tt} - \frac{3b}{(1-\lambda f)}\left(\eta^{10}\right)^{2}\eta^{11} - \frac{3b}{(1-\lambda f)}\left[\left(\xi_0^{(1)}\right)^{2}\eta^{11}\right. \\ & \left. +2\xi_0^{(1)}\eta^{(11)}\eta^{11}\right] - \frac{2}{(1-\lambda f)}\omega_2'(\tau)\eta^{10}_{,tt} - \frac{2}{(1-\lambda f)^{\frac{1}{2}}}\eta^{30}_{,t\tau} + \frac{\lambda f'}{2(1-\lambda f)^{\frac{3}{2}}}\eta^{30}_{,t} - \frac{1}{(1-\lambda f)}\omega_2''(\tau)\eta^{10}_{,t} \end{aligned} \tag{32}$$

$$\begin{aligned} O(\bar{\xi}^{3}\delta^{2}) \ : \ \eta^{32}_{,tt} + \eta^{32} = & -\frac{2}{(1-\lambda f)^{\frac{1}{2}}}\omega_2'(\tau)\eta^{12}_{,tt} + \frac{3b}{(1-\lambda f)}\eta^{10}\left\{\eta^{10}\eta^{12} + \left(\eta^{11}\right)^{2}\right\} - \frac{3b}{(1-\lambda f)} \\ & \times\left[\xi_0^{(1)}\left(\eta^{(11)}\right)^{2} + \eta^{(10)}\eta^{12}\xi_0^{(1)} + \xi_0^{(1)}\left(\eta^{(12)}\right)^{2}\right] - \frac{2}{(1-\lambda f)}\omega_2'(\tau)\eta^{11}_{,tt} - \frac{2}{(1-\lambda f)^{\frac{1}{2}}}\eta^{312}_{,t\tau} + \frac{\lambda f'}{2(1-\lambda f)^{\frac{3}{2}}}\eta^{12}_{,t} \\ & -\frac{1}{(1-\lambda f)}\omega_2''(\tau)\eta^{11}_{,t} - \frac{1}{(1-\lambda f)}\eta^{30}_{,\tau\tau} \end{aligned} \tag{33}$$

etc.

The initial conditions which are evaluated at $(0,\,0)$, are

$$\eta^{ij}(0,0) = 0, \quad i = 1,2,3, \quad . \quad . \quad . \quad : \quad j = 1,2,3, \quad . \quad . \quad . \tag{34}$$

$$\mathrm{O}(\bar{\xi}) \ : \ \eta^{10}_{,t}(0,\,0) = 0 \quad ; \quad \mathrm{O}(\bar{\xi}\delta) \ : \ \eta^{11}_{,t}(0,\,0) + (1-\lambda)^{-\frac{1}{2}}\eta^{10}_{,\tau}(0,\,0) = 0 \tag{35}$$

$$\mathrm{O}(\bar{\xi}\delta^{2}) \ : \ \eta^{12}_{,t}(0,\,0) + (1-\lambda)^{-\frac{1}{2}}\eta^{11}_{,\tau}(0,\,0) = 0 \tag{36}$$

$$\mathrm{O}(\bar{\xi}^{3}) \ : \ \eta^{30}_{,t}(0,\,0) + (1-\lambda)^{-\frac{1}{2}}\left(\omega_2'(0)\,\eta^{10}_{,t}(0,\,0)\right) = 0 \tag{37}$$

$$\mathrm{O}(\bar{\xi}^{3}\delta) \ : \ \eta^{31}_{,t}(0\,,0) + (1-\lambda)^{-\frac{1}{2}}\left\{\omega_2''(0)\eta^{11}_{,t}(0,\,0) + \eta^{30}_{,\tau}(0,\,0)\right\} = 0 \tag{38}$$

$$O\left(\bar{\xi}^3\delta^2\right) \quad : \quad \eta^{32}_{,t}(0,0) + (1-\lambda)^{-\frac{1}{2}}\left\{\omega_2''(0)\eta^{12}_{,t}(0,0) + \eta^{31}_{,\tau}(0,0)\right\} = 0 \tag{39}$$

etc.

We note that terms of order $\bar{\xi}^{2i}$ are not included because such terms varnish automatically on simplification

The solution of Eq. (28) is

$$\eta^{10}(t,\tau) = \alpha_{10}(\tau)\cos t + \beta_{10}(\tau)\sin t + B(\tau) \tag{40}$$

$$\alpha_{10}(0) = -B(0) = -\frac{\lambda}{(1-\lambda)} \quad ; \quad \beta_{10}(0) = 0 \tag{41}$$

On substituting Eq. (41) into Eq. (29), we get

$$\eta^{11}_{,tt} + \eta^{11} = 2(1-\lambda f)^{-\frac{1}{2}}\{\alpha_{10}'\sin t - \beta_{10}'\cos t\} + \frac{\lambda f'}{2(1-\lambda f)^{\frac{3}{2}}}\{-\alpha_{10}\sin t + \beta_{10}\cos t\} \tag{42}$$

To ensure a uniformly valid solution in t, we equate to zero in Eq. (42), the coefficients of $\cos t$ and $\sin t$ and respectively get

$$\beta_{10}' - \frac{\lambda f'\beta_{10}}{4(1-\lambda f)} = 0 \quad ; \quad \alpha_{10}' - \frac{\lambda f'\alpha_{10}}{4(1-\lambda f)} = 0 \tag{43}$$

The solution of Eq. (43), is

$$\beta_{10}(\tau) = 0 \quad ; \quad \alpha_{10}(\tau) = -B(0)\left(\frac{1-\lambda}{1-\lambda f}\right)^{\frac{1}{4}} \tag{44}$$

It easily follows that

$$\eta^{10} = \alpha_{10}(\tau)\cos t + B(\tau) \tag{45}$$

The solution of the remaining equation in Eq. (42) is

$$\eta^{11}(t,\tau) = \alpha_{11}(\tau)\cos t + \beta_{11}(\tau)\sin t \tag{46}$$

$$a_{11}(0) = 0 \quad ; \quad \beta_{11}(0) = \frac{B(0)f'(0)(4-\lambda)}{4\lambda(1-\lambda)^{\frac{3}{2}}}$$

We next, substitute into Eq. (30) and get

$$\eta^{12}_{,tt} + \eta^{12} = 2(1-\lambda f)^{-\frac{1}{2}}\{\alpha_{11}'\sin t - \beta_{11}'\cos t\} + \frac{\lambda f'}{2(1-\lambda f)^{\frac{3}{2}}}\{-\alpha_{11}\sin t + \beta_{11}\cos t\} - \frac{1}{(1-\lambda f)}\{a_{10}''\cos t + B''(\tau)\} \tag{47}$$

To ensure a uniformly valid solution in t, we equate to zero in Eq. (47), the coefficients of $\cos t$ and $\sin t$ and respectively get

$$\beta_{11}' - \frac{\lambda f'\beta_{11}}{4(1-\lambda f)} = -\frac{\alpha_{10}''}{2(1-\lambda f)^{\frac{1}{2}}} \quad ; \quad \alpha_{11}' - \frac{\lambda f'\alpha_{11}}{4(1-\lambda f)} = 0 \tag{48}$$

On solving Eq. (48) we get

$$\beta_{11}(\tau) = (1-\lambda f)^{\frac{1}{4}}\left\{\beta_{11}(0) - \int_0^{\tau}\frac{\alpha_{10}''}{2(1-\lambda f)^{\frac{3}{4}}}ds\right\} \quad ; \quad \beta_{11}(0) = -\frac{B(0)f'(0)(4-\lambda)}{4(1-\lambda f)^{\frac{5}{4}}} \tag{49}$$

Similarly, we get

$$\alpha_{11}(\tau) = 0 \tag{50}$$

Thus, we get

$$\eta^{11} = \beta_{11}(\tau)\sin t \tag{51}$$

The solution of the remaining equation in Eq. (47) is

$$\eta^{12}(t,\tau) = \alpha_{12}(\tau)\cos t + \beta_{12}(\tau)\sin t - \frac{B''(\tau)}{(1-\lambda f)} \tag{52}$$

$$\alpha_{12}(0) = \frac{B''(0)}{(1-\lambda)} \quad ; \quad \beta_{12}(0) = 0$$

After substituting for terms on the right hand side of Eq. (31) and simplifying, we get

$$\begin{aligned}\eta^{30}_{,tt} + \eta^{30} = {} & 2(1-\lambda f)^{-\frac{1}{2}}\omega_2'(\tau)\alpha_{10}\cos t + \frac{b}{(1-\lambda f)}\left[B^3 + \left(\frac{3}{2}B\alpha_{10}^2\right) + 3\left(\frac{1}{4}\alpha_{10}^3 + a_{10}B^2\right)\cos t\right. \\ & \left. + \frac{3}{2}B\alpha_{10}^2\cos 2t + \frac{1}{4}\alpha_{10}^3\cos 3t\right] - \left(\frac{3b}{(1-\lambda f)}\right)\left[\xi_0^{(1)}\left\{\left(B^2 + \frac{1}{2}\alpha_{10}^2\right) + 2B\alpha_{10}\cos t + \frac{1}{2}\alpha_{10}^2\cos 2t\right\} + \left(\xi_0^{(1)}\right)^2(\alpha_{10}\cos t + B)\right]\end{aligned} \tag{53}$$

$$\eta^{30}(0,0) = 0 \quad ; \quad \eta^{30}_{,t}(0,0) + (1-\lambda)^{-\frac{1}{2}}\left(\omega_2'(0)\,\eta^{10}_{,t}(0,0)\right) = 0$$

To ensure a uniformly valid solution in t we equate to zero in Eq. (53) the coefficient of $\cos t$ and get

$$\omega_2'(\tau) = -\frac{1}{2}(1-\lambda)^{-\frac{1}{2}}b\left[\left(\frac{3}{1-\lambda f}\right)\left(\frac{1}{4}\alpha_{10}^2 + B^2\right) - \left(\frac{3}{1-\lambda f}\right)\left\{2B\xi_0^{(1)} + \left(\xi_0^{(1)}\right)^2\right\}\right] \tag{54}$$

where,

$$\omega_2'(0) = -\frac{1}{2}(1-\lambda)^{-\frac{1}{2}}b\left[\frac{15}{4(1-\lambda f)}B^2(0) + 3B(0)\left(2\xi_0^{(1)} + \left(\xi_0^{(1)}\right)^2\right)\right]. \tag{55}$$

We can further write Eq. (55) as

$$\omega_2'(0) = B^2(0)R_2 + B(0)R_3 \quad ; \quad R_2 = -\frac{15b}{8(1-\lambda)^{\frac{3}{2}}} \quad ., \quad R_3 = -3\left(2\xi_0^{(1)} + \left(\xi_0^{(1)}\right)^2\right). \tag{56}$$

The remaining equation in Eq. (53) is re–arranged as

$$\eta^{30}_{,tt} + \eta^{30} = r_1(\tau) + r_2(\tau)\cos 2t + r_3(\tau)\cos 3t\,. \tag{57}$$

$$\eta^{30}(0,0) = 0 \quad ; \quad \eta^{30}_{,t}(0,0) + (1-\lambda)^{-\frac{1}{2}}\left(\omega_2'(0)\,\eta^{10}_{,t}(0,0)\right) = 0$$

where,

$$r_1(\tau) = \left(\frac{b}{(1-\lambda f)}\right)\left(B^3 + \frac{3}{2}B\alpha_{10}^2\right) - \frac{3b}{(1-\lambda f)}\left[\xi_0^{(1)}\left(B^2 + \frac{1}{2}a_{10}^2\right) + \left(\xi_0^{(1)}\right)^2 B\right]. \tag{58}$$

$$r_2(\tau) = \frac{3b}{2(1-\lambda f)}\left(B\alpha_{10}^2 - \alpha_{10}^2\xi_0^{(1)}\right) \quad ; \quad r_3(\tau) = \frac{b\alpha_{10}^3}{4(1-\lambda f)}\,. \tag{59}$$

and where,

$$r_1(0) = -\frac{5bB^3(0)}{2(1-\lambda)} - \frac{3bB^2(0)}{(1-\lambda)}\left\{\frac{3}{2B(0)}\xi_0^{(1)} + \frac{1}{B^2(0)}\left(\xi_0^{(1)}\right)^2\right\} \; ; r_2(0) = \frac{3B^3(0)}{2(1-\lambda)}\left\{1 + \frac{\xi_0^{(1)}}{B(0)}\right\} \; . \quad r_3(0) = -\frac{1}{4(1-\lambda)}bB^3(0) \tag{60}$$

Meanwhile, the following will be needed later

$$r_1'(0) = \frac{B^4(0)f'(0)b}{(1-\lambda)}R_4 \; , \; R_4 = \left[\left(\frac{17}{2} + \frac{3B(0)\lambda}{4}\right) - 3\left\{\left\{\frac{1}{\lambda}\left\{\xi_0^{(1)}\left(\frac{2}{B(0)} - \frac{1-\lambda}{4}\right) + \frac{\left(\xi_0^{(1)}\right)^2}{B^2(0)}\right\} + \left\{\frac{3}{2}\left(\frac{\xi_0^{(1)}}{B(0)}\right) + \frac{\left(\xi_0^{(1)}\right)^2}{B(0)}\right\}\right\}\right\}\right] . \tag{61}$$

$$r_2'(0) = -\frac{3B''(0)f'(0)bR_5}{2(1-\lambda)} \; ; \; R_5 = \left[\left(1 + \frac{3\lambda}{2}\right) - \left\{\lambda + \frac{1}{2}\xi_0^{(1)}(1-\lambda)\right\}\right] \; ; \; r_3'(0) = -\frac{7bB^4(0)f'(0)}{16(1-\lambda)} . \tag{62}$$

The solution of Eq. (57) is

$$\eta^{30}(t,\tau) = \alpha_{30}(\tau)\cos t + \beta_{30}(\tau)\sin t + r_1(\tau) - \frac{1}{3}r_2(\tau)\cos 2t - \frac{1}{8}r_3(\tau)\cos 3t . \tag{63}$$

$$\alpha_{30}(0) = -\frac{b}{(1-\lambda)}\left[\frac{65B^3(0)}{32} - 4B^2(0)\xi_0^{(1)} - 3B(0)\left(\xi_0^{(1)}\right)^2\right] \quad ; \quad \beta_{30}(0) = 0 . \tag{64}$$

On substituting into Eq. (23), we get

$$\begin{aligned}\eta^{31}_{,tt} + \eta^{31} &= 2(1-\lambda f)^{-\frac{1}{2}}\omega_2'(\tau)\beta_{11}\sin t + \frac{3b}{(1-\lambda f)^{\frac{1}{2}}}\left[\left(B^2 + \frac{\alpha_{10}^2}{2}\right) + 2B\alpha_{10}\cos t + \frac{\alpha_{10}^2}{2}\cos 2t\right] \\ &\times\beta_{11}\sin t - \frac{3b}{(1-\lambda f)}\left[\left\{\alpha_{10}\beta_{11}\sin 2t + 2B\beta_{11}\sin t\right\}\xi_0^{(1)} + \xi_0^{(1)}\beta_{11}\sin t\right] + \frac{2}{(1-\lambda f)}\omega_2'(\tau)\alpha_{10}\cos t \\ &-2(1-\lambda f)^{-\frac{1}{2}}\left\{-\alpha_{30}''\sin t + \beta_{30}''\cos t + \frac{2}{3}r_2''(\tau)\sin 2t + \frac{3}{8}r_3''(\tau)\sin 3t\right\} + \frac{\lambda f'}{2(1-\lambda f)^{\frac{3}{2}}}\left\{\frac{2}{3}r_2(\tau)\sin 2t\right. \\ &\left. - \alpha_{30}\sin t + \beta_{30}ost + \frac{3}{8}r_3(\tau)\sin 3t\right\} + \frac{1}{(1-\lambda f)}\omega_2'''(\tau)\alpha_{10}\sin t\end{aligned} \tag{65}$$

$$\eta^{31}(0,0) = 0 \quad ; \quad \eta^{31}_{,t}(0,0) + (1-\lambda)^{-\frac{1}{2}}\left(\omega_2'(0)\eta^{11}_{,t}(0,0) + \eta^{30}_{,\tau}(0,0)\right) = 0$$

To ensure a uniformly valid solution in t, we equate to zero in Eq.(65), the coefficients of $\cos t$ and $\sin t$ to get respectively

$$\beta_{30}' - \frac{\lambda f'\beta_{10}}{4(1-\lambda f)} = \frac{\omega_2'\alpha_{10}}{(1-\lambda f)^{\frac{1}{2}}} \quad ; \quad \alpha_{30}' - \frac{\lambda f'\alpha_{301}}{4(1-\lambda f)} = h_1(\tau) . \tag{66}$$

where,

$$h_1(\tau) = -\frac{1}{2}(1-\lambda)^{\frac{1}{2}}\left[2\omega_2'(\tau)(1-\lambda)^{-\frac{1}{2}}b\beta_{11} + \frac{3b\beta_{11}}{(1-\lambda f)^{\frac{1}{2}}}\left(B^2 + \frac{1}{4}\alpha_{10}^2\right) - \left(\frac{3b}{1-\lambda f}\right)\left\{2B\xi_0^{(1)}\beta_{11} + \beta_{11}\left(\xi_0^{(1)}\right)^2\right\} + \left(\frac{\omega_2''\alpha_{10}}{1-\lambda f}\right)\right] . \tag{67}$$

We shall later need the value of $\omega_2''(0)$ which can easily be evaluated from Eq. (54) to be

$$\omega_2''(0) = -\frac{3f'(0)}{16(1-\lambda)^{\frac{3}{2}}}R_1 \quad ; \quad R_1 = B^2(0)(16-4\lambda) - 8B(0)\xi_0^{(1)}(1-\lambda). \tag{68}$$

In a similar way, the value of $h_1(0)$ gives, from Eq. (67)

$$h_1(0) = \frac{bB(0)f'(0)(1-\lambda)}{8(1-\lambda)}\left[2\left(\frac{B^2(0)R_2 + B(0)R_3}{(1-\lambda)^{\frac{1}{2}}}\right) + \frac{15B^2(0)}{4(1-\lambda)} - 3\left(\frac{2\,\xi_0^{(1)} + \left(\xi_0^{(1)}\right)^2}{(1-\lambda)}\right) + \left(\frac{4(1-\lambda)^{\frac{1}{2}}\omega_2''(0)}{(1-\lambda)f'(0)}\right)\right]. \tag{69}$$

Meanwhile, the solution to Eq. (66) is

$$\beta_{30}(\tau) = (1-\lambda f)^{-\frac{1}{4}}\left\{\int_0^\tau \frac{\omega_2'\alpha_{10}}{(1-\lambda f)^{\frac{1}{4}}}ds\right\} \quad ; \quad \alpha_{30}(\tau) = (1-\lambda f)^{-\frac{1}{4}}\left\{\alpha_{30}(0)(1-\lambda f)^{\frac{1}{4}} + \int_0^\tau h_1(s)\left(1-\lambda f(s)\right)^{\frac{1}{4}}ds\right\}. \tag{70}$$

The remaining equation in Eq. (65) is now gathered to read

$$\eta^{31}_{,tt} + \eta^{31} = r_4(\tau)\sin 2t + r_5(\tau)\sin 3t. \tag{71}$$

$$\eta^{31}(0,0) = 0 \quad ; \quad \eta^{31}_{,t}(0,0) + (1-\lambda)^{-\frac{1}{4}}\left\{\omega_2'(0)\eta^{11}_{,t}(0,0) + \eta^{30}_{,\tau}(0,0)\right\} = 0$$

where,

$$r_4(\tau) = \frac{3b}{(1-\lambda f)}\left\{B\alpha_{10}\beta_{11} + \frac{\alpha_{10}^2\beta_{11}}{4}\right\} - \frac{3b}{(1-\lambda f)}\left\{\alpha_{10}\beta_{11}\xi_0^{(1)}\right\} + \frac{4r_2'(\tau)}{3(1-\lambda f)^{\frac{1}{2}}} - \frac{4f'r_2(\tau)}{3(1-\lambda f)^{\frac{3}{2}}}. \tag{72}$$

$$r_5(\tau) = \frac{3b\alpha_{10}^2\beta_{11}}{4(1-\lambda)} - \frac{3r_3'(\tau)}{4(1-\lambda f)^{\frac{1}{2}}} - \frac{3\lambda f' r_3'(\tau)}{16(1-\lambda f)^{\frac{3}{2}}}. \tag{73}$$

Thus, solving Eq. (71), we get

$$\eta^{31}(t,\tau) = \alpha_{31}(\tau)\cos t + \beta_{31}(\tau)\sin t - \frac{1}{3}r_4(\tau)\sin 2t - \frac{1}{8}r_5(\tau)\sin 3t. \tag{74}$$

$$\alpha_{31}(0) = 0 \quad ; \quad \beta_{31} \neq 0$$

Later, we shall have cause to use $\alpha_{30}'(0)$ which we now evaluate from Eq. (66) as

$$\alpha_{30}'(0) = h_1(0) + \frac{\lambda f'(0)\alpha_3(0)}{4(1-\lambda)} \; i.e\; \alpha_{30}'(0) = h_1(0) - \frac{b\lambda f'(0)}{4(1-\lambda)}\left[\frac{65B^3(0)}{32} - 4B^2(0)\xi_0^{(1)} - 3B(0)\left(\xi_0^{(1)}\right)^2\right]. \tag{75}$$

So far, we write the summary as

$$\eta(t,\tau) = \bar{\xi}\left(\eta^{10} + \delta\eta^{11} + \delta^2\eta^{12} + \ldots\right) + \bar{\xi}^3\left(\eta^{30} + \delta\eta^{31} + \delta^2\eta^{32} + \ldots\right) + \ldots. \tag{76}$$

6. Maximum Displacement

Following, Eq. (24), the condition for maximum displacement ξ_a is

$$\eta_{,t} + (1-\lambda f)^{-\frac{1}{2}}\Big[\Big\{\omega_2'(\tau)\,\overline{\xi}^2 + \omega_3'(\tau)\overline{\xi}^3 + \quad . \quad . \quad . \quad \Big\}\eta_{,t} + \delta\eta_{,\tau}\Big] = 0\cdot \tag{77}$$

We let $\hat{t}_a$, t_a $\overline{t}_a$ and τ_a be the critical values of $\hat{t}$, t, $\overline{t}$ and τ respectively at the maximum displacement and let us assume the following series..

$$\hat{t}_a = \left(\hat{t}_0 + \delta\hat{t}_{01} + \delta^2\hat{t}_{02} + \ldots\right) + \overline{\xi}\left(\hat{t}_{10} + \delta\hat{t}_{11} + \delta^2\hat{t}_{12} + \ldots\right) + \overline{\xi}^2\left(\hat{t}_{20} + \delta\hat{t}_{21} + \delta^2\hat{t}_{22} + \ldots\right) + \ldots . \tag{78}$$

$$t_a = \left(t_0 + \delta t_{01} + \delta^2 t_{02} + \ldots\right) + \overline{\xi}\left(t_{10} + \delta t_{11} + \delta^2 t_{12} + \ldots\right) + \overline{\xi}^2\left(t_{20} + \delta t_{21} + \delta^2 t_{22} + \ldots\right) + \ldots . \tag{79}$$

$$\overline{t}_a = \left(\overline{t}_0 + \delta\overline{t}_{01} + \delta^2\overline{t}_{02} + \ldots\right) + \overline{\xi}\left(\overline{t}_{10} + \delta\overline{t}_{11} + \delta^2\overline{t}_{12} + \ldots\right) + \overline{\xi}^2\left(\overline{t}_{20} + \delta\overline{t}_{21} + \delta^2\overline{t}_{22} + \ldots\right) + \ldots . \tag{80}$$

$$\begin{aligned}\tau_a = \delta\hat{t}_a = \delta\Big\{&\left(\hat{t}_0 + \delta\hat{t}_{01} + \delta^2\hat{t}_{02} + \ .\ .\ .\right) + \overline{\xi}\left(\hat{t}_{10} + \delta\hat{t}_{11} + \delta^2\hat{t}_{12} + \ .\ .\ .\right) + \overline{\xi}^2\big(\hat{t}_{20} + \delta\hat{t}_{21} + \delta^2\hat{t}_{22}\\ &+ ..\ .\ .\ .\big)\Big\} + \ .\ .\ .\end{aligned} \tag{81}$$

We shall now expand the relevant terms of Eq. (77) as follows:

$$\begin{aligned}\xi\eta^{10}_{,t}(t_a,\tau_a) = \overline{\xi}\Big[&\eta^{10}_{,t}(t_0,0) + \Big\{\left(\delta t_{01} + \delta^2 t_{02} + \ .\ .\ .\right) + \overline{\xi}^2\left(t_{20} + \delta t_{21} + \delta^2 t_{22} \ + \ .\ .\ .\right)\Big\}\eta^{10}_{,t} + \delta\Big\{(t_0\\ &+ \delta t_{01} + \delta^2 t_{02} + \ .\ .\ .\ ) + \overline{\xi}^2\left(t_{20} + \delta t_{21} + \delta^2 t_{22} + \ .\ .\ .\right)\Big\}\eta^{10}_{,t\tau} + \ .\ .\ .\ \Big]\Big]\Big|_{(\hat{t},\,0)}\end{aligned} \tag{82}$$

$$\begin{aligned}&(1-\lambda f)^{-\frac{1}{2}}\,\overline{\xi}\delta\eta^{10}_{,\tau} = \delta\overline{\xi}\Big[(1-\lambda)^{-\frac{1}{2}}\eta^{10}_{,\tau} + (1-\lambda)^{-\frac{1}{2}}\eta^{10}_{,\tau t}\left\{\delta t_{01} + \delta^2 t_{02} + (t_{20} + \delta t_{21}).+ \ .\ .\ .\right\}\\ &+\delta\left\{\hat{t}_0 + \delta\hat{t}_{01} + \ .\ .\ .\ . + \overline{\xi}^2\left(\hat{t}_{20} + \delta\hat{t}_{21}\right)\right\}\left((1-\lambda f)^{-\frac{1}{2}}\eta^{10}_{,\tau}\right)_{,\tau} + \ .\ .\ .\ \Big]\Big|_{(\hat{t}\ ,\ 0)} + \ .\ .\ .\ .\ ;\end{aligned} \tag{83}$$

$$(1-\lambda f)^{-\frac{1}{2}}\,\overline{\xi}^3\delta\,\eta^{30}_{,\tau} = (1-\lambda)^{-\frac{1}{2}}\,\overline{\xi}^3\,\delta\eta^{31}_{,\tau}\Big|_{(\hat{t}\,,0)} + \ .\ .\ .$$

$$\overline{\xi}^3\eta^{30}_{,t} = \overline{\xi}^3\Big[\eta^{30}_{,t}(\hat{t}_0,0) + \left\{\delta t_{01} + \delta^2 t_{02} + \ .\ .\ .\ \right\}\eta^{30}_{,tt} + \delta\left\{\hat{t}_0 + \delta\hat{t}_{01} + \ .\ .\ .\ \right\}\eta^{30}_{,\tau}\Big]\Big|_{(\hat{t}\ ,\ 0)}$$

$$\begin{aligned}&\overline{\xi}^3\delta\,\eta^{31}_{,t} = \overline{\xi}^3\,\delta\eta^{31}_{,t}\Big|_{(\hat{t}\,,0)};\ (1-\lambda f)^{-\frac{1}{2}}\omega_2'(\tau)\overline{\xi}^3\eta^{10}_{,t} = \overline{\xi}^3\Big[(1-\lambda)^{-\frac{1}{2}}\omega_2'(0)\eta^{10}_{,t} + (1-\lambda)^{-\frac{1}{2}}\\ &\times\omega_2'(0)\eta^{10}_{,t}\left\{\delta t_{01} + \ .\ .\ .\right\} + \delta\left\{\left((1-\lambda f)^{-\frac{1}{2}}\omega_2'(\tau)\eta^{10}_{,t}\right)'\left(\hat{t}_0 + \delta\hat{t}_{01} + \right)\right\}\Big]\Big|_{(\hat{t},0)} + \ .\ .\ .\end{aligned} \tag{84}$$

$$(1-\lambda f)^{-\frac{1}{2}}\,\omega'(\tau)\,\overline{\xi}^3\eta^{10}_{,t} = (1-\lambda)^{-\frac{1}{2}}\,\omega_2'(0)\delta\overline{\xi}^3\eta^{11}_{,t}\Big|_{(\hat{t}\,,0)} + \ .\ .\ .$$

where the left hand sides of these expansions are evaluated at (t_a,τ_a) while the expansions on the right hand sides are evaluated at $(t_0,0)$

By substituting all these expansions into Eq. (77) and equating the coefficients $\left(\overline{\xi}^i\delta^j\right)$, we get the following:

$$O\left(\overline{\xi}\right)\ :\ \eta^{10}_{,t} = 0\,. \tag{85}$$

$$O\left(\overline{\xi}\delta\right)\quad :\ \eta^{11}_{,t} + (1-\lambda)^{-\frac{1}{2}}\eta^{10}_{,t} + \hat{t}_0\eta^{10}_{,t\tau} + t_{01}\eta^{10}_{,tt} = 0\cdot \tag{86}$$

$$O\left(\bar{\xi}^3\right) \; : \; t_{20}\,\eta^{10}_{,tt} + \eta^{30}_{,t} + (1-\lambda)^{-\frac{1}{2}}\,\omega_2'(0)\eta^{10}_{,t} = 0 \cdot \tag{87}$$

From the $\eta^{10}_{,t} = 0$, we get

$$t_0 = \pi . \tag{88}$$

where we have used the least non - trivial value of t_0

It also follows that

$$t_{01} = -\frac{1}{\eta^{10}_{,tt}}\left[\eta^{11}_{,t} + (1-\lambda)^{\frac{1}{2}}\eta^{10}_{,t} + \hat{t}_0\eta^{10}_{,t\tau}\right] = \frac{f'(0)(1+\lambda)}{4\lambda(1-\lambda)^{\frac{3}{2}}} \quad ; \quad t_{20} = 0 . \tag{89}$$

To determine the maximum displacement ξ_a, we evaluate Eq. (76) at the critical values of the variables, using Eq. (79) – Eq. (81) and get

$$\begin{aligned} \eta(t_a,\tau_a) = \eta_a = \bar{\xi}\left[\eta^{10} + \delta\left(t_{01}\eta^{10}_{,t} + \hat{t}_0\eta^{10}_{,\tau} + \eta^{11}\right) + \ldots\right]\Big|_{(\hat{t}\,,\,0)} + \bar{\xi}^3\Big[t_{20}\eta^{10}_{,t} + \eta^{30} + \delta\big\{\hat{t}_{20}\,\eta^{10}_{,\tau} \\ + t_{21}\eta^{10}_{,tt} + t_{01}t_{20}\eta^{10}_{,tt} + \hat{t}_0 t_{20}\eta^{10}_{,t\tau} + t_{20}\eta^{11}_{,t} + t_{01}\eta^{30}_{,t} + \hat{t}_0\eta^{30}_{,\tau} + \eta^{31}\big\} + \ldots\Big]\Big|_{(\hat{t}\,,\,0)} + \ldots \end{aligned} \tag{90}$$

It is obvious that, on evaluation, most of the terms in Eq. (90), will varnish. There are however certain terms such as $\hat{t}_0$, $\bar{t}_{20}$ etc. in Eq. (90) which are yet to be determined and which we shall next determine from the second of Eq. (22) and get

$$\begin{aligned} \frac{d\bar{t}}{d\hat{t}} = \left(1-\lambda f\left(\delta\hat{t}\right)\right)^{\frac{1}{2}} = \Bigg[\Bigg[1 - \frac{1}{2}\left(\frac{\lambda}{1-\lambda}\right)\left\{f'(0)\delta\hat{t} + \frac{1}{2}f''(0)\left(\delta\hat{t}\right)^2 + \frac{1}{6}f'''(0)\left(\delta\hat{t}\right)^3\right\} - \frac{1}{8}\left(\frac{\lambda}{1-\lambda}\right)^2\Big\{f'(0)\delta\hat{t} \\ + \frac{1}{2}f''(0)\left(\delta\hat{t}\right)^2 + \frac{1}{6}f'''(0)\left(\delta\hat{t}\right)^3\Big\}^2 + \ldots\Bigg] \end{aligned} \tag{91}$$

$$\begin{aligned} \therefore \; \bar{t}_0 = (1-\lambda)^{\frac{1}{2}}\Bigg[\hat{t}_0 - \frac{1}{2}\left(\frac{\lambda}{1-\lambda}\right)\left\{\frac{f'(0)\delta\hat{t}^2}{2} + \frac{f''(0)\delta^2\hat{t}^3}{6} + \frac{f'''(0)\delta^3\hat{t}^4}{24}\right\} - \frac{1}{8}\left(\frac{\lambda}{1-\lambda}\right)^2\Bigg\{\frac{f'(0)\delta\hat{t}^2}{2} \\ + \frac{f''(0)\delta^2\hat{t}^3}{6} + \ldots\Bigg\}^2\Bigg] \end{aligned} \tag{92}$$

Therefore,

$$\bar{t}_a = (1-\lambda)^{\frac{1}{2}}\,\hat{t}_0 \quad ; \quad \bar{t}_{01} = (1-\lambda)^{\frac{1}{2}}\left[\hat{t}_{01} - \frac{\lambda f''(0)\hat{t}_0^2}{2(1-\lambda)}\right] \quad ; \quad \bar{t}_{20} = (1-\lambda)^{\frac{1}{2}}\,\hat{t}_{20} . \tag{93}$$

Similarly, we get

$$\bar{t}_{21} = (1-\lambda)^{\frac{1}{2}}\left[\hat{t}_{21} - \frac{\lambda f'(0)\hat{t}_0\hat{t}_{20}}{2(1-\lambda)}\right]. \tag{94}$$

Now, to determine $\hat{t}_0$ and $\bar{t}_{20}$, we next evaluate the first of Eq. (23), using third of Eq. (53), at the critical values to get

$$t_a = \bar{t}_a + \omega_2'(0)\hat{t}_a\bar{\xi}^2 + \omega_3'(0)\hat{t}_a\bar{\xi}^3 + \ldots \tag{95}$$

On substituting for t_a, $\bar{t}_a$ and $\hat{t}_a$, and equating the relevant coefficients, we get

$$O(1) \; : \; t_0 = \bar{t}_0 = (1-\lambda)^{\frac{1}{2}} \hat{t}_0 = \pi \quad \therefore \; \hat{t}_0 = \frac{\pi}{(1-\lambda)^{\frac{1}{2}}} . \tag{96}$$

$$O(\delta) \; : \; t_{01} = \bar{t}_{01} = (1-\lambda)^{\frac{1}{2}} \left[\hat{t}_{01} - \frac{\lambda f''(0)\hat{t}_0^2}{2(1-\lambda)} \right] = \frac{f'(0)(1-\lambda)}{4(1-\lambda)^{\frac{3}{2}}} \; :. \; \hat{t}_{01} = \frac{f'(0)(1-\lambda)}{4(1-\lambda)^{\frac{3}{2}}} + \frac{\lambda f''(0)\hat{t}_0^2}{2(1-\lambda)} . \tag{97}$$

$$O\left(\bar{\xi}^2\right) \; : \; t_{20} = \bar{t}_{20} + \omega_2'(0)\hat{t}_0 = (1-\lambda)^{\frac{1}{2}} \hat{t}_{20} + \omega_2'(0)\hat{t}_0 \quad \therefore \; \hat{t}_{20} = -\omega_2'(0)\hat{t}_0 (1-\lambda)^{-\frac{1}{2}} + \omega_2'(0)\hat{t}_0 \cdot \tag{98}$$

After evaluating terms in Eq. (90) the only non–varnishing terms are as

$$\eta_a = \bar{\xi}\left[\eta^{10} + \delta \hat{t}_a \eta^{10}_{,\tau} + \; . \; . \; . \right]\Big|_{(t_0\,,\,0)} + \bar{\xi}^3\left[\eta^{20} + \delta\left(\hat{t}_{20}\eta^{10}_{,\tau} + \hat{t}_0\eta^{30}_{,\tau}\right)\right]\Big|_{(t_o\,,\,0)} + \; . \; . \; . \; . \tag{99}$$

On simplifying Eq. (99), we get

$$\eta_a = 2B(0)\bar{\xi}\left(1 + \delta f'(0) A_{11}\right) + \frac{4\bar{\xi}^3 B^3(0) b}{(1-\lambda)}\left[(1 + A_{30}) + \delta f'(0) A_{31}\right]. \tag{100}$$

where,

$$A_{11} = \frac{\pi(1+\lambda)}{8(1-\lambda)^{\frac{5}{2}}} \; ; \; A_{30} = \frac{\left(\xi_0^{(1)}\right)^2}{4B(0)} - \frac{3}{2}\left\{ \frac{3\xi_0^{(1)}}{2B(0)} - \frac{\left(\xi_0^{(1)}\right)^2}{B^2(0)} \right\} \; ; \; A_{31} = \frac{(1-\lambda)}{4B^3(0)}\left[\frac{\hat{t}_{20}(1+\lambda)B(0)}{4(1-\lambda)} - \frac{\alpha_{30}'(0)}{f'(0)} + \frac{B^4(0)}{(1-\lambda)}\left\{ R_4 + \frac{3R_5}{2} - \frac{7}{16} \right\} \right]. \tag{101}$$

and where $\alpha_{30}'(0)$ and $\hat{t}_{20}$ are as in Eqs. (75) and. (98) respectively.

7. Dynamic Buckling Load

For the purpose of determining the dynamic buckling load λ_D, we rewrite Eq. (100) simply as

$$\eta_a = \bar{\xi} e_1 + \bar{\xi}^3 e_3 + \; . \; . \; . \tag{102}$$

where,

$$e_1 = 2B(0)\left(1 + \delta f'(0) A_{11}\right) \quad ; \quad e_3 = \frac{4B^3(0) b}{(1-\lambda)}\left[(1 + A_{30}) + \delta f'(0) A_{31}\right]. \tag{103}$$

As in the first of Eqs. (12) and. (13), we now reverse the series Eq. (102) and let

$$\bar{\xi} = g_1 \eta_a + g_3 \eta_a^3 + \; . \; . \; . \tag{104}$$

where by, we eventually get

$$g_1 = \frac{1}{e_1} \; ; \; g_3 = -\frac{c_3}{e_1^4} . \tag{105}$$

The maximization Eq. (3) (with η_a substituted for ξ_a) gives

$$\eta_{ad}^2 = -\frac{g_1}{3g_3} = \eta_{ad} = \sqrt{\frac{e_1^3}{3e_3}} . \tag{106}$$

where η_{ad} is the value of η_a at dynamic buckling.

On substituting in Eq. (104) for η_{ad}, we get

$$\bar{\xi} = \frac{2}{3\sqrt{3}}\sqrt{\frac{e_1}{e_3}}\,. \tag{107}$$

This gives on simplifications

$$\left(1-\lambda_D\right)^{\frac{3}{2}} = \frac{3\sqrt{6}}{2}\left(b\right)^{\frac{1}{2}}\lambda_D\bar{\xi}\left[\frac{\left(1+A_{30}\right)+\delta f'(0)A_{31}}{1+\delta f'(0)A_{11}}\right]^{\frac{1}{2}}. \tag{108}$$

Certainly, Eq. (108) is implicit in the load parameter λ_D and is asymptotic in nature.

8. Analysis of Results

Equation (89) clearly shows the contribution of each parameter to the dynamic buckling process. To the level of the approximation retained, the dynamic buckling process depends, among other things, on the first derivative of the load function evaluated at the initial time. Using Eq. (19), we can easily relate the static and dynamic buckling loads to get

$$\left(\frac{1-\lambda_D}{1-\lambda_S}\right)^{\frac{3}{2}} = \sqrt{2}\left(\frac{\lambda_D}{\lambda_S}\right)\left[\frac{\left(1+A_{30}\right)+\delta f'(0)A_{31}\lambda_D}{1+\delta f'(0)A_{11}\lambda_D}\right]^{\frac{1}{2}}. \tag{109}$$

The relationship is independent of the imperfection parameter ξ. Thus, given either of λ_D or λ_S, we can easily determine the other value.

The corresponding step loading result for the case, $\delta = 0$, $f\left(\delta\hat{t}\right) \cong 1$ easily yields.

$$\left(1-\lambda_D\right)^{\frac{3}{2}} = \frac{3\sqrt{6}}{2}\left(b\right)^{\frac{1}{2}}\lambda_D\bar{\xi}\left(1+A_{30}\right)^{\frac{1}{2}}. \tag{110}$$

and

$$\left(\frac{1-\lambda_D}{1-\lambda_S}\right)^{\frac{3}{2}} = \sqrt{2}\left(\frac{\lambda_D}{\lambda_S}\right)\left(1+A_{30}\right)^{\frac{1}{2}}. \tag{111}$$

The results Eqs. (110) and (111) are those for a pre–loaded case later trapped by a step load.

However, if there is no pre–load in the step loading case, then, we have

$$\left(1-\lambda_D\right)^{\frac{3}{2}} = \frac{3\sqrt{6}}{2}\left(b\right)^{\frac{1}{2}}\lambda_D\bar{\xi}\,. \tag{112}$$

and

$$\left(\frac{1-\lambda_D}{1-\lambda_S}\right)^{\frac{3}{2}} = \sqrt{2}\left(\frac{\lambda_D}{\lambda_S}\right). \tag{113}$$

The Eqs. (112) and (113) were initially obtained by Budiansky [18] by using phase plane analysis:

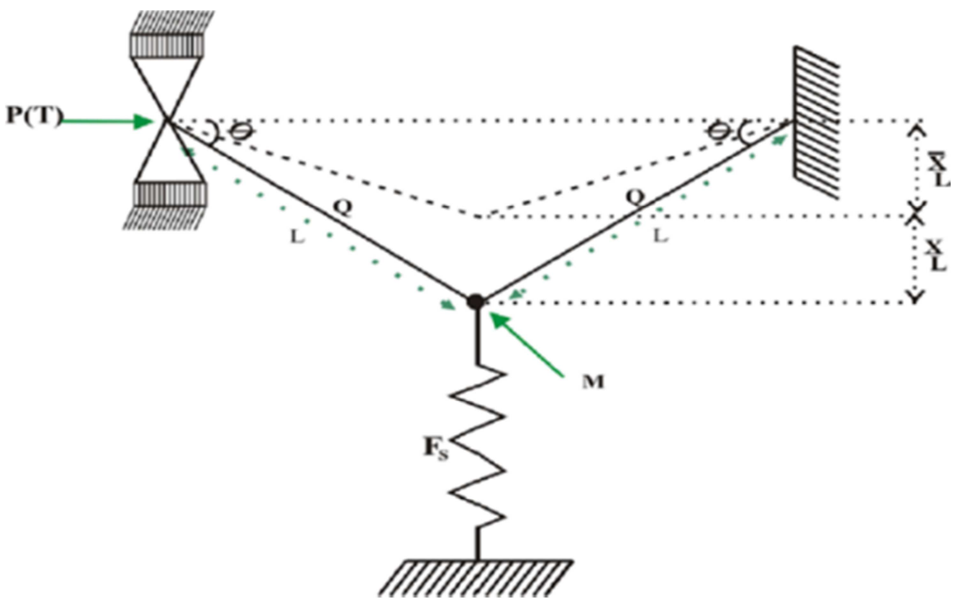

Figure 1. *A Simple Cubic Model Structure.*

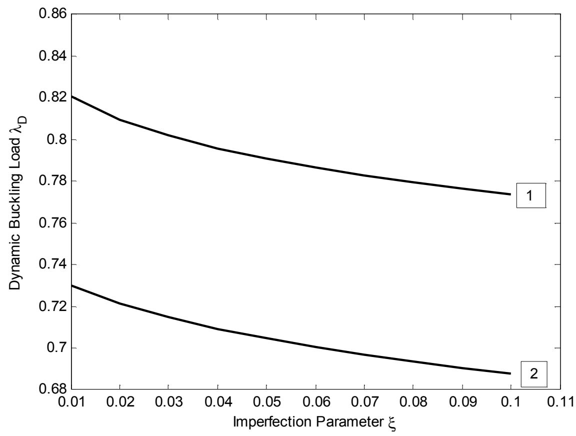

Figure 2. *Dynamic buckling load* λ_D *versus imperfection parameter* $\overline{\xi}$ *calculated by Eq. (108) for different values of* λ_0, δ *and* $f'(0)=-1$ *for* $f(\tau)=e^{-\tau}$. (1) $\lambda_0 = 0.2$, $\delta = 0.03$, (2) $\lambda_0 = 0.2$, $\delta = 0.0$.

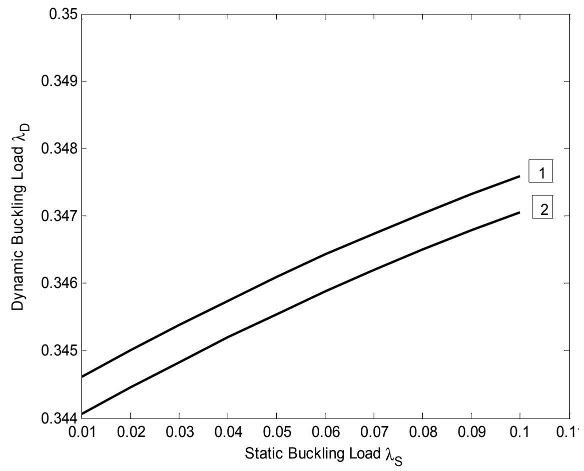

Figure 3. *Dynamic buckling load* λ_D *versus the static buckling load* λ_S *calculated by Eq. (109) for different values of* λ_0, δ *and* $f'(0) = -1$ *for* $f(\tau) = e^{-\tau}$. (1) $\lambda_0 = 0.2$, $\delta = 0.03$, (2) $\lambda_0 = 0.2$, $\delta = 0.0$.

9. Conclusion

Using regular perturbation and asymptotics, we have been able to determine the dynamic buckling load of a Pre – Statically loaded nonlinear cubic elastic model structure struck by a slowly varying dynamic load. The effects of the pre–load and slowly varying dynamic load are determined. The dynamic and static buckling loads are mathematically related and the relationship is independent of the imperfection parameter. Specializations are made to various cases, including the case of no pre–static load and step load.

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