

Boundary Value Problems of Nonlinear Variable Coefficient Fractional Differential Equations

Badawi Hamza Elbadawi Ibrahim^{1, 3}, Qixiang Dong², Zhengdi Zhang^{1, *}

¹Department of Mathematics, Faculty of Science, Jiangsu University, Zhenjiang, China

²School of Mathematical Sciences, Yangzhou University, Yangzhou, China

³Department of Mathematics, Faculty of Education, University of Khartoum, Sudan

Email address:

badawi.12@hotmail.com (B. H. E. Ibrahim), dyzhang@ujs.edu.cn (Zhengdi Zhang)

*Corresponding author

To cite this article:

Badawi Hamza Elbadawi Ibrahim, Qixiang Dong, Zhengdi Zhang. Boundary Value Problems of Nonlinear Variable Coefficient Fractional Differential Equations. *American Journal of Applied Mathematics*. Vol. 7, No. 6, 2019, pp. 157-163. doi: 10.11648/j.ajam.20190706.13.

Received: October 24, 2019; **Accepted:** November 19, 2019; **Published:** December 30, 2019

Abstract: It is recognized that the theory of boundary value problems for fractional order-differential equations is one of the rapidly developing branches of the general theory of differential equations. As far as we know, most of the papers studied the fractional Riemann-Liouville derivative with respect to boundary values that are zero. However, for the purpose of this study, we concern ourselves with Caputo type derivative of the order $\alpha \in (2, 3)$, with respect to boundary values that are nonzero. We establish sufficient conditions for the existence of solutions for boundary value problem of nonlinear variable coefficient of fractional order. On the other hand, the boundary value problem is formulated as follows:

$${}^c D^\alpha u(t) + p(t)f(t, u(t)) + q(t) = 0, u(0) = a, u'(0) = b, u(1) = d$$

Where $a, b, d \in \mathbb{R}$ are constants. In this paper, we investigate the existence and uniqueness of solutions for a class of boundary value problem of the nonlinear variable coefficient of fractional differential equations. The existence of solutions involving Caputo fractional derivatives is discussed under the assumption that the bounded conditions are constants. By means of the Banach contraction mapping principle and Larry-Schauder alternative, the existence of solutions are obtained. Finally, some examples are discussed to illustrate the results, which are generalized to nonlinear fractional derivatives with variable coefficients.

Keywords: Fractional Derivatives, Fixed Point Theorem, Boundary Value Problem

1. Introduction

In recent years, the theory of fractional differential equations has become an important area of study, see [1, 2, 3, 4]. Boundary value problems of fractional differential equations have applications in various fields of science such as physics, mechanics, chemistry, engineering, etc.[7, 6, 5, 8].

Also, it has received great attention and a variety of results concerning the existence of solutions, based on different kinds of analytic techniques, which can be found in [11, 10, 9, 12]. In [13] the authors considered the existence of multiple positive solutions for the following fractional differential equations with a negatively perturbed term

$$\begin{cases} -D^\alpha u(t) = p(t)f(t, u(t)) - q(t), & 0 < t < 1, \\ u(0) = u'(0) = u(1) = 0, \end{cases}$$

Where D^α is the standard Riemann-Liouville derivative, $2 < \alpha \leq 3$ is a real number, $q : (0, 1) \rightarrow [0, \infty]$ is Lebesgue integrable and does not vanish identically on any subinterval of $(0, 1)$. They established the existence results by Krasnoselskii's fixed point theorem in a cone. Cui [14] studied the following boundary value problem

$$\begin{cases} D^p x(t) + p(t)f(t, x(t)) + q(t) = 0, & 0 < t < 1, \\ x(0) = x'(0) = 0, & x(1) = 0, \end{cases}$$

where D^p is the standard Riemann-Liouville derivative, $2 < p \leq 3$ is a real number, $q : (0, 1) \rightarrow R$ is continuous and Lebesgue integrable, $p : (0, 1) \rightarrow [0, \infty)$ is continuous and does not vanish identically on any subinterval of $(0, 1)$. Under the assumption that $f(t, x)$ is a Lipschitz continuous function, by using u_0 -positive operator, he deduced that the Lipschitz constant is related to the first eigenvalues corresponding to the relevant operators.

In this paper we study the following boundary value problems for fractional differential equation (BVP in short)

$$\begin{cases} {}^c D^\alpha u(t) + p(t)f(t, u(t)) + q(t) = 0, & 0 < t < 1, \\ u(0) = a, & u'(0) = b, & u(1) = d, \end{cases} \quad (1)$$

where ${}^c D^\alpha$ is Caputo fractional derivatives with $2 < \alpha \leq 3$, $a, b, d \in R$ are constants, $q : (0, 1) \rightarrow R$ is continuous and Lebesgue integrable and $p : (0, 1) \rightarrow [0, \infty)$ is continuous and does not vanish identically on any subinterval of $(0, 1)$.

2. Preliminaries

In this section, we state some definitions and results that we are going to use throughout this paper, we investigate the problem (1) with non-homogenous boundary value conditions. By using Banach contraction mapping principle and Larry-Schauder alternative, the existence of solutions of the BVP (1) is obtained.

Definition 2.1. Let $p > 0$ be a fixed number. The Riemann-Liouville fractional integral of order $p > 0$ of a function $h \in C([a, b])$ is defined by

$$I_a^p h(t) = \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1} h(s) ds, \quad t \in [a, b]$$

provided the right side is point-wisely defined, where $\Gamma(\cdot)$ denotes the well-known gamma function, i.e., $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$.

Definition 2.2. Let $p \geq 0$ and $n = [p] + 1$. If $h \in AC^n[a, b]$ then the Caputo fractional derivative of order p of h at the point t is defined by

$${}^c D_a^p h(t) = \frac{1}{\Gamma(n-p)} \int_a^t (t-s)^{n-p-1} h^{(n)}(s) ds, \quad t \in [a, b]$$

${}^c D_a^p$ is also called the Caputo fractional differential operator.

For simplicity, when $a = 0$, we denote ${}^c D_0^\alpha$ and I_0^α by ${}^c D^\alpha$ and I^α , respectively.

Lemma 2.3. Let $p, q > 0$ and $n = [p] + 1$. Then the following relations hold.

$${}^c D^p t^{q-1} = \frac{\Gamma(q)}{\Gamma(q-p)} t^{q-p-1}, \quad q > n$$

and

$${}^c D^p t^k = 0, \quad k = 0, 1, 2, \dots, n-1.$$

Lemma 2.4. Let $p > 0, h(t) \in C(0, 1)$. The homogenous fractional differential equation

$${}^c D^\alpha h(t) = 0$$

has a solution

$$h(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1},$$

where $c_i \in R, i = 0, 1, 2, \dots, n-1$ are some constants.

Lemma 2.5. Let $\alpha, \beta \geq 0$ and $h \in L_1[a, b]$. Then $I^\alpha I^\beta h(t) = I^{\alpha+\beta} h(t) = I^\beta I^\alpha h(t)$ and ${}^c D^\alpha I^\alpha h(t) = h(t)$ for all $t \in [a, b]$.

Lemma 2.6. (Leray-Schauder nonlinear alternative). Let F be a Banach space and Ω a bounded open subset of $F, 0 \in \Omega$ and

$T : \bar{\Omega} \rightarrow F$ be a completely continuous operator. Then, either

(i) there exists $u \in \partial\Omega, \lambda > 1$ such that $T(u) = \lambda u$, or

(ii) there exists a fixed point $u^* \in \bar{\Omega}$.

Lemma 2.7. Let $a, b, c \in R$ and $y \in C[0, 1]$. The unique solution of the boundary value problem

$$\begin{cases} {}^c D^\alpha u(t) + y(t) = 0, & 0 < t < 1, \\ u(0) = a, & u'(0) = b, & u(1) = d, \end{cases} \quad (2)$$

is given by

$$u(t) = h(t) + \int_0^1 G(t, s) y(s) ds, \quad (3)$$

where $h(t) = (d-b-a)t^2 + bt + a$, and $G(t, s)$ is the Green's function given by

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^2(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^2(1-s)^{\alpha-1}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (4)$$

Proof. Applying Lemma 2.4 and 2.5, the Eq (2) is equivalent to the integral equation

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds - c_0 - c_1 t - c_2 t^2 \quad (5)$$

The boundary condition $u(0) = a$, gives $c_0 = -a$. Differentiating (5), we get

$$u'(t) = -\frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} y(s) ds - c_1 - 2c_2 t.$$

Since $u'(0) = b$, we deduce that $c_1 = -b$. Further, condition $u(1) = d$ implies that

$$c_2 = a + b - d - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds.$$

Substituting c_0, c_1, c_2 into Eq (5), we obtain

$$u(t) = (d-b-a)t^2 + bt + a + \frac{1}{\Gamma(\alpha)} \int_0^1 t^2(1-s)^{\alpha-1} y(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds, \quad (6)$$

or

$$u(t) = (d-b-a)t^2 + bt + a + t^2 I^\alpha y(1) - I^\alpha y(t)$$

Thus we get

$$u(t) = h(t) + \frac{1}{\Gamma(\alpha)} \int_0^1 G(t,s) y(s) ds$$

The proof is completed.

3. Existence Results

In this section, Now we study the existence of solutions to BVP (1). Let us denote by $C([a, b], R)$ the Banach space of all continuous functions $u : [0, 1] \rightarrow R$ endowed with supremum norm $\|u\| = \max_{t \in [0, 1]} |u(t)|$.

Definition 3.1. A function $u : [0, 1] \rightarrow R$ is said to be a solution to (I), if u satisfies

$$u(t) = h(t) + \int_0^1 G(t,s) [p(s)f(s, u(s)) + q(s)] ds \quad (7)$$

for $t \in [0, 1]$.

Define an operator $T : C([0, 1], R) \rightarrow C([0, 1], R)$ by

$$Tu(t) = h(t) + \int_0^1 G(t,s) [p(s)f(s, u(s)) + q(s)] ds$$

for $u \in C([0, 1], R)$ and $t \in [0, 1]$. Then we transform the existence of solutions to the fixed point problem. We first list the following hypothesis.

(H1) $f : [0, 1] \times R \rightarrow R$ is continuous.

(H2) There exist nonnegative function $g \in L^1([0, 1], R_+)$ such that

$$|f(t, u) - f(t, v)| \leq g(t)|u - v|$$

for all $u, v \in R$ and $t \in [0, 1]$.

(H3) There exist nonnegative function $\phi \in L^1([0, 1], R_+)$, and a continuous nondecreasing function $\psi : [0, \infty) \rightarrow (0, \infty)$ such that

$$|f(t, u)| \leq \phi(t)\psi(|u|)$$

for all $(t, u) \in [0, 1] \times R$.

Lemma 3.2. Let $\alpha > 0$ and $f \in L_1([a, b], R_+)$, Then for all $t \in [a, b]$, we have

$$I_a^{\alpha+1} f(t) \leq \|I_a^\alpha f\|_{L_1}.$$

Proof. Let $f \in L_1([a, b], R_+)$. Then

$$\begin{aligned} \|I_a^\alpha f\|_{L_1} &= \int_0^1 I^\alpha f(r) dr \geq \frac{1}{\Gamma(\alpha)} \int_a^t \int_a^r (r-s)^{\alpha-1} f(s) ds dr \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t \left(\int_s^t (r-s)^{\alpha-1} f(s) dr \right) ds \\ &= \frac{1}{\alpha \Gamma(\alpha)} \int_a^t (t-s)^\alpha f(s) ds = I_a^{\alpha+1} f(t). \end{aligned}$$

Theorem 3.3. Suppose that the condition (H1) and (H2) are satisfied. If

$$N < 1 \tag{8}$$

then the BVP (1) has a unique solution in $C([0, 1], R)$, where $N = 2\|I^{\alpha-1}pg\|$.

Proof. Define an operator $T : C([0, 1], R) \rightarrow C([0, 1], R)$ by

$$Tu(t) = h(t) + \int_0^1 G(t, s)[p(s)f(s, u(s)) + q(s)] ds$$

for $u \in C([0, 1], R)$ and $t \in [0, 1]$. Then $u \in C([0, 1], R)$ is a solution to the BVP (1) if and only if u is a fixed point of T .

For $u, v \in C([0, 1], R)$, applying (6), we obtain

$$\begin{aligned} |Tu(t) - Tv(t)| &\leq \int_0^1 G(t, s) |p(s)[f(s, u(s)) - f(s, v(s))]| ds \\ &= [I^\alpha p(1)(f(1, u(1)) - f(1, v(1))) - [I^\alpha p(t)(f(t, u(t)) - f(t, v(t)))]. \end{aligned}$$

The condition (H2) implies that

$$|Tu(t) - Tv(t)| \leq \max_{0 \leq t \leq 1} |u(t) - v(t)| \left[t^2 I^\alpha |p(1)|g(1) + I^\alpha |p(t)|g(t) \right].$$

It follows from Lemma 3.2 that

$$\|Tu - Tv\| \leq \|u - v\| \left[\|I^{\alpha-1}pg\|_{L_1} + \|I^{\alpha-1}pg\|_{L_1} \right] = 2\|I^{\alpha-1}pg\| \|u - v\|$$

For $t \in [0, 1]$. Hence

$$\|Tu - Tv\| \leq N \|u - v\|.$$

(H3).

Theorem 3.4. Suppose that (H1) and (H3) are satisfied. If

$$\limsup_{r \rightarrow +\infty} M \frac{\psi(r)}{r} < 1,$$

The assumption (8) shows that T is a contraction. By Banach contraction principle, T has a unique fixed point in $C([0, 1], R)$, which is the solution to the BVP (1). The proof is completed.

Next, we prove an existence result by using Larry-Schouder's nonlinear alternative. For simplicity, let $k = \max_{0 \leq t \leq 1} |h(t)| = |(d - b - a) + b + a| = |d|$, $M_1 = 2\|I^{\alpha-1}p\phi\|_{L_1}$, $M_2 = 2\|I^{\alpha-1}q\|_{L_1} + k$, and $M = \max\{M_1, M_2\}$, where ϕ is the function appearing in condition

then the BVP (1) has at least one solution in $C([0, 1], R)$.

Proof. First let us prove that T is completely continuous. It is clear that T is continuous since f and G are continuous. Since $\limsup_{r \rightarrow +\infty} M \frac{\psi(r)}{r} < 1$, there exists a number $r > 0$ such that $M(\psi(r) + 1) < r$. Let $B_r = \{u \in C([0, 1], R) : \|u\| \leq r\}$. Then B_r is a bounded subset in $C([0, 1], R)$. For any $u \in B_r$, we have

$$\begin{aligned}
 |Tu(t)| &= \left| \int_0^1 G(t,s)[p(s)f(s,u(s)+q(s))ds + h(t) \right| \\
 &\leq \int_0^1 |G(t,s)|[|p(s)|\phi(s)\psi(\|u\|) + |q(s)|] + |h(t)| \\
 &\leq \psi(r) \int_0^1 |G(t,s)|ds |p(s)|\phi(s) + \int_0^1 |G(t,s)|ds |q(s)| + |h(t)| \\
 &\leq \psi(r) \left[I^\alpha |p(1)|\phi(1) + I^\alpha |p(s)|\phi(s) \right] + I^\alpha |q(1)| + I^\alpha |q(s)| + |h(t)| \\
 &\leq \psi(r) \left[\|I^{\alpha-1}p\phi\|_{L^1} + \|I^{\alpha-1}p\phi\|_{L^1} \right] + \|I^{\alpha-1}q\|_{L^1} + \|I^{\alpha-1}q\|_{L^1} + k \\
 &\leq \psi(r) 2\|I^{\alpha-1}p\phi\|_{L^1} + 2\|I^{\alpha-1}q\|_{L^1} + k \\
 &\leq (\psi(r)M_1 + M_2) \leq M(\psi(r) + 1)
 \end{aligned}$$

Hence $T(B_r)$ is uniformly bounded. For all $t_1, t_2 \in [0, 1], t_1 < t_2$ and $u \in B_r$, we have,

$$\begin{aligned}
 |Tu(t_1) - Tu(t_2)| &\leq \int_0^1 |G(t_1,s) - G(t_2,s)| |p(s)f(s,u(s))| ds + |h(t_1) - h(t_2)| \\
 &\leq \int_0^1 |G(t_1,s) - G(t_2,s)| |p(s)|\phi(s)\psi(\|u\|) ds + |h(t_1) - h(t_2)| \\
 &\leq \frac{\|p\|\psi(r)}{\Gamma(\alpha)} \left[\int_0^{t_1} |G(t_1,s) - G(t_2,s)|\phi(s) ds + \int_{t_1}^{t_2} |G(t_1,s) - G(t_2,s)|\phi(s) ds \right. \\
 &\quad \left. + \int_{t_2}^1 |G(t_1,s) - G(t_2,s)|\phi(s) ds \right] + |h(t_1) - h(t_2)| \\
 &\leq \frac{\|p\|\psi(r)}{\Gamma(\alpha)} \left[\int_0^{t_1} |t_1^2(1-s)^{\alpha-1} - (t_1-s)^{\alpha-1} - t_2^2(1-s)^{\alpha-1} + (t_2-s)^{\alpha-1}| ds \right. \\
 &\quad \left. + \int_{t_1}^{t_2} |t_1^2(1-s)^{\alpha-1} - t_2^2(1-s)^{\alpha-1} + (t_2-s)^{\alpha-1}| ds \right. \\
 &\quad \left. + \int_{t_2}^1 |t_1^2(1-s)^{\alpha-1} - t_2^2(1-s)^{\alpha-1}| ds \right] + |h(t_1) - h(t_2)| \\
 &\leq \frac{\|p\|\psi(r)}{\Gamma(\alpha)} \int_0^{t_1} \left[|t_1^2 - t_2^2|(1-s)^{\alpha-1} + (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right] ds \\
 &\quad + \int_{t_1}^{t_2} |t_1^2 - t_2^2|(1-s)^{\alpha-1} + (t_2-s)^{\alpha-1} ds \\
 &\quad + \int_{t_2}^1 |t_1^2 - t_2^2|(1-s)^{\alpha-1} ds + |h(t_1) - h(t_2)| \\
 &\leq \frac{\|p\|\psi(r)}{\Gamma(\alpha+1)} \left[|t_1^2 - t_2^2|(1 - |1-t_1|^\alpha) + |t_2^\alpha - t_1^\alpha| - |t_2 - t_1|^\alpha \right. \\
 &\quad \left. + |t_1^2 - t_2^2|(|1-t_1|^\alpha - |1-t_2|^\alpha) + |t_2 - t_1|^\alpha \right. \\
 &\quad \left. + |t_1^2 - t_2^2||1-t_2|^\alpha \right] + |h(t_1) - h(t_2)|.
 \end{aligned}$$

It follows that

$$|Tu(t_1) - Tu(t_2)| \leq \frac{\|p\|\psi(r)}{\Gamma(\alpha+1)} \left[|t_1^2 - t_2^2| + |t_2^\alpha - t_1^\alpha| \right] + |h(t_1) - h(t_2)|$$

It is easy to see that $|Tu(t_1) - Tu(t_2)|$ tend to 0 as $t_1 - t_2 \rightarrow 0$, and the convergence is independent of $u \in B_r$. This show that $T(B_r)$ is equicontinuous. By Arzela- Ascoli Theorem we deduce that T is completely continuous.

Now let $\Omega = \{u \in B : \|u\| < r\}$. Then Ω is an open and bounded subset in B and $0 \in \Omega$. If there is a $u \in \partial\Omega$ such that $u = \lambda Tu$ for some $\lambda \in (0, 1)$ and each $t \in [0, 1]$, then we have

$$|u(t)| = \lambda |Tu(t)| \leq |Tu(t)| \leq M(\psi(r) + 1) < r.$$

This is contradictory to the fact that $u \in \partial\Omega$. Hence Lemma 2.6 (Leray-Schauder nonlinear alternative) allows us to conclude that T has a fixed point $u^* \in \bar{\Omega}$. Therefore the BVP (1) has at least a solution $u^* \in B$. This completes the proof.

4. Example

Example 4.1. Consider the following fractional boundary value problem

$$\begin{cases} {}^c D^{\frac{11}{4}} u(t) = \Gamma(\frac{11}{4}) \left[\frac{t^2}{(1-t)^{\frac{3}{4}}} + \frac{1}{(1+t)} \right], & 0 < t < 1, \\ u(0) = a, \quad u'(0) = b, \quad u(1) = d, \end{cases}$$

In this case we have

$$f(t, x) = \frac{t^2}{5} u + t + 3, \quad 2 < \alpha = \frac{11}{4} < 3$$

$$p(t) = \frac{\Gamma(\frac{11}{4})}{(1-t)^{\frac{3}{4}}}, \quad q(t) = \frac{\Gamma(\frac{11}{4})}{(1+t)}.$$

$$|f(t, x) - f(t, y)| \leq \frac{t^2}{5} |x - y|$$

therefore

$$|f(t, x) - f(t, y)| \leq g(t) |x - y|, \quad \forall x, y \in R$$

and

$$g(t) = \frac{t^2}{5}$$

$$\|I^{\alpha-1} pg\|_{L^1} = I^\alpha p(t)g(t) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} p(s)g(s)ds = \frac{1}{5} \int_0^1 (1-s)s^2 ds = \frac{\Gamma(2)\Gamma(3)}{5\Gamma(5)} = 0.01667 < 1.$$

Example 4.2. Consider the following problem

$$\begin{cases} {}^c D^{\frac{7}{3}} u(t) = \frac{1}{\sqrt{1+t}} \frac{(1+t^3)e^{-t}u^3}{216} + \frac{t^2}{(1-t)}, & 0 < t < 1, \\ u(0) = a, u'(0) = b, \quad u(1) = d, \end{cases}$$

has at least one solution . Applying Theorem (3.4), we have $\alpha = \frac{7}{3}$ and

$$\begin{aligned} |f(t, x)| &= \frac{1}{\sqrt{1+t}} \frac{(1+t^3)e^{-t}x^3}{216} + \frac{t^2}{(1-t)} \\ &\leq \frac{1}{\sqrt{1+t}} (1+t^3) \left(\frac{|x|}{6} \right)^3 + \frac{t^2}{(1-t)} \\ &\leq p(t)\phi(t)\psi(|x|) + q(t). \end{aligned}$$

where $p(t) = \frac{1}{\sqrt{1+t}}$, $\phi(t) = (1+t^3)$, $\psi(|x|) = \left(\frac{|x|}{6} \right)^3$, $q(t) = \frac{t^2}{(1-t)}$.

Let us evaluate $M(\psi(r) + 1)$, some computations lead to

$$\begin{aligned} \|I^{\alpha-1} p\phi\|_{L^1} &= I^\alpha p(t)\phi(t) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} p(s)\phi(s)ds = \frac{1}{\Gamma(\frac{7}{3})} \int_0^1 (1-s)^{\frac{4}{3}} \frac{1}{(1+s)^{\frac{1}{2}}} (1+s^3)ds \\ &\leq \frac{2}{\Gamma(\frac{7}{3})} \int_0^1 (1-s)^{\frac{4}{3}} ds \leq 0.7199. \end{aligned}$$

And

$$\|I^{\alpha-1}q\|_{L^1} = \frac{1}{\Gamma(\frac{7}{3})} \int_0^1 (1-s)^{\frac{4}{3}} \frac{s^2}{(1-s)} ds = \frac{\Gamma(\frac{4}{3})\Gamma(3)}{\Gamma(\frac{7}{3})\Gamma(\frac{13}{3})} = 0.2223.$$

In fact

$$h(t) = (d - b - a)t^2 + bt + a$$

we take $a = b = d = 0$, there for

$$k = |h(t)| = 0, M_1 = 2\|I^{\alpha-1}p\phi\|_{L^1} = 1.4398$$

$$M_2 = 2\|I^{\alpha-1}q\|_{L^1} + k = 0.4446, \text{ then } M = 1.8844$$

we see that $(H4)$ is equivalent to

$$1.8844 \left(\left(\frac{r}{6} \right)^3 + 1 \right) - r < 0 \text{ for } r = 2.$$

5. Conclusion

In this work we studied the existence and uniqueness of solutions for a class of boundary value problem of nonlinear variable coefficient of fractional differential equations. Employing Banach contraction theorem and Larry-Schauder alternative, the existence of solutions were obtained. Some examples were discussed to illustrate our results which generalized to nonlinear Fractional derivatives with variable coefficients.

Funding

This work was supported by Key Program of the National Natural Science Foundation of China (Grant No. 11872189) and the National Natural Science Foundation of China (Grant No. 11472116).

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