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Numerical Study of the Boundary Layer Flow Problem over a Flat Plate by Finite Difference Method

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Abstract: The present study involves a numerical investigation of laminar boundary layer flow over a flat plate, controlled by the Prandtl equations. The flow is governed by a dimensionless third-order system of nonlinear ordinary differential equations. The finite difference method is employed to solve the system, which serves as an approximation technique. The study explores the properties of the finite difference method and discusses its efficacy in solving the boundary layer flow problem. Additionally, we discuss an inverse problem related to the Falkner-Skan equation, aiming to obtain precise values for the second derivative's initial value. This inverse problem is successfully resolved using an appropriate initial value procedure. The results obtained from the finite difference method and the inverse problem resolution is compared with those from cubic spline interpolation, proposed by Alavi and Aminikhan. By doing so, the reliability and accuracy of present approach is demonstrated. Overall, this study contributes to a better understanding of boundary layer flow and presents a viable numerical technique for tackling similar fluid dynamics problems. The findings shed light on the significance of choosing appropriate numerical methods for solving complex systems of equations in fluid mechanics.

Keywords: Boundary Layer Flow, Falkner-Skan Equation, Finite Difference Method

1. Introduction

Finite difference approximation is the easiest and most ancient approach for the solution of different equations. Most physical and engineering procedures are expressed in terms of boundary value problem. When boundary value problem cannot be resolved analytically then numerical schemes are applied such as Rung Kutta method, Euler method and Finite difference method. The emergence of finite difference method in numerical analysis started in the early 1950's. The finite difference methods theoretical result has been attained in terms of accuracy balance and convergence for Sturn Liouville differential equation. Mukhtarov et al. [1] applied numerical scheme to find the numerical solution of Sturn Liouville differential equation. Finite difference method is based on the differential quotient that is used as substitute for derivative in a given equation. Their comparison between the

numerical and exact solutions provides an accurate numerical result with the theoretical findings. Pandey, [2] discussed the accuracy and uniqueness of the seventh order boundary value problem through the finite difference method. Bhagyamma et al., [3] presented a novel approach combining modified decomposition method and finite difference scheme to obtain solution of singular point boundary value problem.

The Falkner-Skan equation, introduced in 1931, is significant in fluid mechanics, especially in the bounded viscous flow problems deduced from Navier-Stokes two-dimensional incompressible fluid equation for one-sided flow using the same analysis and its solution defines the type of outer laminar boundary layer in the presence of a nonconductive pressure gradient of the stream wise flow. Regardless of the difficulties of Falkner-Skan equation, it may be necessary to solve them; these problems are not linear and are generally of the third-degree. Analytical solutions on the Falkner-Skan scale

with special cases, indicating the existence of a solution and the difference or availability of a numerical solution to a particular boundary layer condition are discussed by Duque-Daza et al. [4]. Some studies contain the solution to the Falkner-Skan equation as well as its uniqueness and have discussed the result in absence of upper and lower boundary. According to Duque-Daza et al. [4], wall shear pressure has no dimension. The flow over a flat plate is described by the Prandtl equation, which is solved using the finite difference approach. The practical work of Ludwig Prandtl was successful in 1904. He setup a result that, in many systems the effect of viscosity is limited by weakness. The region nearest to the body and rest part of the flow area, which can be a good measure, is handled as inviscid, so it can be calculated as a potential flow theory. He suggested that when an overflowing low-viscosity liquid enters a thin layer, the flow field is not audible both inside and beyond the layer boundary and outside the boundary layer. The presence of an object near the wall causes a viscous affect, which increases the velocity of the boundary layer, merging forms. As it approaches the wall, the speed tends to zero due to the slick surface, and from a distance on the wall, it is adapted to the external velocity the slip velocity of wall smoothness in this model is dictated by the external speed which would be given by a theory of potential flow. Alavi and Aminikhah [5] studied the problem of boundary layer flow over a flat plate by orthogonal cubic spline basis functions and examined the behavior of wall temperature and the free stream difference with heat conduction.

In mathematics, significant attention has been directed towards two-point boundary value problems due to their crucial relevance across various scientific and engineering domains. These problems frequently arise in fields such as optimal control, fluid mechanics, quantum mechanics and geophysics. Numerous logical and computational approaches have been explored in the literature to address differential equations. Some of these techniques include the differential transform method, Bernoulli polynomials, Runge-Kutta 4th order method and cubic spline method. Shalini & Sathyavathy [6] analyzed the second order boundary value sine wave problem using Laplace transform and finite difference methods. They concluded that the wave elongation enhances the convergence of results when compared to closed-form solutions obtained through explicit methods. Ahmad et al. [7] discussed the classical approximation technique to analyze the velocity profile linked with the Falkner-Skan boundary layer problem, focused on solving the boundary layer equation for a model scenario with a significant region of strong reverse flow. The problem delves into the behavior of a viscous fluid flowing over a semi-infinite flat plate against an adverse pressure gradient. The investigation

optimizes dimensionless velocity profiles for reverse wedge flow, showcasing the results graphically across varying values of the wedge angle parameter β (from 0 to 2.5). The solution is determined using the weighted residual method (WRM), and a comparison is drawn between WRM and the homotopy perturbation method specifically for $\beta = 0$.

Addressing the nonlinear Falkner-Skan boundary-value problem involves resolving nonlinear differential equations particularly, finding the solutions that characterize boundary layer flows along a wedge. Asaithambi [8] aimed to reveal the underlying mathematical insights governing this challenging fluid dynamics problem. Kokurin et al. [9] used finite-difference method for tackling fractional differential equations of order 1/2. These equations involve derivatives of non-integer order, which are fundamental in describing various complex phenomena. The effort focused on developing numerical techniques that could efficiently handle these unique equations, contributing to the computational tools available for solving fractional-order problems. Sharma et al. [10] employed finite difference approach to investigate the numerical aspects of fractional boundary layer flow over a stretching sheet with varying thickness. The focus is on understanding the behavior of fluid flow within this complex system, where the sheet is elongated and its thickness changes. Sharma et al. [10] applied numerical technique to analyze the intricate characteristics of the flow induced by both forced and free convective boundary layer flows of a magnetic fluid over a flat plate focused towards the understanding of fractional-order phenomena in boundary layers. Tzirtzilakis et al. [11] considered the influence of a localized magnetic field on the fluid flow, aiming to comprehend the interplay between magnetic effects and convective flows. Xie et al. [12] worked over the problem involving high-order compact finite difference schemes for addressing systems of third-order boundary value problems. Their focus was on devising numerical methods that offer enhanced accuracy and efficiency in solving complex equations of this nature and aimed for advancing computational tools that can contribute to the realm of solving intricate boundary value problems with higher-order systems.

This study replicates the work conducted by Alavi and Aminikhah [5] utilizing the cubic spline method to investigate the solution of Falkner–Skan boundary-layer problem. However, we approach the same problem using the finite difference method and subsequently compare our results with those obtained through the cubic spline method. Through this comparative analysis, we aim to offer insight into the accuracy and effectiveness of the finite difference approach in relation to the cubic spline method for this particular problem.
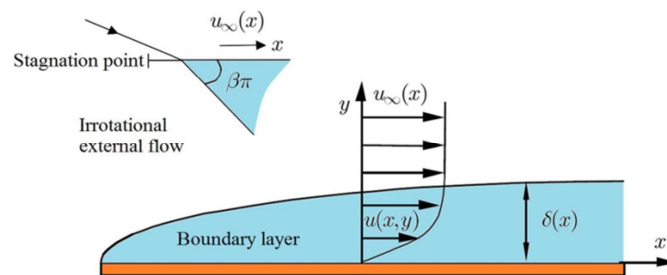

Figure 1. *Two-Dimensional boundary layer formulation.*

2. Flow Analysis and Mathematical Formulation

Assume that we have a flow occurring over a flat plat, as illustrated in Figure 1. This flow possesses certain unchanging attributes, such as being incompressible and maintaining consistent characteristics throughout the process. Additionally, there are no external forces acting on the flow. This specific setup gives rise to a set of equations that plays a crucial role in describing the behavior of this type of flow.

The boundary layer equations [13-19] of conservation of mass, conservation of linear momentum are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{3}$$

Here u represents the velocity component in the horizontal x direction, v symbolizes the velocity component in the vertical y direction. p stands for pressure, which indicates the force exerted by the fluid on its surroundings. ρ denotes density, ν represents kinematic viscosity.

Suppose the wall is impenetrable. In this situation, the velocity of the fluid must satisfy two conditions: firstly, it should have no velocity perpendicular to the surface (zero transverse velocity) at the surface itself, and secondly, it should adhere to the no-slip rule, which means the horizontal velocity at the surface u is zero. As we move away from the surface within the boundary layer, the velocity of the fluid needs to approach the upstream velocity $u_\infty(x)$ while considering the thickness of the boundary layer x. Given that the pressure is assumed to stay constant within the boundary layer, the equation (3) can't be used in this case. Instead, equation (2) yields.

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{4}$$

2.1. Similarity Transformation

Similarity transformations [20-26] are commonly used to analyze and solve certain types of flow problems. These transformations help to simplify the governing equations of fluid motion, making them more amenable for analysis or numerical solution. One of the most well-known applications of similarity transformations in fluid dynamics is in boundary layer theory [27-33]. In the present case we consider the similarity function as

$$\Psi = \sqrt{c\lambda}f(\eta)x^{\frac{m+1}{2}} \tag{5}$$

2.2. Von Mises Transformation

For Von Mises transformation consider the relations

$$\left.\frac{\partial}{\partial x}\right|_y = \left.\frac{\partial}{\partial x}\right|_\eta + \left.\frac{\partial}{\partial \eta}\right|_x \left.\frac{\partial \eta}{\partial x}\right|_y, \tag{6}$$

and

$$\left.\frac{\partial}{\partial y}\right|_x = \left.\frac{\partial}{\partial \eta}\right|_x \left.\frac{\partial \eta}{\partial y}\right|_x, \tag{7}$$

then

$$\Psi = \sqrt{c\lambda}f(\eta)x^{\frac{m+1}{2}}, \tag{8}$$

gives

$$\left.\frac{\partial \Psi}{\partial x}\right|_y = \sqrt{c\lambda}f(\eta)\frac{m+1}{2}x^{\frac{m-1}{2}} + x^{m-1}f'yc\frac{m-1}{2}, \tag{9}$$

and

$$\left.\frac{\partial \Psi}{\partial y}\right|_x = f'u_\infty, \tag{10}$$

The velocity components (u, v) from stream function Ψ are

$$u = u_\infty f',$$

$$v = \sqrt{\frac{u_\infty \gamma}{x}}\frac{m+1}{2}\left(-f - \frac{m-1}{m+1}\eta f'\right),$$

Using Von Mises transformation, we get

$$\frac{\partial u}{\partial x} = \frac{mu_\infty}{x}f' + \frac{u_\infty}{x}\eta f''\frac{m-1}{2}, \tag{11}$$

$$\frac{\partial u}{\partial y} = cx^m f''\sqrt{\frac{c}{\gamma}}x^m = u_\infty\sqrt{\frac{u_\infty}{\gamma x}}f'', \tag{12}$$

$$\frac{\partial^2 u}{\partial y^2} = u_\infty\frac{u_\infty}{\gamma x}f''', \tag{13}$$

The flow outside the boundary layer can be considered as inviscid.

From Bernoulli's equation we have

$$p_\infty + \frac{\rho u_\infty^2}{2} = c, \tag{14}$$

So $\frac{dp_\infty}{dx} = -\rho u_\infty\frac{du_\infty}{dx}, \tag{15}$

inside the boundary layer whatever pressure gradient will be there that will be equal to the outside pressure gradient so

$$\frac{1}{\rho}\frac{dp}{dx} = -\frac{m}{x}u_\infty^2, \tag{16}$$

For these values Eq. (4) now gets the form

$$f'''(\eta) + \frac{m+1}{2}f(\eta)f'(\eta) + m(1 - f'^2) = 0, \tag{17}$$

This 3rd order nonlinear ordinary differential equation is known as Falkner-Skan equation. Its boundary conditions are

$$f(0) = f'(0) = 0, \lim_{\eta \to \infty} f'(\eta) = 1, \tag{18}$$

Eq. (17) is the well-known Falker-Skan Equation where

$f(\eta)$ defines the shape of the velocity profile.

2.3. Energy Equation

For energy equation, as a special case consider surface or the wall having a uniform temperature, so temperature will remain constant [34-40]. Then

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}, \qquad (19)$$

With boundary conditions

$$T(w,0) = T_w, T(x,\infty) = T_\infty, \qquad (20)$$

In the above equation T_w and T_∞ denotes the wall and the external temperature.

The exact solution of the energy equation would be of the form

$$T = T_w + \theta(T_\infty - T_w), \qquad (21)$$

Applying the Von Mises transformation over Eq. (19) gives

$$\theta''(\eta) + \frac{m+1}{2} Pr f'(\eta)\theta'(\eta) - n\, Pr f'(\eta)\theta(\eta) = 0, \qquad (22)$$

with boundary conditions

$$\theta(0) = 1, \lim_{\eta\to\infty}\theta(\eta) = 0, \qquad (23)$$

where $Pr = \frac{v}{\alpha}$ is the dimensionless Prandtl number. The function $\theta(\eta)$ defines shape of temperature profiles [41-44].

3. Finite Difference Method Solution

The finite difference method is presented in this section before being utilized to resolve the mentioned boundary layer problems. By replacing the derivatives in the differential equation with their approximations, which are the approximate finite difference values, one may solve a two-point boundary value problem using the finite difference approach. Specifically, third order boundary value problems (BVPs) on the interval by assuming

$$f'''(x) = u[x, f(x), f'(x), f''(x)], I = [x_0, x_n] \qquad (24)$$

subject to the boundary conditions

$$f(x_0) = \eta_0, f'(x_0) = \theta_0, f'(x) = \mu_0, \qquad (25)$$

In order to solve a boundary value problem, we segment the range $[x_0, x_n]$ into equal subintervals of width h. Each subinterval has neighboring points, and the prime notation signifies differentiation with respect to x. Constants $\eta_0, \theta_0,$ and μ_0 are given. we convert the continuous problem into discrete steps, allowing us to apply boundary conditions

and approximate the solution point by point. So that,

$$x_i = x_0 + ih,\ i = 1,\ 2,\ 3,...,\ N. \qquad (26)$$

The corresponding values of $f(x)$ at each point are obtained by,

$$f(x_i) = f_i = f(x_i + ih), \qquad (27)$$

Using Taylor series expansion second order central difference formula

$$f'(x) = \frac{f_{i+1} - f_{i-1}}{2h} + o(h), \qquad (28)$$

$$f''(x) = \frac{f_{i-1} - 2f_i + f_{i+1}}{h^2} + o(h^2), \qquad (29)$$

$$f'''(x) = \frac{f_{i+2} - 2f_{i+1} + 2f_{i-1} - f_{i-2}}{2h} + o(h^2), \qquad (30)$$

Rearranging the central difference formulas mentioned in equations (28) to (30) and substituting them into equation (22), we simplify equation (22) into a set of linear equations. To solve this set of equations, we apply the finite difference method while considering the boundary conditions provided by equation (23). By using smaller sub-intervals, the solution can be improved to achieve the desired level of accuracy.

4. Result and Discussion

The flow of fluid over a flat plate is analyzed working with the Von Mises Transformation. This method involves solving a set of mathematical equations that describe the behavior of the flow. To solve these equations, we have used the numeical technique called finite difference method (FDM), which is a way to approximate the solutions. When we change the values of certain parameters, like Pr, the behavior of the fluid flow changes as well. We've conducted different cases by varying these parameters and obtained numerical values and graphs as results. The focus was to compare these results with the outcomes of cubic spline interpolation, which is another mathematical technique used for approximating curves. We've organized our findings into different tables and figures to make it easier to understand the comparisons;

Table 1, Table 2, Table 3, and Table 4 represents a comparison of special cases of present solutions with already available work of Alavi and Aminikhah [5]. In these tables, we've set specific values for the parameters n and Pr to be 0.5, and the parameter β is taken as variable varying from 0 to 1. For each case, we've calculated and displayed values for different functional values of $f(\eta), f'(\eta), \theta(\eta)$ and $\theta'(\eta)$. The comparison of these results with the outcomes from cubic spline interpolation in [5] are in good agreement.

***Table 1.** Comparison of $f(\eta)$ for different β using n=0.5, Pr=0.5.*

η	$\beta = 0$		$\beta = 0.5$		$\beta = 1$	
	Present	cubic spline method [5]	Present	cubic spline method [5]	Present	cubic spline method [5]
0	0	0	0	0	0	0
1	0.028872	0.028867	0.072185	0.072145	0.025200	0.025207
2	0.115124	0.115110	0.264830	0.264860	0.095132	0.095211

η	β = 0		β = 0.5		β = 1	
	Present	cubic spline method [5]	Present	cubic spline method [5]	Present	cubic spline method [5]
3	0.258010	0.258010	0.545280	0.545370	0.335671	0.335692
4	0.455450	0.455460	0.884880	0.884720	0.664761	0.664815
5	0.703470	0.703420	1.261214	1.261409	1.042231	1.042241
6	0.996320	0.996390	1.658735	1.658799	1.240931	1.240946
7	1.326840	1.326940	2.067033	2.067112	1.443381	1.443450
8	1.687110	1.687240	2.273242	2.273265	1.648123	1.648175
9	2.069110	2.069270	2.480327	2.480343	1.854477	1.854509
10	2.465810	2.465900	2.687951	2.68761	1.992539	1.992566

Table 2. *Comparison of $f'(\eta)$ for different β using n=0.5, Pr=0.5.*

η	β = 0		β = 0.5		β = 1	
	Present	cubic spline method [5]	Present	cubic spline method [5]	Present	cubic spline method [5]
0	0	0	0	0	0	0
1	0.1383	0.1384	0.2763	0.2764	0.2352	0.2353
2	0.2756	0.2756	0.5006	0.5008	0.4284	0.4284
3	0.4096	0.4098	0.6741	0.6741	0.7031	0.7035
4	0.5364	0.5365	0.8005	0.8006	0.8608	0.8609
5	0.6517	0.6519	0.8863	0.8865	0.9416	0.9417
6	0.7513	0.7518	0.9400	0.9400	0.9638	0.9639
7	0.8322	0.8323	0.9709	0.9711	0.9782	0.9784
8	0.8937	0.8939	0.9871	0.9873	0.9873	0.9876
9	0.9370	0.9374	0.9917	0.9919	0.9929	0.9929
10	0.9652	0.9653	0.9948	0.9948	0.9952	0.9956

Table 3. *Comparison of $\theta(\eta)$ for different β using n=0.5, Pr=0.5.*

η	β = 0		β = 0.5		β = 1	
	Present	cubic spline method [5]	Present	cubic spline method [5]	Present	cubic spline method [5]
0	1.0000	1.0000	1.0000	1.0000	1.000	1.0000
1	0.8505	0.8506	0.8188	0.8189	0.8879	0.8879
2	0.7066	0.7068	0.6446	0.6447	0.7784	0.7785
3	0.5726	0.5726	0.4910	0.4912	0.5750	0.5753
4	0.4522	0.4523	0.3607	0.3609	0.4018	0.4019
5	0.3475	0.3476	0.2551	0.2552	0.2648	0.2649
6	0.2596	0.2598	0.1734	0.1735	0.2100	0.2102
7	0.1884	0.1886	0.1132	0.1133	0.1639	0.1639
8	0.1326	0.1327	0.0708	0.0709	0.1258	0.1259
9	0.0905	0.0906	0.0551	0.0551	0.0950	0.0952
10	0.0598	0.0599	0.0424	0.0426	0.0780	0.0781

Table 4. *Comparison of $\theta'(\eta)$ for different β using n=0.5, Pr=0.5.*

η	β = 0		β = 0.5		β = 1	
	Present	cubic spline method [5]	Present	cubic spline method [5]	Present	cubic spline method [5]
0	-0.3610	-0.3610	-0.4445	-0.4445	-0.5400	-0.5400
1	-0.2708	-0.2707	-0.4303	-0.4301	-0.5337	-0.5336
2	-0.1518	-0.1517	-0.3934	-0.3933	-0.4554	-0.4553
3	-0.0864	-0.0863	-0.3419	-0.3419	-0.3734	-0.3732
4	-0.0426	-0.0425	-0.2833	-0.2831	-0.2846	-0.2845
5	-0.0181	-0.0180	-0.2241	-0.2240	-0.2416	-0.2415
6	-0.0086	-0.0084	-0.1690	-0.1690	-0.2014	-0.2012
7	-0.0028	-0.0025	-0.1216	-0.1215	-0.1646	-0.1645
8	-0.0015	-0.0014	-0.0834	-0.0831	-0.1320	-0.1320
9	-0.0004	-0.0001	-0.0678	-0.0677	-0.1128	-0.1126
10	-0.0001	0.0000	-0.0545	-0.0543	-0.0954	-0.0953

Figure 2 shows the velocity gradient $f''(\eta)$ for different values of β. The graph indicates a good agreement for the present solutions with the already available solutions in [5]. Figure 3 represents the velocity profile $f'(\eta)$ for different values of β. The graph indicates that an increase in the parameter β the velocity profile increases. Figure 4 shows this behavior of temperature profile $\theta(\eta)$ for different values of the Prandtl numbers *Pr* while keeping $\beta=0$. From figure it is noted that by increasing the Prandtl numbers *Pr* the temperature profile $\theta(\eta)$ decreases. Figure 5 shows this behavior of temperature profile $\theta(\eta)$ for different values of the Prandtl numbers *Pr* while keeping $\beta=1$. From figure it is again noted that by increasing the Prandtl numbers *Pr* the temperature profile $\theta(\eta)$ decreases. It is also noted that the thermal boundary layer thickness [41-44] reduces quickly for $\beta=0$ as compared to when $\beta=1$. Figure 6

represent the temperature gradient $\theta'(\eta)$ for different values of Prandtl numbers *Pr* when *β=0* Figure 7 represent the temperature gradient $\theta'(\eta)$ for different values of Prandtl numbers *Pr* when *β=1*. From these two figures it is the change in temperature gradient $\theta'(\eta)$ with respect to the Prandtl numbers *Pr* is higher for *β=0* as compared to when *β=1*.
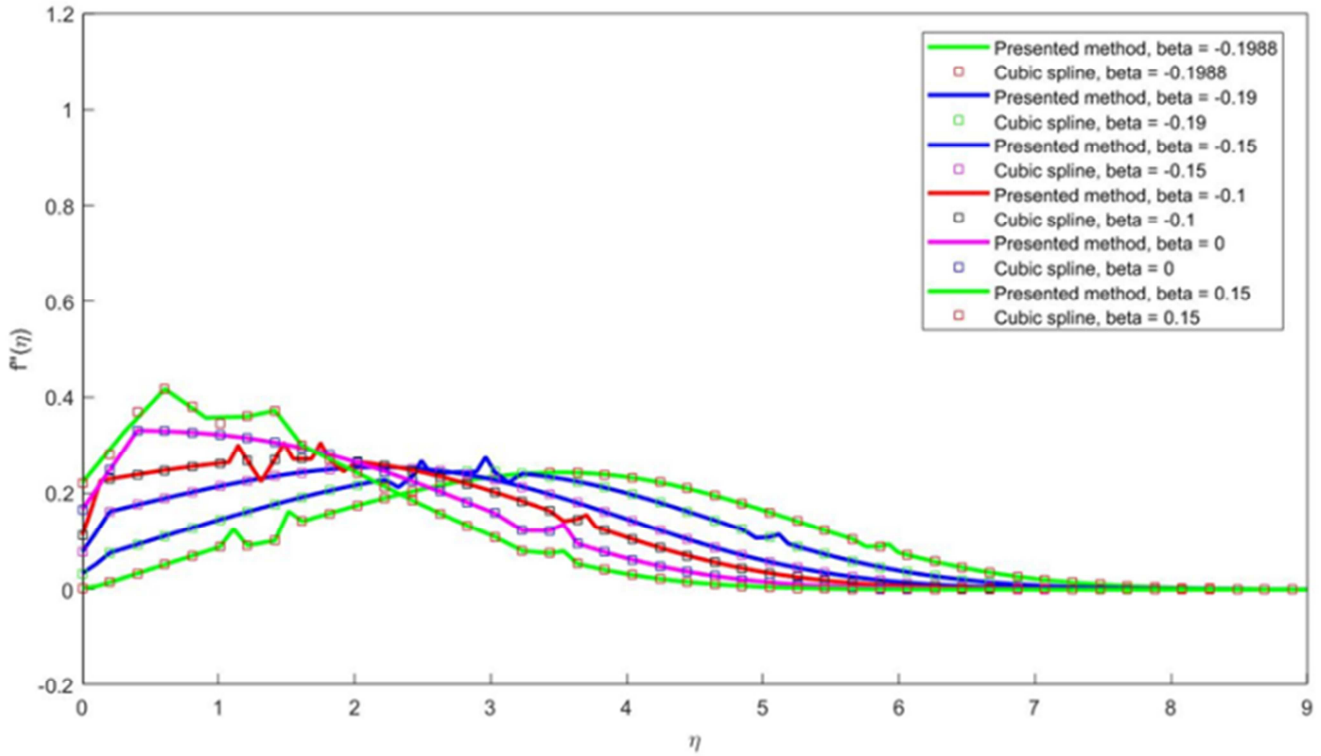

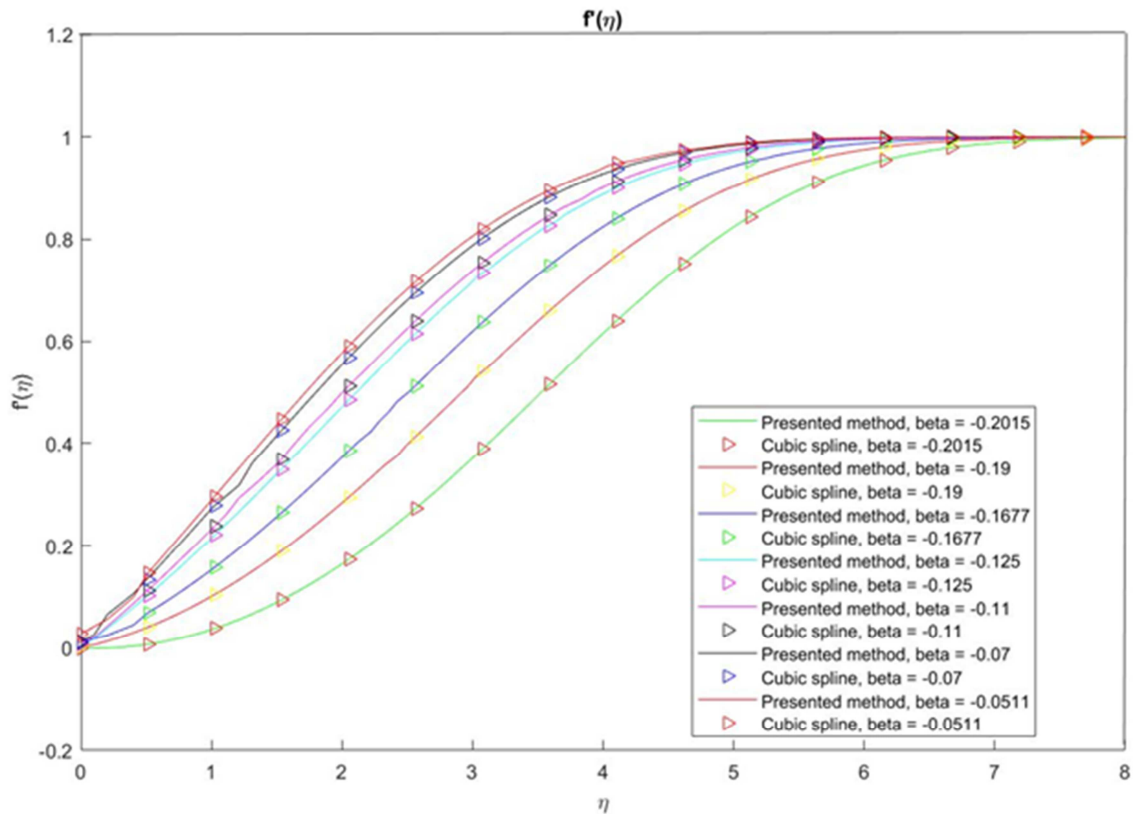**Figure 2.** *Comparison of $f''(\eta)$ for different values of β.*

Figure 3. *Comparison of $f'(\eta)$ for different values of β.*

Figure 4. Comparison of $\theta(\eta)$ for different values of Pr when $\beta = 0$.
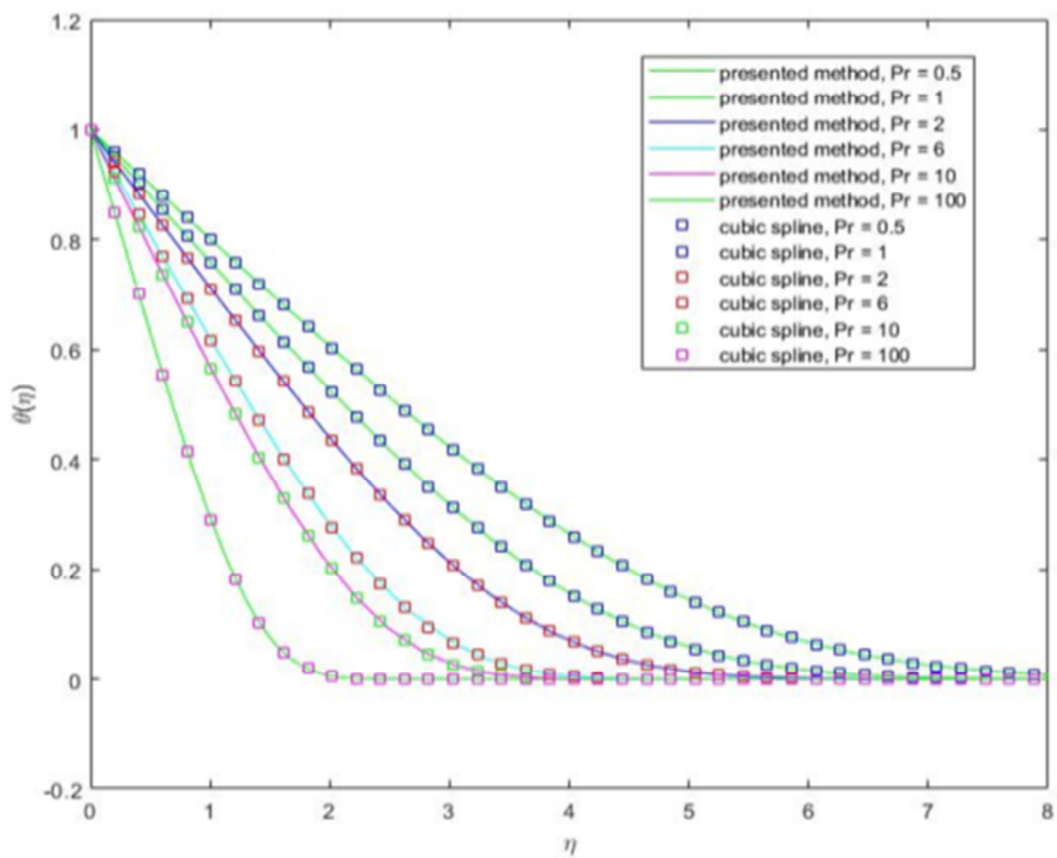

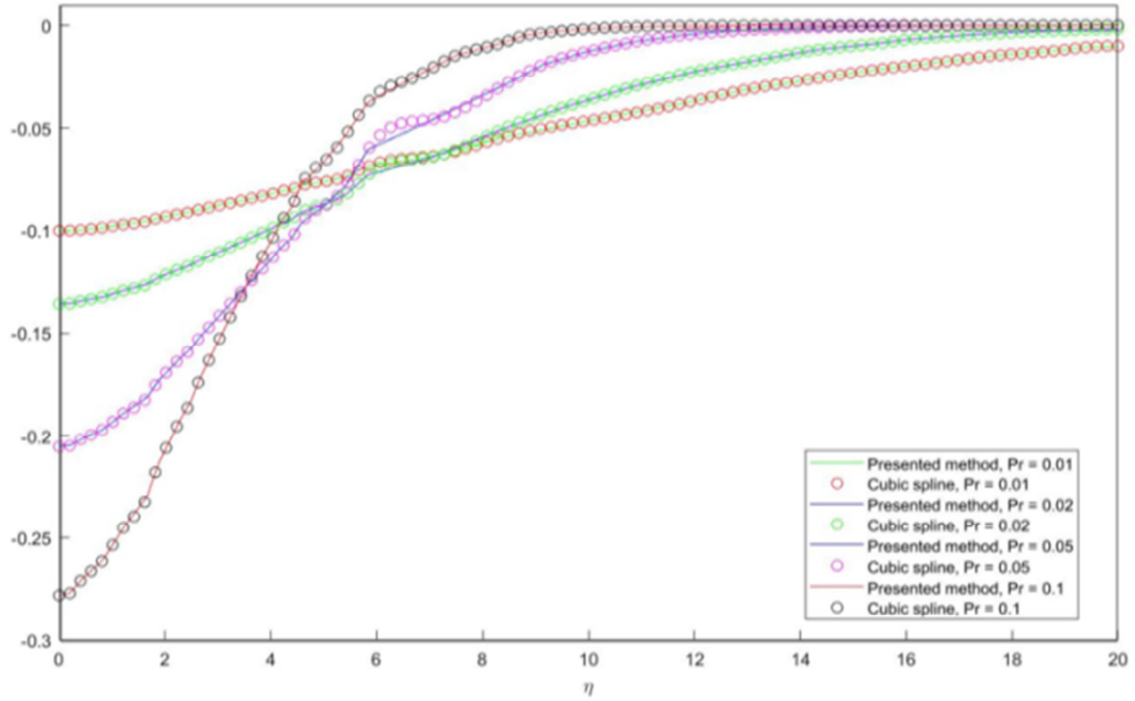**Figure 5.** Comparison of $\theta(\eta)$ for different values of Pr when $\beta = 1$.
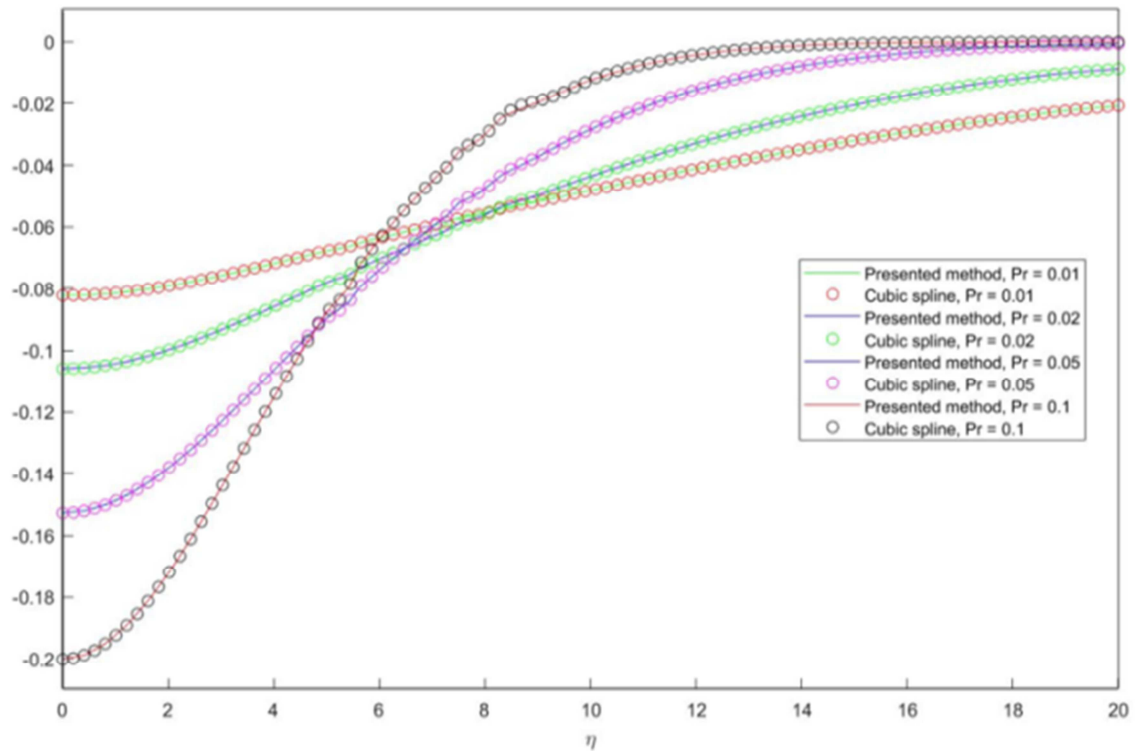
Figure 6. *Comparison of* $\theta'(\eta)$ *for different values of Pr when* $\beta = 0$.

Figure 7. *Comparison of* $\theta'(\eta)$ *for different values of Pr when* $\beta = 1$.

5. Conclusion

In this paper, we addressed the problem of laminar boundary layer flow over a flat plate, which can be described by a third-order system of nonlinear ordinary differential equations. The Finite Difference Method was used to numerically solve this problem. The paper also discussed the improvement of the initial values for the Falkner-Skan second derivative equations to achieve more accurate results. We compared numerical results obtained through the Finite Difference Method with those obtained using the cubic spline interpolation approach. The presented method was found to provide satisfactory and accurate solutions for the laminar

boundary layer flow problem. Overall, the paper highlights the application of the Finite Difference Method to solve the laminar boundary layer flow problem and demonstrates its accuracy by comparing it with the results obtained through the cubic spline approach.

Conflicts of Interest

The authors declare that they have no conflicts of interests.

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