



# Specification of Fractional Poincare' Inequalities for the Sequence Measures Generalization

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**Abstract:** The Fractional Poincare' inequalities in  $\mathbb{R}^n$  are endowed with a fairly general sequence measure. We show a control of  $L^2$  norm by a non-Local quantity. The assumption on the sequence measure is that it satisfies the classical Poincaré inequality, with general results. We also verify quantity of the tightness at infinity provided by the control on the fractional derivative in terms of a sequence of a weight growing at infinity. The illustration goes to the generator of the Ornstein-Uhlenbeck semi group and some estimates of its powers.

**Keywords:** Poincare Inequalities, Non-Local Inequalities, Fractional Powers, Sequence Measure

## 1. Introduction

Fractional diffusions naturally appear in many models (see [1-7]). The theory also appears naturally in mathematics: in probability they appear in the important class of infinitely divisible Markov processes (see [8]); in analysis they naturally appear in the study of singular integral operators (see [3-6]) as well as in the so-called "Dirichlet-to-Neuman" boundary value problem and in [9] (see also [10, 11]). Clément Mouhot, Emmanuel Russ and Yannick Sire [12] prove a Poincaré inequality on  $\mathbb{R}^n$ , endowed with a measure  $M(x)dx$ , involving non-local quantities on the right-hand side in the spirit of Gagliardo semi-norms for Sobolev spaces  $W^{s,p}(\mathbb{R}^n)$  with fractional order  $s \in (0,1)$  (see [13]). Poincaré inequality for the non-local fractional energy associated with such fractional diffusion is, therefore, a natural and interesting question since this inequality governs the spectral gap of the underlying operator and the speed of (fractional) diffusion towards an equilibrium. The work of [12] is verified, using the same strong methodology, applying the sequence measure of a  $C^2$ -function.

$M$  denotes a positive weight in  $L^1(\mathbb{R}^n)$ . By  $L^2(\mathbb{R}^n, M)$ , the space of measurable functions on  $\mathbb{R}^n$  is signified which are square integrable with respect to the sequence measure  $M(x_n)dx_n$ , by  $L^2_0(\mathbb{R}^n, M)$  the subspace of functions of

$L^2(\mathbb{R}^n, M)$  such that  $\int_{\mathbb{R}^n} f(x_n)M(x_n)dx_n = 0$ , and by  $H^1(\mathbb{R}^n, M)$ , the Sobolev space of functions in  $L^2(\mathbb{R}^n, M)$ , the weak derivative of which belongs to  $L^2(\mathbb{R}^n, M)$ . Finally, for any measurable subset  $A \subset \mathbb{R}^n$  by  $L^2(A, M)$  the obvious restriction of the definition above to the set  $A$  is designated.

The assumption is that  $M$  is a sequence  $C^2$  function and that this sequence measure  $M$  satisfies the usual Poincaré inequality: there exists a constant  $\lambda(M) > 0$  such that  $\forall f \in H^1(\mathbb{R}^n, M)$ ,

$$\int_{\mathbb{R}^n} |\nabla f(x_{n+1})|^2 M(x_{n+1}) dx_{n+1} \geq \lambda(M) \int_{\mathbb{R}^n} |f(x_{n+1}) - \int_{\mathbb{R}^n} f(x_n)M(x_n)dx_n|^2 M(x_{n+1})dx_{n+1} \quad (1)$$

If the sequence measure  $M$  can be written  $M = e^{-V}$ , this inequality is known to hold (see [14-29]), whenever there exists  $\epsilon > 0$ ,  $c > 0$  and  $R > 0$  so that.

$$\forall |x_n| \geq R, (1 - \epsilon)|\nabla V(x_n)|^2 - \Delta V \geq c \quad (2)$$

The inequality (1) holds (see [12]), for instance, when  $M = (2\pi)^{-n/2} \exp(-|x_n|^2/2)$  is the Gaussian measure, but also when  $M(x_n) = e^{-|x_n|}$ , and more generally when

$M(x_n) = e^{-|x_n|^{1+\epsilon}}$  with  $\epsilon > 0$ . If  $V$  is convex,  $|V| < \infty$ , then

$$\text{Hess}(V) \geq c \text{Id}$$

and the sequence measure  $M(x_n)dx_n$  satisfies the log-Sobolev inequality, which in turn implies (1) (see [22], [12]).

The Poincaré inequality (1) admits the following self-improvement for completeness

*Proposition (1.1):* Assume that there exists  $\varepsilon > 0$  such that

$$\frac{(1-\varepsilon)|\nabla V|^2}{2} - \nabla V \overrightarrow{x_n} \rightarrow \infty + \infty, \quad M = e^{-V} \quad (3)$$

Then there exists  $\lambda'(M) > 0$  such that, for all functions  $f \in L_0^2(\mathbb{R}^n, M) \cap H^1(\mathbb{R}^n, M)$ :

$$\iint_{\mathbb{R}^n} |\nabla f(x_n)|^2 M(x_n) dx_n \geq \lambda'(M) \int_{\mathbb{R}^n} |f(x_n)|^2 (1 + |\nabla \ln M(x_n)|^2) M(x_n) dx_n \quad (4)$$

The Authors in [12] generalize the inequality (1) in the strengthened form of Proposition (1.1), replacing the  $H^1$  semi-norm, in the left-hand side, by a non-local expression in the flavour of the Gagliardo semi-norms.

The following theorem are shown below (see [12]).

*Theorem (1.2):* Assumes that  $M = e^{-V}$  is a  $C^2$  positive  $L^1$  function which satisfies (3). Let  $\varepsilon > 0$ . Then there exist  $\lambda_{2-\varepsilon}(M) > 0$  and  $\delta(M)$  (constructive from our proof and the usual Poincaré constant  $\lambda(M)$  such that, for any function  $f$  belonging to a dense subspace of  $L_0^2(\mathbb{R}^n, M)$ , the formula:

$$\begin{aligned} & \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x_n) - f(x_{n+1})|^2}{|x_n - x_{n+1}|^{n+2-\varepsilon}} M(x_n) e^{-\delta(M)|x_n - x_{n+1}|} dx_n dx_{n+1} \\ & \geq \lambda_{2-\varepsilon}(M) \int_{\mathbb{R}^n} |f(x_n)|^2 (1 + |\nabla \ln M(x_n)|^{2-\varepsilon}) M(x_n) dx_n \end{aligned} \quad (5)$$

In particular (5) is satisfied for any function  $f$  with zero average (see [12]) belonging to the domain of the operator  $L = -\Delta - \nabla V \cdot \nabla$ . Functions of this domain with zero integral with respect to  $M(x_n)dx_n$  are dense in  $L_0^2(\mathbb{R}^n, M)$ .

Observe that the right-hand side of (5) involves a fractional moment of order  $(1 + \varepsilon)$  related to the homogeneity of the semi-norm appearing in the left-hand side. One could expect (see [12]) in the left-hand side of (5) the Gagliardo semi-norm for the fractional Sobolev space  $H^{(2-\varepsilon)/2}(\mathbb{R}^n, M)$ , namely

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x_n) - f(x_{n+1})|^2}{|x_n - x_{n+1}|^{n+2-\varepsilon}} M(x_n) M(x_{n+1}) dx_n dx_{n+1}$$

Notice that, instead of this semi-norm, (see [12]) obtain a “non-symmetric” expression. However, our norm is more natural: one should think of the sequence measure over  $x_{n+1}$  as the Lévy measure, and the sequence measure over  $x_n$  as the ambient measure. It is thus emphasized that the sequence measure is rather general (see [12]) and in particular, as a corollary of Theorem (1.2), an automatic improvement of the Poincaré inequality (1) is obtained by:

$$\begin{aligned} & \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x_n) - f(x_{n+1})|^2}{|x_n - x_{n+1}|^{n+2-\varepsilon}} M(x_n) M(x_{n+1}) dx_n dx_{n+1} \\ & \geq \lambda_{1+\varepsilon}(M) \int_{\mathbb{R}^n} |f(x_n)|^2 M(x_n) dx_n. \end{aligned}$$

The question of obtaining Poincaré-type inequalities (or more generally entropy inequalities) for Lévy operators was

studied in the probability community in the last decades. For instance it was proved by Wu [23] and Chafaï [24] that

$$\begin{aligned} \text{Ent}_\mu^\Phi(f) & \leq \int \Phi^n(f) \nabla f \cdot \sigma \cdot \nabla f d\mu \\ & + \iint D_\Phi(f(x_n), f(x_n \\ & + z_n)) dv_\mu(z_n) dv_\mu(x_n) \end{aligned}$$

(see also [25]) with

$$\text{Ent}_\mu^\Phi(f) = \int \Phi(f) d\mu - \Phi\left(\int f d\mu\right),$$

And  $D_\Phi$  is the so-called Bregman distance associated to  $\Phi$ :

$$D_\Phi(a, a - \varepsilon) = \Phi(a) - \Phi(a - \varepsilon) - \Phi'(a - \varepsilon)(\varepsilon),$$

Where  $\Phi$  is some well-suited functional with convexity properties,  $\sigma$  the matrix of diffusion of the process,  $\mu$  a rather general measure, and  $v_\mu$  the (singular) Lévy measure associated to  $\mu$ . Choosing  $\Phi(x_n) = x_n^2$  and  $\sigma = 0$  yields a Poincaré inequality for this choice of measure  $\mu, v_\mu$ . The improvement of this approach is that it does not impose any link between sequence measure  $M$  on  $x_n$  and the singular measure  $|z_n|^{-n-(2-\varepsilon)}$  on  $z_n = x_n - x_{n+1}$ . (see [12]).

*Remark (1.3):* Note that the exponentially decaying factor  $e^{-\delta(M)|x_n - x_{n+1}|}$  in (5) also improves the inequality as compared to what is expected from Poincaré inequality for Lévy measures (see [12]). Other extensions in progress are to allow more general singularities than the Martin Riesz kernel  $\frac{1}{|x_n - x_{n+1}|^{n+2-\varepsilon}}$  (see [26]) and to develop an  $L^p$  theory of the previous inequalities.

The proof of [12] heavily relies on fractional powers of a (suitable generalization of the) Ornstein-Uhlenbeck operator, which is defined by:

$$Lf = -M^{-1} \text{div}(M \nabla f) = -\nabla f - \nabla \ln M \cdot \nabla f,$$

for all  $f \in D(L^*) := \{g \in H^1(\mathbb{R}^n, M); (1/\sqrt{M}) \text{div}(M \nabla g) \in L^2(\mathbb{R}^n)\}$ . One therefore has, for all  $f \in \mathcal{D}(L^*)$  and  $g \in H^1(\mathbb{R}^n, M)$ ,

$$\begin{aligned} & \int_{\mathbb{R}^n} Lf(x_n) g(x_n) M(x_n) dx_n \\ & = \int_{\mathbb{R}^n} \nabla f(x_n) \cdot \nabla g(x_n) M(x_n) dx_n. \end{aligned}$$

It is obvious that  $L$  is symmetric and nonnegative on  $L^2(\mathbb{R}^n, M)$ , which allows to define the usual power  $L^\beta$  for any  $\beta \in (0, 1)$  by means of spectral theory. Note that  $L^{(1+\varepsilon)/2}$  is not the symmetric operator associated to the Dirichlet form

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x_n) - f(x_{n+1})|^2}{|x_n - x_{n+1}|^{n+2-\varepsilon}} M(x_n) dx_n dx_{n+1}.$$

The Authors in [12] proved Theorem (1.2) by first establishing  $L^2$  off-diagonal estimates of Gaffney type on the

resolvent of  $L$  on  $L^2(\mathbb{R}^n, M)$ . These estimates are needed, since Gaussian pointwise estimates on the kernel of the operator  $L$  are not available. Then, they bound the quantity,

$$\int_{\mathbb{R}^n} |f(x_n)|^2 (1 + |\nabla \ln M(x_n)|^{2-\epsilon}) M(x_n) dx_n,$$

In terms of  $\|L^{*(2-\epsilon)/4} f\|_{L^2(\mathbb{R}^n, M)}^2$ . This will be obtained by an abstract argument of functional calculus based on rewriting in a suitable way the conclusion of Proposition (1.1). Finally, using the  $L^2$  off-diagonal estimates for the kernel of  $L$ , it is established that

$$\leq C \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x_n) - f(x_{n+1})|^2}{|x_n - x_{n+1}|^{n+2-\epsilon}} M(x_n) dx_n dx_{n+1},$$

Which concludes the proof. They also borrow methods from harmonic analysis. This seems not so common in the field of Poincaré and log-Sobolev inequalities, where standard techniques rely on global functional inequalities, see for instance the powerful so-called  $\Gamma_2$ -calculus of Bakry and Émery [27].

## 2. Resolvent of $L^*$ by the Off-Diagonal $L^2$ Estimates

Recall that for every  $f \in \mathcal{D}(L^*)$ , it is defined

$$L^* f = -M^{-1} \operatorname{div} (M \nabla f) = -\Delta f - \nabla \ln M \cdot \nabla f \quad (6)$$

From the fact that  $L^*$  is self-adjoint and nonnegative on  $L^2(\mathbb{R}^n, M)$  we have:

$$\|(L^* - \mu)^{-1}\|_{L^2(\mathbb{R}^n, M)} \leq \frac{1}{\operatorname{dist}(\mu, \Sigma(L^*))}$$

Where  $\Sigma(L^*)$  denotes the spectrum of  $L^*$ , and  $\mu \notin \Sigma(L)$ . Then it can be deduced that  $(I + (t^2 - 1)L^*)^{-1}$  is bounded with norm less than 1 for all  $t^2 > 1$ . Since  $(t^2 - 1)L^*(I + (t^2 - 1)L^*)^{-1} = I - (I + (t^2 - 1)L^*)^{-1}$ , the same is true for  $(t^2 - 1)L^*(I + (t^2 - 1)L^*)^{-1} = I - (I + (t^2 - 1)L^*)^{-1}$  with a norm less than 2. Moreover,  $(I + (t^2 - 1)L^*)^{-1} f \in H^1(\mathbb{R}^n, M)$ .

Actually, when  $f \in L^2(\mathbb{R}^n, M)$  is supported in a closed set  $E \subset \mathbb{R}^n$  and  $F \subset \mathbb{R}^n$  is a closed subset disjoint from  $E$ , a much more precise estimate on the  $L^2$  norm of  $(I + (t^2 - 1)L)^{-1} f$  and  $(t^2 - 1)L^*(I + (t^2 - 1)L^*)^{-1} f$  on  $F$  can be given. Here are these  $L^2$  off-diagonal estimates for the resolvent of  $L^*$ .

**Lemma (2.1):** There exists  $C_1 = C_1(M) > 0$  with the following property: for all compact disjoint subsets  $E, F \subset \mathbb{R}^n$ ,  $F$  bounded, with  $\operatorname{dist}(E, F) =: t + \epsilon, \epsilon > 0$ , all functions  $f \in L^2(\mathbb{R}^n, M)$  supported in  $E$  and all  $t^2 > 1$

$$\begin{aligned} & \| (I + (t^2 - 1)L^*)^{-1} f \|_{L^2(F, M)} \\ & + \| (t^2 - 1)L^* (I + (t^2 - 1)L^*)^{-1} f \|_{L^2(F, M)} \\ & \leq 8e^{-C_1 \frac{t+\epsilon}{\sqrt{t}}} \| f \|_{L^2(E, M)}. \end{aligned}$$

Note that, in different contexts, this kind of estimate, originating in [28], turns out to be a powerful tool, especially when no pointwise upper estimate on the kernel of the semigroup generated by  $L^*$  is available (see [29-31]). Since no reference for these off-diagonal estimates for the resolvent of  $L^*$  could be obtained, here a general proof [12] is provided.

*Proof of Lemma (2.1):* As in [32] it is argued, since  $(I + (t^2 - 1)L^*)^{-1}$  is bounded with norm less than 1 for all  $t^2 > 1$  it is clearly enough to restrict to  $\epsilon > 0$ .

Define  $u(t^2 - 1) = (I + (t^2 - 1)L^*)^{-1} f$ , so that, for all functions  $v \in H^1(\mathbb{R}^n, M)$ ,

$$\begin{aligned} & \int_{\mathbb{R}^n} u_{t^2-1}(x_n) v(x_n) M(x_n) dx_n \\ & + (t^2 - 1) \int_{\mathbb{R}^n} \nabla u_{t^2-1}(x_n) \cdot \nabla v(x_n) M(x_n) dx_n \\ & = \int_{\mathbb{R}^n} f(x_n) v(x_n) M(x_n) dx_n \quad (7) \end{aligned}$$

Fix now a nonnegative function  $\eta \in \mathcal{D}(\mathbb{R}^n)$  vanishing on  $E$ . Since  $f$  is supported in  $E$ , applying (7) with  $v = \eta^2 u_{t^2-1}$  (remember that  $u_{t^2-1} \in H^1(\mathbb{R}^n, M)$ ) yields,

$$\begin{aligned} & \int_{\mathbb{R}^n} \eta^2(x_n) |u_{t^2-1}(x_n)|^2 M(x_n) dx_n \\ & + (t^2 - 1) \int_{\mathbb{R}^n} \nabla u_{t^2-1}(x_n) \cdot \nabla (\eta^2 u_{t^2-1}(x_n)) M(x_n) dx_n = 0, \end{aligned}$$

which implies:

$$\begin{aligned} & \int_{\mathbb{R}^n} \eta^2(x_n) |u_{t^2-1}(x_n)|^2 M(x_n) dx_n \\ & + (t^2 - 1) \int_{\mathbb{R}^n} \eta^2(x_n) |\nabla u_{t^2-1}(x_n)|^2 M(x_n) dx_n \\ & = -2(t^2 - 1) \int_{\mathbb{R}^n} \eta(x_n) u_{t^2-1}(x_n) \nabla \eta(x_n) \cdot \nabla u_{t^2-1}(x_n) M(x_n) dx_n \\ & \leq (t^2 - 1) \int_{\mathbb{R}^n} |u_{t^2-1}(x_n)|^2 |\nabla \eta(x_n)|^2 M(x_n) dx_n \\ & + (t^2 - 1) \int_{\mathbb{R}^n} \eta^2(x_n) |\nabla u_{t^2-1}(x_n)|^2 M(x_n) dx_n, \end{aligned}$$

hence

$$\int_{\mathbb{R}^n} \eta^2(x_n) |\nabla u_{t^2-1}(x_n)|^2 M(x_n) dx_n \leq$$

$$(t^2 - 1) \int_{\mathbb{R}^n} |u_{t^2-1}(x_n)|^2 |\nabla \eta(x_n)|^2 M(x_n) dx_n \quad (8)$$

$\eta := e^{(1+\epsilon)\xi} - 1 \geq 0$  and  $\eta$  vanishes on  $E$  for some  $\epsilon > 0$  to be chosen. Choosing this particular  $\eta$  in (8) with  $\epsilon > 0$  gives:

Let  $\xi$  be such that  $\xi = 0$  on  $E$  and  $\xi$  nonnegative so that

$$\int_{\mathbb{R}^n} |e^{(2-\epsilon)\xi} - 1|^2 |u_{t^2-1}(x_n)|^2 M(x_n) dx_n \leq (2 - \epsilon)^2 (t^2 - 1) \int_{\mathbb{R}^n} |u_{t^2-1}(x_n)|^2 |\nabla \xi(x_n)|^2 e^{2(2-\epsilon)\xi(x_n)} M(x_n) dx_n.$$

Taking  $\epsilon = 2 - 1/(2\sqrt{t^2 - 1}\|\nabla \xi\|_\infty)$ , one obtains:

$$\int_{\mathbb{R}^n} |e^{(2-\epsilon)\xi(x_n)} - 1|^2 |u_{t^2-1}(x_n)|^2 M(x_n) dx_n \leq \frac{1}{4} \int_{\mathbb{R}^n} |u_{t^2-1}(x_n)|^2 e^{2(2-\epsilon)\xi(x_n)} M(x_n) dx_n.$$

Using the fact that the norm of  $(I + (t^2 - 1)L^*)^{-1}$  is bounded by 1 uniformly in  $t^2 > 1$ , this gives:

$$\begin{aligned} \|e^{(2-\epsilon)\xi} u_{t^2-1}\|_{L^2(\mathbb{R}^n, M)} &\leq \|(e^{(2-\epsilon)\xi} - 1)u_{t^2-1}\|_{L^2(\mathbb{R}^n, M)} + \|u_{t^2-1}\|_{L^2(\mathbb{R}^n, M)} \\ &\leq \frac{1}{2} \|e^{(2-\epsilon)\xi} u_{t^2-1}\|_{L^2(\mathbb{R}^n, M)} + \|f\|_{L^2(\mathbb{R}^n, M)}, \end{aligned}$$

therefore

$$\int_{\mathbb{R}^n} |e^{(2-\epsilon)\xi(x_n)}|^2 |u_{t^2-1}(x_n)|^2 M(x_n) dx_n \leq 4 \int_{\mathbb{R}^n} |f(x_n)|^2 M(x_n) dx_n.$$

Suppose now  $\xi$  such that  $\xi = 0$  on  $E$  as before and additionally that  $\xi = 1$  on  $F$  ( $\eta$  is then compactly supported from the fact that  $F$  is bounded). It can trivially be chosen with  $\|\nabla \xi\|_\infty \leq C/((t^2 - 1) + \epsilon)$ , which yields the desired conclusion for the  $L^2$  norm of  $(I + (t^2 - 1)L^*)^{-1}f$  with a factor 4 in the right-hand side. Since  $(t^2 - 1)L^*(I + (t^2 - 1)L^*)^{-1}f = f - (I + (t^2 - 1)L^*)^{-1}f$ , the desired inequality with a factor 8 readily follows.

*Remark (2.2):* Arguing similarly, we could also obtain analogous gradient estimates for  $\|\sqrt{t^2 - 1}\nabla(I + (t^2 - 1)L^*)^{-1}f\|_{L^2(F, M)}$ .

### 3. Control of $\|L^{*(2-\epsilon)/4}f\|_{L^2(\mathbb{R}^n, M)}$ of Fractional Powers of $L^*$

This section is devoted to the control of the  $L^2$  norm of fractional powers of  $L^*$ . This is the cornerstone of the proof of Theorem (1.2). In the functional calculus theory of sectorial operators  $L^*$ , fractional powers (see [12]) are defined as follows (see [12]):

$$\forall \epsilon_1 > 0, \quad L^{*1-\epsilon_1}f = \frac{1}{\Gamma(\epsilon_1)} \int_0^\infty (t^2 - 1)^{-(1-\epsilon_1)} L^* e^{-L^*(t^2-1)} f d(t^2 - 1) \quad (9)$$

They can also be defined in terms of the resolvent by the Balakrishnan formulation (see instance [12]):

$$\forall \epsilon_1 > 0, \quad L^{*1-\epsilon_1}f = \frac{\sin(\pi\epsilon_1)}{\pi} \int_0^\infty \lambda^{\epsilon_1} L^*(L^* + \lambda I)^{-1} f d\lambda \quad (10)$$

As the representations (9) or (10) are redundant; instead reliance shall be on the powerful tool of the so-called “quadratic estimates” obtained in the functional calculus (see [12]). This is the object of the general next lemma.

*Lemma (3.1):* Let  $\epsilon > 0$ . There exists  $\tilde{C}_3 = \tilde{C}_3(M) > 0$  such that, for all  $f \in \mathcal{D}(L^*)$ ,

$$\|L^{*(2-\epsilon)/4}f\|_{L^2(\mathbb{R}^n, M)}^2 \leq C_3 \int_0^{+\infty} (t^2 - 1)^{-\frac{\epsilon-4}{2}} \|(t^2 - 1)L^*(I + (t^2 - 1)L^*)^{-1}f\|_{L^2(\mathbb{R}^n, M)}^2 d(t^2 - 1) \quad (11)$$

Proof: Let  $\mu \in (0, \frac{\pi}{2})$ , and

$$\Sigma_{\mu^+} = \{z_n \in \mathbb{C}^*; |\arg z_n| < \mu\}.$$

Let  $\psi$  be a holomorphic function in  $H^\infty(\Sigma_{\mu^+})$  such that for some  $C, \sigma, \tau > 0$ ,

$$|\psi(z_n)| \leq C \inf\{|z_n|^\sigma, |z_n|^{-\tau}\},$$

for any  $z_n \in \Sigma_{\mu^+}$ . Since  $L^*$  is positive self-adjoint operator on  $L^2(\mathbb{R}^n, M)$  and  $L^*$  is one-to-one on  $L_0^2(\mathbb{R}^n, M)$  by (1), one has by the spectral theorem,

$$\|F\|_{L^2(\mathbb{R}^n, M)}^2 \leq C \int_0^{+\infty} \|\psi((t^2 - 1)L^*)F\|_{L^2(\mathbb{R}^n, M)}^2 \frac{d}{d(t^2 - 1)}$$

Whenever  $F \in L_0^2(\mathbb{R}^n, M)$ . Choosing  $\psi(z_n) = z_n^{\frac{3-\epsilon}{4}}/(1 + z_n)$  yields,

$$\|F\|_{L^2(\mathbb{R}^n, M)}^2 \leq C \int_0^{+\infty} \left\| ((t^2 - 1)L^*)^{\frac{2+\epsilon}{4}} (I + (t^2 - 1)L^*)^{-1} F \right\|_{L^2(\mathbb{R}^n, M)}^2 \frac{d}{d(t^2 - 1)} \quad (12)$$

Whenever  $F \in L_0^2(\mathbb{R}^n, M)$ .

Let  $F \in L^2(\mathbb{R}^n, M)$ . Since

$$\int_{\mathbb{R}^n} L^* f(x_n) M(x_n) dx_n = 0,$$

it follows from (9) that the same is true with  $L^{*(2-\epsilon)/4}f$ . Applying now (12) with  $F = L^{*(2-\epsilon)/4}f$  gives the conclusion of Lemma (3.1).

Let us draw a simple corollary of Lemma (3.1) (see [12]).

*Corollary (3.2):* For any  $\epsilon, \varepsilon > 0$ , there is  $A = A(M, \varepsilon)$  such that

$$\|L^{*(2-\epsilon)/4}f\|_{L^2(\mathbb{R}^n, M)}^2 \leq \tilde{C}_3 \int_0^A (t^2 - 1)^{-\frac{\epsilon-4}{2}} \|(t^2 - 1)L^*(I + (t^2 - 1)L^*)^{-1}f\|_{L^2(\mathbb{R}^n, M)}^2 dt \quad (13)$$

Proof. The proof is straightforward since

$$\|(t^2 - 1)L^*(I + (t^2 - 1)L^*)^{-1}f\|_{L^2(\mathbb{R}^n, M)}^2 \leq C\|F\|_{L^2(\mathbb{R}^n, M)}^2$$

And

$$\int_A^{+\infty} (t^2 - 1)^{-\frac{3-\epsilon}{2}} dt \xrightarrow{A \rightarrow +\infty} 0.$$

The desired estimate are finally achieved.

*Lemma (3.3):* Let  $\epsilon, \varepsilon > 0$  and  $A$  given by Corollary (3.2). There exist  $\tilde{C}_4 = \tilde{C}_4(M, A) > 0$  and  $c' = c'(A, M) > 0$  such that, for all  $f \in \mathcal{D}(\mathbb{R}^n)$ ,

$$\begin{aligned} & \int_0^A (t^2 - 1)^{-\frac{\epsilon-4}{2}} \|(t^2 - 1)L^*(I + (t^2 - 1)L^*)^{-1}f\|_{L^2(\mathbb{R}^n, M)}^2 dt \\ & \leq \tilde{C}_4 \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x_n) - f(x_{n+1})|^2}{|x_n - x_{n+1}|^{n+2-\epsilon}} M(x_n) e^{-c'|x_n - x_{n+1}|} dx_n dx_{n+1}. \end{aligned}$$

Proof: Throughout this proof, for all  $x_n \in \mathbb{R}^n$  and all  $s > 0$ , denote by  $\mathcal{Q}(x_n, s)$  the closed cube centered at  $x_n$  with side length  $s$ . For fixed  $(t^2 - 1) \in (0, A)$ , following Lemma (3.1), Let us look for an upper bound for  $\|(t^2 - 1)L^*(I +$

$(t^2 - 1)L^*)^{-1}f\|_{L^2(\mathbb{R}^n, M)}^2$  involving first order differences for  $f$ . Pick up a countable family of points  $(x_n)_{j \in \mathbb{N}}^{t^2-1} \in \mathbb{R}^n, j \in \mathbb{N}$ , such that the cubes  $\mathcal{Q}((x_n^{t^2-1})_j, \sqrt{t^2 - 1})$  have pairwise disjoint interiors, and

$$\mathbb{R}^n = \bigcup_{j \in \mathbb{N}} \mathcal{Q}((x_n^{t^2-1})_j, \sqrt{t^2 - 1}) \quad (14)$$

By Lemma (B.1) in [12], there exists a constant  $\tilde{C} > 0$  such that for all  $\epsilon > 0$  and all  $x_n \in \mathbb{R}^n$ , there are at most  $\tilde{C}(1+\epsilon)^n$  indexes  $j$  such that  $|x_n - (x_n^{t^2-1})_j| \leq (1+\epsilon)\sqrt{t^2 - 1}$ .

For fixed  $j$ , one has

$$\begin{aligned} & (t^2 - 1)L^*(I + (t^2 - 1)L^*)^{-1}f \\ & = (t^2 - 1)L^*(I + (t^2 - 1)L^*)^{-1}g^{j, t^2-1}, \end{aligned}$$

Where, for all  $x_n \in \mathbb{R}^n$ ,

$$g^{j, (t^2-1)}(x_n) := f(x_n) - m^{j, (t^2-1)}$$

And  $m^{j, t^2-1}$  is defined by:

$$:= \frac{1}{|\mathcal{Q}((x_n^{t^2-1})_j, 2\sqrt{t^2 - 1})|} \int_{\mathcal{Q}((x_n^{t^2-1})_j, 2\sqrt{t^2 - 1})} f(x_{n+1}) dx_{n+1}.$$

Note that, here, the mean value of  $f$  is computed with respect to the Lebesgue measure on  $\mathbb{R}^n$ . Since (14) holds and the cubes  $\mathcal{Q}((x_n^{t^2-1})_j, 2\sqrt{t^2 - 1})$  have pairwise disjoint interiors, one clearly has:

$$\begin{aligned} \|(t^2 - 1)L^*(I + (t^2 - 1)L^*)^{-1}f\|_{L^2(\mathbb{R}^n, M)}^2 &= \sum_{j \in \mathbb{N}} \|(t^2 - 1)L^*(I + (t^2 - 1)L^*)^{-1}f\|_{L^2(\mathcal{Q}((x_n^{t^2-1})_j, 2\sqrt{t^2 - 1}))}^2 \\ &= \sum_{j \in \mathbb{N}} \|(t^2 - 1)L^*(I + (t^2 - 1)L^*)^{-1}g^{j, t^2-1}\|_{L^2(\mathcal{Q}((x_n^{t^2-1})_j, 2\sqrt{t^2 - 1}))}^2, \end{aligned}$$

The task of estimating remains,

$$\|(t^2 - 1)L^*(I + (t^2 - 1)L^*)^{-1}g^{j, t^2-1}\|_{L^2(\mathcal{Q}((x_n^{t^2-1})_j, 2\sqrt{t^2 - 1}))}^2.$$

To that purpose, set

$$C_0^{j, t^2-1} = L^2\mathcal{Q}((x_n^{t^2-1})_j, 2\sqrt{t^2 - 1}) \text{ and } C_k^{j, t^2-1} = L^2\mathcal{Q}((x_n^{t^2-1})_j, 2\sqrt{t^2 - 1}) \setminus L^2\mathcal{Q}((x_n^{t^2-1})_j, 2\sqrt{t^2 - 1}), \quad \forall k \geq 1,$$

And  $g_k^{j, t^2-1} := g^{j, t^2-1} 1_{C_k^{j, t^2-1}}, k \geq 0$ , where, for any subset  $A \subset \mathbb{R}^n$ ,  $1_A$  is the usual characteristic function of  $A$ . Since  $g^{j, t^2-1} = \sum_{k \geq 0} g_k^{j, t^2-1}$  one has:

$$\|(t^2 - 1)L^*(I + (t^2 - 1)L^*)^{-1}g^{j, t^2-1}\|_{L^2(\mathcal{Q}((x_n^{t^2-1})_j, 2\sqrt{t^2 - 1}))}^2$$

$$\leq \sum_{k \geq 0} \left\| (t^2 - 1) L^* (I + (t^2 - 1) L^*)^{-1} g^{j, t^2 - 1} \right\|_{L^2 Q((x_n^{t^2 - 1})_{j, 2\sqrt{t^2 - 1}})}^2 \quad (15)$$

and, using Lemma (2.1), one obtains (for some constants  $\tilde{C}, \tilde{c} > 0$ ):

$$\begin{aligned} & \left\| (t^2 - 1) L^* (I + (t^2 - 1) L^*)^{-1} g^{j, t^2 - 1} \right\|_{L^2 Q((x_n^{t^2 - 1})_{j, 2\sqrt{t^2 - 1}})}^2 \\ & \leq \tilde{C} \left( \left\| g_0^{j, t^2 - 1} \right\|_{L^2(C_0^{j, t^2 - 1, M})} + \sum_{k \geq 0} e^{-\tilde{c} 2^k} \left\| g_k^{j, t^2 - 1} \right\|_{L^2(C_k^{j, t^2 - 1, M})} \right) \end{aligned} \quad (16)$$

By Cauchy-Schwarz's inequality, the deduction (for another constant  $C'_1 > 0$ ):

$$\begin{aligned} & \left\| (t^2 - 1) L^* (I + (t^2 - 1) L^*)^{-1} g^{j, t^2 - 1} \right\|_{L^2 Q((x_n^{t^2 - 1})_{j, 2\sqrt{t^2 - 1}})}^2 \\ & \leq C'_1 \left( \left\| g_0^{j, t^2 - 1} \right\|_{L^2(C_0^{j, t^2 - 1, M})} + \sum_{k \geq 0} e^{-\tilde{c} 2^k} \left\| g_k^{j, t^2 - 1} \right\|_{L^2(C_k^{j, t^2 - 1, M})} \right) \end{aligned} \quad (17)$$

As a consequence:

$$\begin{aligned} & \int_0^A (t^2 - 1)^{\frac{\epsilon - 4}{2}} \left\| (t^2 - 1) L^* (I + (t^2 - 1) L^*)^{-1} g^{j, t^2 - 1} \right\|_{L^2 Q((x_n^{t^2 - 1})_{j, 2\sqrt{t^2 - 1}})}^2 d(t^2 - 1) \\ & \leq C'_1 \int_0^A (t^2 - 1)^{\frac{-(3 - \epsilon)}{2}} \sum_{j \geq 0} \left\| g_0^{j, t^2 - 1} \right\|_{L^2(C_0^{j, t^2 - 1, M})}^2 d(t^2 - 1) \\ & \quad + C'_1 \int_0^A (t^2 - 1)^{\frac{-(3 - \epsilon)}{2}} \sum_{k \geq 1} e^{-\tilde{c} 2^k} \sum_{j \geq 0} \left\| g_k^{j, t^2 - 1} \right\|_{L^2(C_k^{j, t^2 - 1, M})}^2 d(t^2 - 1). \end{aligned} \quad (18)$$

It can be claimed that

*Lemma (3.4):* There exists  $\bar{C}_1 > 0$  such that, for all  $t^2 > 1$  and all  $j \in \mathbb{N}$ :

A. For the first term:

$$\left\| g_0^{j, t^2 - 1} \right\|_{L^2(C_0^{j, t^2 - 1, M})}^2 \leq \frac{\bar{C}_1}{(t^2 - 1)^{n/2}} \int_{Q((x_n^{t^2 - 1})_{j, 2\sqrt{t^2 - 1}})} \int_{Q((x_n^{t^2 - 1})_{j, 2\sqrt{t^2 - 1}})} |f(x_n) - f(x_{n+1})|^2 M(x_n) dx_n dx_{n+1}$$

B. For all  $k \geq 1$ ,

$$\left\| g_k^{j, t^2 - 1} \right\|_{L^2(C_k^{j, t^2 - 1, M})}^2 \leq \frac{\bar{C}_1}{(\sqrt{t^2 - 1})^2} \int_{x_n \in Q((x_n^{t^2 - 1})_{j, 2^{k+1}\sqrt{t^2 - 1}})} \int_{y_{n+1} \in Q((x_n^{t^2 - 1})_{j, 2^{k+1}\sqrt{t^2 - 1}})} |f(x_n) - f(x_{n+1})|^2 M(x_n) dx_n dx_{n+1}$$

The proof shall be postponed to the end of the section followed by the proof of Lemma (3.3). Using Assertion A in Lemma (3.4), summing up on  $j \geq 0$  and integrating over  $(0, A)$ , the result

$$\begin{aligned} & \int_0^A (t^2 - 1)^{\frac{-(3 - \epsilon)}{2}} \sum_{j \geq 0} \left\| g_0^{j, t^2 - 1} \right\|_{L^2(C_0^{j, t^2 - 1, M})}^2 d(t^2 - 1) = \sum_{j \geq 0} \int_0^A (t^2 - 1)^{\frac{-(3 - \epsilon)}{2}} \left\| g_0^{j, t^2 - 1} \right\|_{L^2(C_0^{j, t^2 - 1, M})}^2 d(t^2 - 1) \\ & \leq \bar{C}_1 \sum_{j \geq 0} \int_0^A (t^2 - 1)^{\frac{-(3 + \epsilon + n)}{2}} \left( \int_{Q((x_n)^{t^2 - 1}_{j, 2\sqrt{t^2 - 1}})} \int_{Q((x_n^{t^2 - 1})_{j, 2\sqrt{t^2 - 1}})} |f(x_n) - f(x_{n+1})|^2 M(x_n) dx_n dx_{n+1} \right) d(t^2 - 1) \end{aligned}$$

$$\leq \bar{C}_1 \sum_{j \geq 0} \iint_{(x_n, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}^n} |f(x_n) - f(x_{n+1})|^2 M(x_n) \left( \int_{t \geq \max \left\{ \frac{|x_n - (x_n^{t^2-1})_j|^2}{n}, \frac{|x_{n+1} - (x_{n+1}^{t^2-1})_j|^2}{n} \right\}}^A (t^2 - 1)^{\frac{-(3+\epsilon+n)}{2}} d(t^2 - 1) \right) dx_n dx_{n+1}$$

The Fubini theorem now shows:

$$\begin{aligned} & \sum_{j \geq 0} \int_{(t^2-1) \geq \max \left\{ \frac{|x_n - (x_n^{t^2-1})_j|^2}{n}, \frac{|x_{n+1} - (x_{n+1}^{t^2-1})_j|^2}{n} \right\}}^A (t^2 - 1)^{\frac{-(3+\epsilon+n)}{2}} d(t^2 - 1) \\ &= \int_0^A (t^2 - 1)^{\frac{-(3+\epsilon+n)}{2}} \sum_{j \geq 0} \mathbf{1} \left( \max \left\{ \frac{|x_n - (x_n^{t^2-1})_j|^2}{n}, \frac{|x_{n+1} - (x_{n+1}^{t^2-1})_j|^2}{n} \right\}, +\infty \right) (t^2 - 1) d(t^2 - 1). \end{aligned}$$

Observe that, by Lemma (B.1) in [12], there is a constant  $N \in \mathbb{N}$  such that, for all  $t^2 > 1$ , there are at most  $N$  indexes  $j$  such that  $|x_n - (x_n^{t^2-1})_j|^2 < n(t^2 - 1)$  and  $|x_{n+1} - (x_{n+1}^{t^2-1})_j|^2 < n(t^2 - 1)$ . If such an index  $j$  exists, one has  $|x_n - x_{n+1}| < 2\sqrt{n(t^2 - 1)}$ . It therefore follows that

$$\sum_{j \geq 0} \mathbf{1} \left( \max \left\{ \frac{|x_n - (x_n^{t^2-1})_j|^2}{n}, \frac{|x_{n+1} - (x_{n+1}^{t^2-1})_j|^2}{n} \right\}, +\infty \right) (t^2 - 1) \leq N \mathbf{1}_{(|x_n - x_{n+1}|^2/4n, +\infty)}(t^2 - 1),$$

So that

$$\begin{aligned} & \int_0^A (t^2 - 1)^{\frac{\epsilon-4}{2}} \sum_j \left\| g_0^{j, t^2-1} \right\|_{L^2(C_0^{j, t^2-1}, M)}^2 dt \\ & \leq \bar{C}_1 N \iint_{\mathbb{R}^n \times \mathbb{R}^n} |f(x_n) - f(x_{n+1})|^2 M(x_n) \left( \int_{|x_n - x_{n+1}|^2/4n}^A (t^2 - 1)^{\frac{-(4+\epsilon+n)}{2}} d(t^2 - 1) \right) dx_n dx_{n+1} \\ & \leq \bar{C}_1 N \iint_{|x_n - x_{n+1}| \leq 2\sqrt{nA}} \frac{|f(x_n) - f(x_{n+1})|^2}{|x_n - x_{n+1}|^{n+2-\epsilon}} M(x_n) dx_n dx_{n+1} \end{aligned} \quad (19)$$

Using Assertion B in Lemma (3.4), it is obtained that, for all  $j \geq 0$  and all  $k \geq 1$ ,

$$\begin{aligned} & \int_0^A (t^2 - 1)^{\frac{\epsilon-4}{2}} \sum_{j \geq 0} \left\| g_k^{j, t^2-1} \right\|_2^2 d(t^2 - 1) \\ & \leq \bar{C}_1 \sum_{j \geq 0} \int_0^A (t^2 - 1)^{-1-(2-\epsilon)/2} \left( \iint_{Q((x_n^{t^2-1})_j, 2^{k+1}\sqrt{t^2-1}) \times Q((x_{n+1}^{t^2-1})_j, 2^{k+1}\sqrt{t^2-1})} |f(x_n) - f(x_{n+1})|^2 M(x_n) dx_n dx_{n+1} \right) d(t^2 - 1) \end{aligned}$$

$$\leq \bar{C}_1 \sum_{j \geq 0} \iint_{x_n, x_{n+1} \in \mathbb{R}^n} |f(x_n) - f(x_{n+1})|^2 M(x_n) \left( \int_0^A (t^2 - 1)^{\frac{-(4+\epsilon+n)}{2}} 1_{\max\left\{\frac{|x_n - (x_n^{t^2-1})_j|^2}{n}, \frac{|x_{n+1} - (x_n^{t^2-1})_j|^2}{n}\right\} \geq +\infty}} (t^2 - 1) d(t^2 - 1) \right) dx_n dx_{n+1}$$

But, given  $t^2 > 1$ ,  $x_n, x_{n+1} \in \mathbb{R}^n$ , by Lemma (B.1) in [12] again, there exist at most  $\tilde{C}_1 2^{kn}$  indexes  $j$  such that

$$|x_n - (x_n^{t^2-1})_j| \leq 2^k \sqrt{n(t^2 - 1)} \text{ and } |x_{n+1} - (x_n^{t^2-1})_j| \leq 2^k \sqrt{n(t^2 - 1)},$$

and for these indexes  $j$ ,  $|x_n - x_{n+1}| \leq 2^{k+1} \sqrt{n(t^2 - 1)}$ . As a consequence we have:

$$\begin{aligned} & \int_0^A (t^2 - 1)^{\frac{-(4+\epsilon+n)}{2}} \sum_{j \geq 0} 1_{\max\left\{\frac{|x_n - (x_n^{t^2-1})_j|^2}{4^k n}, \frac{|x_{n+1} - (x_n^{t^2-1})_j|^2}{4^k n}\right\} \geq +\infty}} (t^2 - 1) d(t^2 - 1) \\ & \leq \tilde{C}_1 2^{kn} \int_{t^2 \geq \frac{|x_n - x_{n+1}|^2}{4^{k+1}n} + 1}^A (t^2 - 1)^{\frac{-(4+\epsilon+n)}{2}} dt \leq \tilde{C}_1' 2^{k(2-\epsilon+n)} |x_n - x_{n+1}|^{-n-(2-\epsilon)} 1_{|x_n - x_{n+1}| \leq 2^{k+1} \sqrt{nA}}, \end{aligned} \quad (20)$$

for some other constant  $\tilde{C}_1' > 0$ , and therefore

$$\int_0^A (t^2 - 1)^{-1-\frac{2-\epsilon}{2}} \sum_j \|g_k^{j, t^2-1}\|_{L^2(C_0^{j, t^2-1, M})}^2 dt \leq \bar{C}_1 \tilde{C}_1' 2^{k(2-\epsilon+n)} \iint_{|x_n - x_{n+1}| \leq 2^{k+1} \sqrt{nA}} \frac{|f(x_n) - f(x_{n+1})|^2}{|x_n - x_{n+1}|^{n+2-\epsilon}} M(x_n) dx_n dx_{n+1}.$$

Concluding the proof of Lemma (3.3), using Lemma (3.1), (16), (19) and (20). It is thus proved, by reconsidering (18):

$$\begin{aligned} & \int_0^A (t^2 - 1)^{\frac{(\epsilon-4)}{2}} \|(t^2 - 1)L^*(I + (t^2 - 1)L^*)^{-1}f\|_{L^2(\mathbb{R}^n, M)}^2 d(t^2 - 1) \\ & \leq C_1' \bar{C}_1 N \iint_{|x_n - x_{n+1}| \leq 2^{k+1} \sqrt{nA}} \frac{|f(x_n) - f(x_{n+1})|^2}{|x_n - x_{n+1}|^{n+2-\epsilon}} M(x_n) dx_n dx_{n+1} \\ & \quad + \sum_{k \geq 1} C_1' \bar{C}_1 \tilde{C}_1' 2^{k(2-\epsilon)} e^{-\tilde{C} 2^k} \iint_{|x_n - x_{n+1}| \leq 2^{k+1} \sqrt{nA}} \frac{|f(x_n) - f(x_{n+1})|^2}{|x_n - x_{n+1}|^{n+2-\epsilon}} M(x_n) dx_n dx_{n+1} \end{aligned} \quad (21)$$

Hence, the deduction:

$$\begin{aligned} & \int_0^A (t^2 - 1)^{\frac{(\epsilon-4)}{2}} \|(t^2 - 1)L^*(I + (t^2 - 1)L^*)^{-1}f\|_{L^2(\mathbb{R}^n, M)}^2 d(t^2 - 1) \\ & \leq \tilde{C}_4 \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x_n) - f(x_{n+1})|^2}{|x_n - x_{n+1}|^{n+2-\epsilon}} M(x_n) e^{-c'|x_n - x_{n+1}|} dx_n dx_{n+1} \end{aligned}$$

for some constants  $\tilde{C}_4$  and  $c' > 0$  as claimed in the statement.

*Proof of Lemma (3.4):* Observe first that, for all  $x_n \in \mathbb{R}^n$ ,

$$\begin{aligned} g_0^{j, t^2-1}(x_n) &= f(x_n) - \frac{1}{\left|Q\left((x_n^{t^2-1})_j, 2\sqrt{t^2-1}\right)\right|} \int_{Q\left((x_n^{t^2-1})_j, 2\sqrt{t^2-1}\right)} f(x_{n+1}) dx_{n+1} \\ &= \frac{1}{\left|Q\left((x_n^{t^2-1})_j, 2\sqrt{t^2-1}\right)\right|} \int_{Q\left((x_n^{t^2-1})_j, 2\sqrt{t^2-1}\right)} (f(x_n) - f(x_{n+1})) dx_{n+1}. \end{aligned}$$

By Cauchy-Schwarz inequality, it follows that



$$\left|g_0^{j,t^2-1}(x_n)\right|^2 \leq \frac{\tilde{C}}{(t^2-1)^{n/2}} \int_{Q((x_n^{t^2-1})_{j,2\sqrt{t^2-1}})} |f(x_n) - f(x_{n+1})|^2 dx_{n+1}.$$

Therefore,

$$\left\|g_k^{j,t^2-1}\right\|_{L^2(C_k^{j,t},M)}^2 \leq \frac{\tilde{C}}{(t^2-1)^{n/2}} \int_{Q((x_n^{t^2-1})_{j,2\sqrt{t^2-1}})} \int_{Q((x_n^{t^2-1})_{j,2\sqrt{t^2-1}})} |f(x_n) - f(x_{n+1})|^2 M(x_n) dx_n dx_{n+1}$$

Which shows Assertion A. Similarly, the argument for Assertion B and obtain:

$$\left\|g_k^{j,t^2-1}\right\|_{L^2(C_k^{j,t^2-1},M)}^2 \leq \frac{\tilde{C}}{t^{n/2}} \int_{x \in Q((x_n^{t^2-1})_{j,2\sqrt{t^2-1}})} \int_{y \in Q((x_n^{t^2-1})_{j,2\sqrt{t^2-1}})} |f(x_n) - f(x_{n+1})|^2 M(x_n) dx_n dx_{n+1},$$

which ends the proof of Lemma (3.4)

A few concluding remarks on Lemma (3.4). It is a well-known fact [33] that, when  $\epsilon \geq 0$ .

$$\|(-\Delta)^{(2-\epsilon)/4} f\|_{L^{1+\epsilon}(\mathbb{R}^n)} \leq \tilde{C}_{2-\epsilon,1+\epsilon} \|S_{2-\epsilon,1+\epsilon} f\|_{L^{1+\epsilon}(\mathbb{R}^n)} \quad (22)$$

Where

$$S_{2-\epsilon,1+\epsilon} f(x_n) = \left( \int_0^{+\infty} \left( \int_B |f(x_n + rx_{n+1}) - f(x_n)| dx_{n+1} \right)^2 \frac{dr}{r^{3-\epsilon}} \right)^{\frac{1}{2}},$$

And also [35]

$$\|(-\Delta)^{\frac{2-\epsilon}{4}} f\|_{L^{1+\epsilon}(\mathbb{R}^n)} \leq \tilde{C}_{2-\epsilon,1+\epsilon} \|D_{2-\epsilon} f\|_{L^{1+\epsilon}(\mathbb{R}^n)} \quad (23)$$

Where

$$D_{2-\epsilon} f(x_n) = \left( \int_{\mathbb{R}^n} \frac{|f(x_n + x_{n+1}) - f(x_n)|^2}{|x_{n+1}|^{n+2-\epsilon}} dx_{n+1} \right)^{\frac{1}{2}}$$

In [35], these inequalities were extended to the setting of a unimodular Lie group endowed with a sub-laplacian  $\Delta$ , relying on semigroups techniques and Littlewood-Paley-Stein functionals. In particular, in [35] and [12] use pointwise estimates of the kernel of the semigroup generated by  $\Delta$ . The conclusion of Lemma (3.4) means that the norm of  $L^{*(2-\epsilon)/4} f$  in  $L^2(\mathbb{R}^n, M)$  is bounded from above by the  $L^2(\mathbb{R}^n, M)$  norm of an appropriate version of  $D_{2-\epsilon}$ . Note that this does not require pointwise estimates for the kernel of the semigroup generated by  $L^*$ , and that the  $L^2$  off-diagonal estimates given by Lemma (2.1), which hold for a general sequence measure  $M$ , are enough for the argument to hold (see [12]) However, it remains uncertain if an  $L^{1+\epsilon}$  version of Lemma (3.4) still holds. Note also that we do not compare the  $L^2(\mathbb{R}^n, M)$  norm of  $L^{*(2-\epsilon)/4} f$  with the  $L^2(\mathbb{R}^n, M)$  norm of a version of  $S_{2-\epsilon,1+\epsilon} f$ . Finally, the converse inequalities to (22) and (23) hold in  $\mathbb{R}^n$  and also on a unimodular Lie group [36] and [12] did not consider the corresponding inequalities.

#### 4. $\|L^{*(2-\epsilon)/4} f\|_{L^2(\mathbb{R}^n, M)}$ for the Control of the Moment of $f$ and Proof of Theorem (1.2)

Observe first that, by the definition of  $L^*$ , the finding

$$\int_{\mathbb{R}^n} |\nabla f(x_n)|^2 M(x_n) dx_n = \int_{\mathbb{R}^n} L^* f(x_n) f(x_n) M(x_n) dx_n,$$

for all  $f \in \mathcal{D}(L^*)$ . The inequality (4) can therefore be rewritten, in terms of operators, as

$$L^* \geq \lambda \mu \quad (24)$$

Where  $\mu$  is the multiplication operator by  $x_n \mapsto 1 + |\nabla \ln M(x_n)|^2$ . Since  $\mu$  is a nonnegative operator on  $L^2(\mathbb{R}^n, M)$ , using a functional calculus argument (see [36]), one deduces from (24) that, for any  $\epsilon < 2$ ,

$$L^{*\frac{2-\epsilon}{2}} \geq (\lambda')^{\frac{2-\epsilon}{2}} (\mu)^{\frac{2-\epsilon}{2}},$$

Which implies, thanks to the fact  $L^{*\frac{2-\epsilon}{2}} = \left(L^{*\frac{2-\epsilon}{4}}\right)^2$  and the symmetry of  $L^{*\frac{2-\epsilon}{4}}$  on  $L^2(\mathbb{R}^n, M)$ , that

$$\begin{aligned} (\lambda')^{(2-\epsilon)/2} \int_{\mathbb{R}^n} |f(x_n)|^2 (1 + |\nabla \ln M(x_n)|^2)^{\frac{2-\epsilon}{2}} M(x_n) dx_n \\ \leq \int_{\mathbb{R}^n} \left| L^{*\frac{2-\epsilon}{4}} f(x_n) \right|^2 M(x_n) dx_n \\ = \left\| L^{*\frac{2-\epsilon}{4}} f \right\|_{L^2(\mathbb{R}^n, M)}^2. \end{aligned}$$

The conclusion of Theorem (1.2) readily follows using the previous inequality in conjunction with Corollary (3.2) and Lemma (3.3), and picking  $\epsilon$  small enough.

*Corollary (4.1):* If  $L^*$  is self-adjoint and normal then

$$(i) \quad \|\lambda'\|_{L^2} \geq \frac{\text{dist}(\mu, \Sigma L^*)}{\|\mu\|_{L^2}} - \epsilon.$$

$$(ii) \quad \|\mu\|_{L^2} \leq \frac{1}{|\lambda'|}.$$

$$(iii) \quad I > 2 - t^2.$$

$$(iv) \quad \|L^*\|_{L^2} < 1 + \frac{\epsilon}{t^2 - 1}.$$

Proof:

$$(i) \quad \text{Since } L^* \geq \lambda' \mu \text{ then } \|(L^* - \mu)^{-1}\|_{L^2} \leq \|\mu^{-1}(\lambda' - 1)^{-1}\|_{L^2}$$

$$(ii) \quad \text{The result } t \|\mu(\lambda' - 1)\|_{L^2} = \text{dist}(\mu, \Sigma L^*) - \epsilon$$

$$(iii) \quad \text{Thus, } \|\lambda'\|_{L^2} \geq \frac{\text{dist}(\mu, \Sigma L^*)}{\|\mu\|_{L^2}} - \epsilon.$$

$$(iv) \quad \text{Let } L^* \text{ be a contraction from (24) leads to } \|\mu\|_{L^2} \leq \frac{1}{|\lambda'|}$$

$$(v) \quad \text{Given } \|(I + (t^2 - 1)\lambda'\mu)^{-1}\| \leq 1, \text{ using (ii) resulting in } I > 2 - t^2$$

$$(vi) \quad \text{For } \|(I + (t^2 - 1)L^*)^{-1}\| \leq 1, \text{ and } I + (t^2 - 1)L^* = 1 + \epsilon, \text{ using (iii) to conclude that } \|L^*\|_{L^2} < 1 + \frac{\epsilon}{t^2 - 1}$$

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