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The Metric Dimension of Subdivisions of Lilly Graph, Tadpole Graph and Special Trees

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Abstract: Consider a robot that is navigating in a space represented by a graph and wants to know its current location. It can send a signal to find out how far it is from each set of fixed landmarks. We study the problem of computing the minimum number of landmarks required, and where they should be placed so that the robot can always determine its location. The set of nodes where the landmarks are located is called the metric basis of the graph, and the number of landmarks is called the metric dimension of the graph. On the other hand, the metric dimension of a graph G is the smallest size of a set B of vertices that can distinguish each vertex pair of G by the shortest-path distance to some vertex in B. The finding of the metric dimension of an arbitrary graph is an NP-complete problem. Also, the metric dimension has several applications in different areas, such as geographical routing protocols, network discovery and verification, pattern recognition, image processing, and combinatorial optimization. In this paper, we study the metric dimension of subdivisions of several graphs, including the Lilly graph, the Tadpole graph, and the special trees star tree, bistar tree, and coconut tree.

Keywords: Resolving Set, Metric Dimension, Tadpole Graph, Bistar Tree, Coconut Tree

1. Introduction

Let $G=(V, E)$ be a connected graph and $d(u,v)$ be the shortest path between two vertices $u,v \in V(G)$. An ordered vertex set $B=\{x_1,x_2,\ldots,x_k\} \subseteq V(G)$ is a resolving set of G if the representation

$$r(v|\, B) = (d(v, x_1), d(v, x_2),\ldots,d(v, x_k))$$

is unique for every $v \in V(G)$. The metric dimension of G, denoted dim(G), is the cardinality of minimum resolving set of G.

Slater [1, 2] introduced the notion of a minimum resolving set as a locating set of G and uses the cardinality of B as a locating number to uniquely identify the location of an intruder in a network. Harary and Melter in [3] introduced independently the notion of minimum resolving set as a metric basis of G and the cardinality of B as the metric dimension of G and since then it has been used in several applications such as robot navigation in networks [4, 5], application to pharmaceutical chemistry Chartrand et al. [6], application to pattern recognition Melter et al. [7], and application to wireless sensor network localization [8].

Despite Khuller et al [4] have shown that the problem of determining the metric dimension of a graph is NP-complete, many scholars have improved an upper bound for the metric dimension of several graphs or determined their exact values. Chartrand et al. [6] showed the metric dimension of the path graph P_n is 1, $n \geq 2$, cycle graph C_n is 2, $n \geq 3$, complete graph K_n is n-1, $n \geq 2$ and star graph $K_{1,n-1}$ is n-2, $n \geq 3$. Susilowati et al. [9] determined the metric dimension of the k-subdivision of the path graph P_n is 1, $n \geq 2$, cycle graph C_n is 2, $n \geq 3$, complete graph K_n is n-2, $n \geq 4$, star graph S_n is n-2, $n \geq 3$, ladder graph L_n is 2, $n \geq 3$ and other special graphs. Borchert et al. [10] computed the metric dimension of the circulant graphs C_n (±1, ±2) and demonstrated that if $n \equiv 1$ (mod 4), then dim(C_n (±1,±2)) = 4. Imran et al. [11] investigated the metric dimension of the barycentric subdivision of Möbius ladders, the generalized Petersen multigraphs $P(2n, n)$ and proved that they have metric dimension 3 when n is even and 4 when n is odd. Nawaz et al. [12] demonstrated that the total graph of path power of three

$T(P_n^3)$ and four $T(P_n^4)$ has unbounded metric dimension. Nazeer et al. [13] demonstrated that the metric dimension of a two-middle path graph *Two−Mid*(p_q), $q \geq 3$ is 2, three-middle path graph *Three−Mid*(P_q), $q \geq 3$ is 3, three-total P_q *Three−T*(P_q), $q \geq 3$ is 3, reflection middle tower path graph *RL*(*Tower$_s$*), $s \geq 3$ is 2, middle tower path graph *Middle Tower$_s$*, $s=2$ is 1, *Middle Tower$_s$*, $s \geq 3$ is 2, symmetrical planar pyramid graph *SPPs* is 2 and reflection symmetrical planar pyramid graph *RL*(*SPPs*) is 2. Ahmad et al. [14] determined the metric dimension of kayak paddles graph *KP*(ℓ, m, n) and cycles C_n with chord and proved that both families possess a metric dimension of 2. Mulyono et al. [15] determined the metric dimension of friendship graph F_n, $n \geq 2$ is n lollipop graph $L_{m,n}$, $m \geq 3$, $n \geq 1$ is m-1 and the Petersen graph $P_{n,m}$, $m = 1$, n is odd, $n \geq 3$ is 2 and $m = 1$, n is even, $n \geq 3$ is 3. Garces et al. [16] proved that the metric dimension of truncated wheels TW_n, $n = 3$ or $n = 6$ is 3, $n = 4$ or $n = 5$ is 2 and $n \geq 7$ is $\left\lceil\frac{n}{2}\right\rceil - 1$. Siddique et al. [17] showed the metric dimensions of antiweb-gear graphs AWJ_{2n}, $n \geq 15$ is $\left\lceil\frac{n+1}{3}\right\rceil$ and m-level wheel graphs $W_{n,m}$, $n \geq 7$, $m \geq 3$ is $\left\lfloor\frac{2n+2}{5}\right\rfloor + (m-1)\left\lfloor\frac{2n+4}{5}\right\rfloor$. Jäger et al. [18] proved that the metric dimension of $Z_n \times Z_n \times Z_n$, $n \geq 2$ is $\left\lfloor\frac{3n}{2}\right\rfloor$. Tomescu et al. [19] determined that the metric dimension of Jahangir graph J_{2n}, $n \geq 4$ is $\left\lfloor\frac{2n}{3}\right\rfloor$. Tomescu et al. [20] determined that the metric dimension of necklace graph N_{en} is 3 when n is odd and 2 when n is even. The metric dimension of convex polytopes has been studied [21-24].

In this paper, we determine the metric dimension of Lilly graph and its subdivision, tadpole graph and its subdivision, the subdivisions of the special trees star tree, bistar tree, coconut tree, Y-tree, F-tree and n-centipede tree.

Definition 1.1 [25] The subdivision of a graph is the graph obtained by subdividing each edge of a graph G. It is denoted by $S(G)$.

Example 1.2:

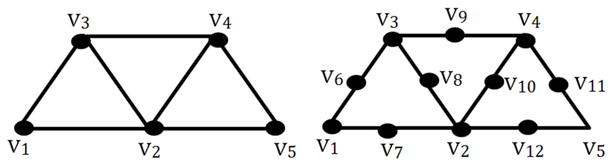

Figure 1. *Line graph G and its subdivision S(G).*

We show that dim (G) = 2 and dim($S(G)$) = 3. The set $B = \{v_1, v_3\}$ is a resolving set of G. The representations for the vertices of G with respect to B are r(v_1| B)=(0,1), r(v_2| B) = (1,1), r(v_3| B)=(1,0), r(v_4| B) =(2,1), r(v_5| B) =(2,2) is unique.

Clearly. Dim (G) = 2. The minimum resolving set of $S(G)$ is $B = \{v_1, v_3, v_4\}$. The representations for the vertices of $S(G)$ with respect to B are

r(v_1| B)=(0,2,4), r(v_2| B) = (2,2,2), r(v_3| B)=(2,0,2), r(v_4| B)=(4,2,0), r(v_5| B)=(4,4,2), r(v_6| B)=(1,1,3), r(v_7| B)=(1,3,3), r(v_8| B)=(3,1,3), r(v_9| B)=(3,1,1), r(v_{10}| B)=(3,3,1), r(v_{11}| B)=(5,3,1), r(v_{12}| B)=(3,3,3).

Clearly. Dim ($S(G)$) = 3.

2. Metric Dimension of Lilly Graph L_n and Its Subdivision $S(L_n)$

In this section, we compute the metric dimension of Lilly graph and its subdivision. The metric dimension of Lilly graph and its subdivision have different metric dimension.

Theorem 2.1 For Lilly graph L_n, $n \geq 2$, dim (L_n) = $\frac{n+3}{2}$.

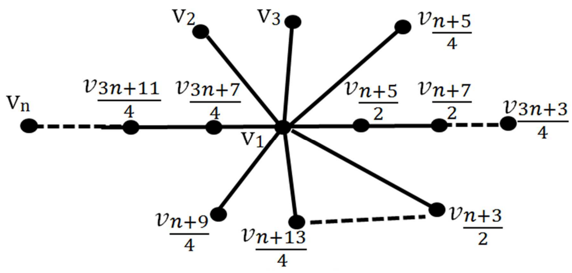

Figure 2. *Lilly graph L_n.*

Proof. The resolving set in general form is $B = \{v_2, v_3, v_4, \ldots, v_{\frac{n+5}{2}}\} \subset V(L_n)$. The representations of vertices $v_i \in V(L_n)$ in regard to B are as follow:

$$r(v_1|B) = (1, 1, 1, \ldots, 1, 1)$$

$$r(v_2|B) = (0, 2, 2, \ldots, 2, 2)$$

$$r(v_3|B) = (2, 0, 2, \ldots, 2, 2)$$

$$r(v_4|B) = (2, 2, 0, \ldots, 2, 2)$$

$$\vdots \quad = \quad \vdots$$

$$r(v_{\frac{n+5}{2}}|B) = (2, 2, 2, \ldots, 2, 0)$$

$$r(v_{\frac{n+7}{2}}|B) = (i - \tfrac{n+3}{2} + 1, i - \tfrac{n+3}{2} + 1, \ldots, i - \tfrac{n+3}{2} + 1, i - \tfrac{n+3}{2} - 1)$$

$$\vdots \quad = \quad \vdots$$

$$r(v_{\frac{3n+3}{4}}|B) = (i - \tfrac{n+3}{2} + 1, i - \tfrac{n+3}{2} + 1, \ldots, i - \tfrac{n+3}{2} + 1, i - \tfrac{n+3}{2} - 1)$$

$$r(v_{\frac{3n+7}{4}}|B) = (i - \tfrac{3n-1}{4}, i - \tfrac{3n-1}{4}, \ldots\ldots\ldots\ldots, i - \tfrac{3n-1}{4})$$

$$\vdots \quad = \quad \vdots$$

$$r(v_n|B) = (i - \tfrac{3n-1}{4}, i - \tfrac{3n-1}{4}, \ldots\ldots\ldots\ \ldots\ldots, i - \tfrac{3n-1}{4})$$

The vertices in graph L_n have unique representations, as seen above. B is resolving set, but not necessarily the lower bound. So the upper bound is dim (L_n) $\leq \frac{n+3}{2}$.

So, We demonstrate that dim (L_n) $\geq \frac{n+3}{2}$. Let $B = \{v_2, v_3, \ldots,$

$v_{\frac{n+5}{2}}\}$ be a resolving set with $|B| = \frac{n+3}{2}$ Assume that B_1 is another minimal resolving set, or $|B_1| < \frac{n+3}{2}$.

If we select an ordered set $B_1 \subseteq B - \{v_i, v_j\}$, $1 \le i,j \le \frac{n+3}{2}$, $i \neq j$, so that there exist two vertices $v_i, v_j \in L_n$, then $r(v_i|B) = r(v_j|B) = (2, 2, 2,..., 2, 2)$. The assumption that B_1 is a resolving set is invalid. As a result, dim $(L_n) \ge \frac{n+3}{2}$ is the lower bound. In conclusion dim $(L_n) = \frac{n+3}{2}$.

Theorem 2.2 If $S(L_n)$, $n \ge 3$ is a subdivision of Lilly graph, then dim $(S(L_n)) = \frac{n+3}{4}$.

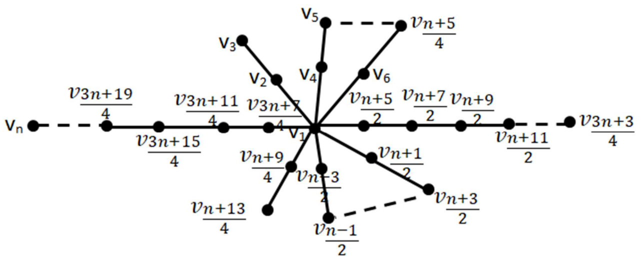

Figure 3. *Subdivision of Lilly graph $S(L_n)$.*

Proof. The resolving set in general form is $B = \{v_2, v_3, v_4,..., v_{\frac{n+7}{4}}\} \subset V(S(L_n))$. The representations of vertices $v_i \in V(S(L_n))$ in regard to B are as follow:

$r(v_1|B) = (1,1,1,..........., 1,1)$

$r(v_2|B) = (0, 2, 2,.........., 2,2)$

$r(v_3|B) = (2, 0, 2,2,........, 2,2)$

$r(v_4|B) = (2, 2, 0,2,2,......., 2,2)$

$\vdots \qquad = \qquad \vdots$

$r(v_{\frac{n+3}{4}}|B) = (2, 2, 2,......., 2, 0)$

$r(v_{\frac{n+7}{4}}|B) = (1, 3, 3,..............., 3)$

$r(v_{\frac{n+11}{4}}|B) = (3, 1, 3, 3,..........., 3)$

$\vdots \qquad = \qquad \vdots$

$r(v_{n-10}|B) = (3,3,3,..........., 3, 1)$

$r(v_{n-9}|B) = (2,2,............., 2, 2)$

$r(v_{n-8}|B) = (3,3,............., 3, 3)$

$\vdots \qquad = \qquad \vdots$

$r(v_{n-4}|B) = (7, 7,............., 7, 7)$

$r(v_{n-3}|B) = (4, 4,................, 4, 2)$

$r(v_{n-2}|B) = (5, 5,................, 5, 3)$

$r(v_{n-1}|B) = (6, 6,................, 6, 4)$

$r(v_n|B) = (7, 7,................, 7, 5)$

3. Metric Dimension of Tadpole Graph

In this section, we compute the metric dimension of Tadpole graph and its subdivision. The metric dimension of tadpole graph and its subdivision have the same constant metric dimension 2.

Theorem 3.1 If $T_{n,m}$, $n \ge 3$, $m \ge 1$ is a tadpole graph, then dim $(T_{n,m})$=2.

Proof The tadpole graph $T_{n,m}$ has two cases.

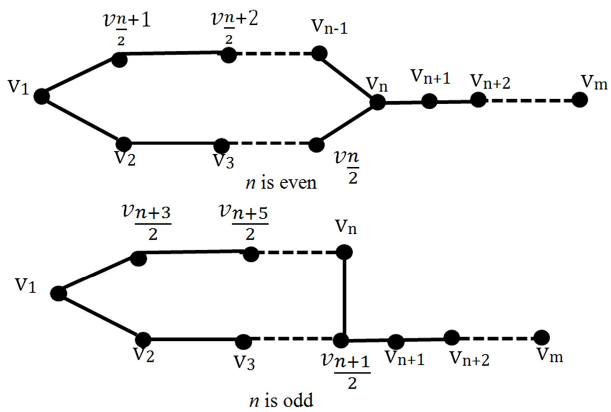

Figure 4. *Tadpole graph $T_{n,m}$.*

The general form of resolving set of the two cases is $B = \{v_1, v_2\} \subset V(T_{n,m})$.

Case 1. When n is even. The representations of vertices $v_i \in V(T_{n,m})$ with regard to B are

$$r(v_i|B) = \begin{cases} (0,1), i = 1; \\ (i-1, i-2), 2 \le i \le \frac{n}{2}; \\ \left(i - \frac{n}{2}, i - \frac{n}{2} + 1\right), \frac{n}{2} + 1 \le i \le n-1; \\ \left(i - \frac{n}{2}, i - \frac{n}{2} - 1\right), n \le i \le m. \end{cases}$$

Case 2. When n is odd. The representations of vertices $v_i \in V(T_{n,m})$ in regard to B are

$$r(v_i|B) = \begin{cases} (0,1), i = 1; \\ (i-1, i-2), 2 \le i \le \frac{n+1}{2}; \\ (i - \frac{n+1}{2}, i - \frac{n-1}{2}), \frac{n+3}{2} + 1 \le i \le n-1; \\ \left(i - \frac{n+1}{2}, i - \frac{n+1}{2}\right), n \le i \le m. \end{cases}$$

As all representations are distinct, dim $(T_{n,m}) = 2$.

Theorem 3.2 If $S(T_{n,m})$, $n \ge 6$, $m \ge 2$, is a subdivision of tadpole graph $T_{n,m}$, then dim$(S(T_{n,m}))$=2.

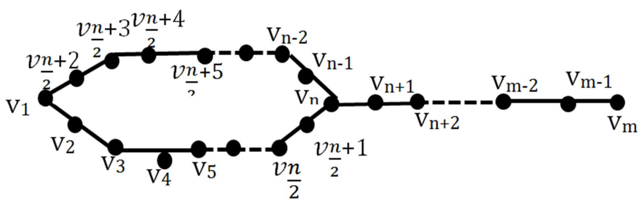

Figure 5. *Subdivision of tadpole graph S ($T_{n,m}$).*

Proof. The resolving set in general form is $B = \{v_1, v_2\} \subset V(S(T_{n,m}))$. The representations of vertices $v_i \in V(S(T_{n,m}))$ in regard to B are as follow:

$$r(v_i|B) = \begin{cases} (0,1), \mathrm{i} = 1; \\ (i-1, i-2), 2 \le \mathrm{i} \le \frac{n}{2}+1; \\ \left(i-\frac{n}{2}-1, i-\frac{n}{2}\right), \frac{n}{2}+2 \le \mathrm{i} \le m. \end{cases}$$

Since all vertices have unique representations, we obtain dim $(S(T_{n,m}))= 2$.

4. Metric Dimension of Subdivisions of Special Trees

In this section, we compute the metric dimension of subdivisions of star tree, bistar tree and coconut tree. The metric dimension of star tree and its subdivision have different metric dimension, subdivision of bistar tree $S\ (BT_{n,n})$ has metric dimension $\frac{n-7}{2}$ and subdivision of coconut tree $S(CT(m,n))$ has metric dimension m.

Theorem 4.1 If $S\ (ST_{n,n})$, $n \ge 6$ is a subdivision of star tree then dim $(S(ST_{n,n})) = \frac{n-3}{2}$.

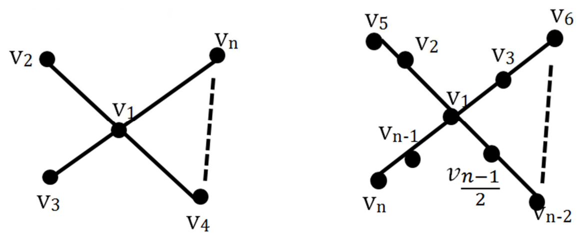

Figure 6. *Star tree $ST_{n,n}$ and its subdivision S ($ST_{n,n}$).*

Proof. In [6] it was shown that dim $(ST_{n,n})= n-2$, $n \ge 3$ and [9] it was computed the metric dimension of k-subdivision of star graph with more than one edge dim $S(ST_{n,n})= n-2$, $n \ge 4$. In this theorem, we compute the metric dimension of the subdivision of a star graph.

The resolving set in general form is $B=\{v_2, v_3, \ldots, v_{\frac{n-1}{2}}\} \subset V\ (S(ST_{n,n}))$. The representations of vertices $v_i \in V(S(ST_{n,n}))$ in regard to B are as follow:

$r(v_1|B) = (1, 1, \ldots, 1)$

$r(v_2|B) = (0, 2, 2, \ldots, 2)$

$r(v_3|B) = (2, 0, 2, 2, \ldots, 2)$

$\vdots \quad = \quad \vdots$

$r(v_{\frac{n-1}{2}}|B) = (2, 2, \ldots, 0)$

$r(v_{\frac{n+1}{2}}|B) = (1, 3, 3, \ldots, 3)$

$r(v_{\frac{n+3}{2}}|B) = (3, 1, 3, \ldots, 3)$

$\vdots \quad = \quad \vdots$

$r(v_{n-2}|B) = (3, 3, \ldots, 1)$

$r(v_{n-1}|B) = (2, 2, \ldots, 2)$

$r(v_n|B) = (3, 3, \ldots, 3)$

As seen above, the representations of vertices $V\ (S(ST_{n,n}))$ have unique representations, B is resolving set, but not necessarily the lower bound. An upper bound is dim $(S(ST_{n,n})) \le \frac{n-3}{2}$. So, we prove that dim $(S(ST_{n,n})) \ge \frac{n-3}{2}$. Let $B = \{v_2, v_3, \ldots, v_{\frac{n-1}{2}}\}$ be a resolving set with $|B| = \frac{n-3}{2}$. Assume that B_1 is another minimal resolving set, or $|B_1| < \frac{n-3}{2}$.

If we select an ordered set $B_1 \subseteq B - \{v_i, v_j\}$, $1 \le i, j \le \frac{n-3}{2}$, $i \ne j$, so there exist two vertices $v_i, v_j \in (S(ST_{n,n}))$ such that $r(v_i|B) = r(v_j\ |B) = (3, 3, \ldots, 3)$. The assumption that B_1 is a resolving set is invalid. As a result, dim $(S\ (ST_{n,n})) \ge \frac{n-3}{2}$ is the lower bound. In conclusion dim $(S(ST_{n,n})) = \frac{n-3}{2}$.

Theorem 4.2 If $S\ (BT_{n,n})$, $n \ge 3$ is a subdivision of bistar tree then dim $(S(BT_{n,n})) = \frac{n-7}{2}$.

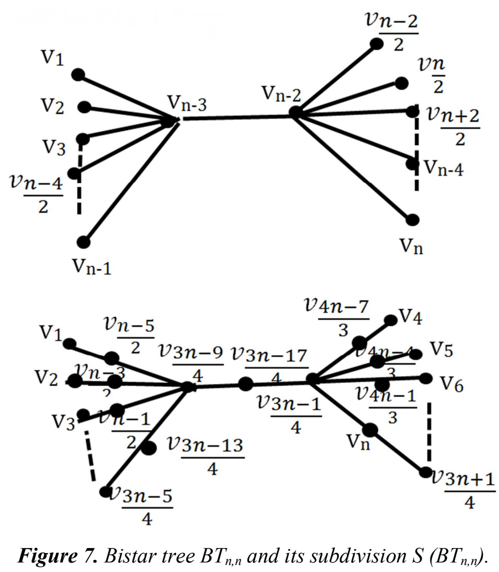

Figure 7. *Bistar tree $BT_{n,n}$ and its subdivision S ($BT_{n,n}$).*

Proof. The resolving set in general form is $B = \{v_1, v_2, \ldots, v_{\frac{n-7}{2}}\} \subset V\ (S(BT_{n,n}))$. The representations of vertices $v_i \in V(S(BT_{n,n}))$ in regard to B are as follow:

$r(v_1|B) = (0, 4, 4 \ldots, 4, 6, \ldots, 6)$

$$r(v_2|B) = (4, 0, 4, 4, \ldots, 6, \ldots, 6)$$

$$\vdots \quad = \quad \vdots$$

$$r(v_{\frac{n-7}{2}}|B) = (6, \ldots, 6, 4, 4, \ldots, 4, 0)$$

$$r(v_{\frac{n-5}{2}}|B) = (1, 3, \ldots, 3, 3, 5, \ldots, 5)$$

$$r(v_{\frac{n-3}{2}}|B) = (3, 1, 3, \ldots, 3, 3, 5, \ldots, 5)$$

$$\vdots \quad = \quad \vdots$$

$$r(v_{\frac{3n-21}{4}}|B) = (3, 3, \ldots, 3, 1, 5, \ldots, 5)$$

$$r(v_{\frac{3n-17}{4}}|B) = (3, 3, \ldots, 3)$$

$$r(v_{\frac{3n-13}{4}}|B) = (3, 3, \ldots, 3, 5, 5, \ldots, 5)$$

$$r(v_{\frac{3n-9}{4}}|B) = (2, 2, \ldots, 2, 4, 4, \ldots, 4)$$

$$r(v_{\frac{3n-5}{4}}|B) = (4, 4, \ldots, 4, 6, 6, \ldots, 6)$$

$$r(v_{\frac{3n-1}{4}}|B) = (4, 4, \ldots, 4, 2, 2, \ldots, 2)$$

$$r(v_{\frac{3n+3}{4}}|B) = (6, 6, \ldots, 6, 4, 4, \ldots, 4)$$

$$r(v_{\frac{4n-7}{3}}|B) = (5, \ldots, 5, 1, 3, \ldots, 3)$$

$$\vdots \quad = \quad \vdots$$

$$r(v_{n-1}|B) = (5, \ldots, 5, 3, \ldots, 3, 1)$$

$$r(v_n|B) = (5, \ldots, 5, 3, \ldots, 3, 3)$$

As seen above, the representations of vertices $V(S(BT_{n,n}))$ have unique representations, B is resolving set, but not necessarily the lower bound. An upper bound is dim $(S(BT_{n,n})) \leq \frac{n-7}{2}$. So, we prove that dim $(S(BT_{n,n})) \geq \frac{n-7}{2}$. Let $B = \{v_1, v_2, \ldots, v_{\frac{n-7}{2}}\}$ be a resolving set with $|B| = \frac{n-7}{2}$. Assume that B_1 is another minimal resolving set, or $|B_1| < \frac{n-7}{2}$.

If we select an ordered set $B_1 \subseteq B - \{v_i, v_j\}$, $1 \leq i, j \leq \frac{n-7}{2}$, $i \neq j$, so there exist two vertices $v_i, v_j \in (S(BT_{n,n}))$ such that $r(v_i|B) = r(v_j|B) = (5, \ldots, 5, 3, \ldots, 3)$. The assumption that B_1 is a resolving set is invalid. As a result, dim $(S(BT_{n,n})) \geq \frac{n-7}{2}$ is the lower bound. In conclusion dim $(S(BT_{n,n})) = \frac{n-7}{2}$.

Theorem 4.3 If $(S(CT(m,n))$, $m \geq 3$ and $n \geq 5$ is a subdivision of Coconut tree then $\dim(S(CT(m,n)) = m$.

Proof. The resolving set in general form is $B = \{v_1, v_2, v_3, \ldots, v_m\}$ $V(S(CT(m,n))$. The representations of vertices $v_i \in V(S(CT(m, n))$ in regard to B are as follow:

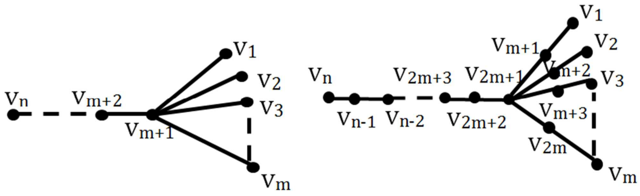

Figure 8. *Coconut tree CT(m,n) and it subdivision S (CT(m,n)).*

$$r(v_1|B) = (0, 4, 4, \ldots, 4, 4)$$

$$r(v_2|B) = (4, 0, 4, \ldots, 4, 4)$$

$$r(v_3|B) = (4, 4, 0, \ldots, 4, 4)$$

$$\vdots \quad = \quad \vdots$$

$$r(v_m|B) = (4, 4, 4, \ldots, 4, 0)$$

$$r(v_{m+1}|B) = (1, 3, 3, \ldots, 3, 3)$$

$$r(v_{m+2}|B) = (3, 1, 3, \ldots, 3, 3)$$

$$r(v_{m+3}|B) = (3, 3, 1, \ldots, 3, 3)$$

$$\vdots \quad = \quad \vdots$$

$$r(v_{2m}|B) = (3, 3, 3, \ldots, 3, 1)$$

$$r(v_{2m+1}|B) = (2, 2, 2, \ldots, 2, 2)$$

$$\vdots \quad = \quad \vdots$$

$$r(v_n|B) = (6, 6, 6, \ldots, 6, 6)$$

The representations of vertices in graph $S(CT(m,n))$ are distinct as shown above, B is resolving set, but not always the lower bound. An upper bound is dim $S(CT(m, n)) \leq m$. As a result, we show that dim $S(CT(m, n)) \geq m$. Let $B = \{v_1, v_2, v_3, \ldots, v_m\}$ be a resolving set with $|B| = m$. Assume that B_1 is another minimum resolving set, or $|B_1| < m$.

If we choose an ordered set $B_1 \subseteq B - \{v_i, v_j\}$, $1 \leq i, j \leq m$, $i \neq j$, so that there exist two vertices $v_i, v_j \in S(CT(m, n)$ such that $r(v_i|B) = r(v_j|B) = (3, 3, 3, \ldots, 3, 3)$. B_1 is not a resolving set, which is contrary to the assumption. As a result, dim $S(CT(m, n)) \geq m$ is the lower bound.

As a result, dim $S(CT(m, n)) = m$.

5. Conclusion

The metric dimension of Lilly graph and its subdivision have different metric dimensions. The tadpole graph and its subdivisions have the same constant metric dimension 2. The metric dimension of star tree and its subdivision have different metric dimension, subdivision of bistar tree $S(BT_{n,n})$ has metric dimension $\frac{n-7}{2}$ and subdivision of coconut tree $S(CT(m,n))$ has metric dimension m.

References

[1] P. J. Slater, “Leaves of Trees,” Congressus Numerantium, Vol. 14, pp. 549-559, 1975.

[2] P. J. Slater, “Dominating and Reference Sets in Graphs,” Journal of Mathematical Physics, Vol. 22, no. 4, pp. 445-455, 1998.

[3] F. Harary and R. A. Melter, “On the Metric Dimension of a Graph,” Ars Combinatoria, Vol. 2, pp. 191-195, 1976.

[4] S. Khuller, B. Raghavachari and A. Rosenfeld, “Localization in Graphs,” Technical Report CS-TR-3326, Universityof Maryland at CollegePark, 1994.

[5] R. Manjusha and AS. Kuriakose, “Metric dimension and uncertainty of traversing robots in a network”, International Journal on Applications of Graph Theory in Wireless Ad hoc Networks and Sensor Networks (GRAPH-HOC), Vol. 7, no. 2/3, pp. 1-9, 2015.

[6] G. Chartrand, L. Eroh, M. A. Johnson and O. R. Oellermann, “Resolvability in graphs and the metric dimension of a graph”, Discrete Applied Mathematics, Vol. 105, no. 1-3, pp. 99–113, 2000.

[7] R. A. Melter and I. Tomescu, “Metric Bases in Digital Geometry,” Computer Vision, Graphics, and Image Processing, Vol. 25, no. 1, pp. 113-121, 1984.

[8] M. Idrees, H. Ma, M. Wu, A. R. Nizami, M. Munir and S. Ali, “Metric Dimension of Generalized Möbius Ladder and its Application to WSN Localization”, Journal of Advanced Computational Intelligence and Intelligent Informatics, Vol. 24, pp. 3-11, 2020.

[9] L. Susilowati, S. Zahidah, R. D. Nastiti and M. I. Utoyo, “The metric dimension of k-subdivision graphs”, In Journal of Physics: Conference Series (Vol. 1494, No. 1, p. 012015). IOP Publishing, 2020.

[10] A. Borchert and S. Gosselin,“The metric dimension of circulant graphs and Cayley hypergraphs”, Utilitas Mathematica, vol. 106, pp. 125-147, 2018.

[11] M. Imran, M. K. Siddiqui and R. Naeem, “On the metric dimension of generalized petersen multigraphs”, IEEE Access, Vol. 6, pp. 74328-74338, 2018.

[12] S. Nawaz, M. Ali, M. A. Khan and S. Khan, “Computing Metric Dimension of Power of Total Graph”, IEEE Access, vol. 9, pp. 74550-74561, 2021.

[13] S. Nazeer, M. Hussain, F. A. Alrawajeh and S. Almotairi, “Metric Dimension on Path-Related Graphs”, Mathematical Problems in Engineering. 2021.

[14] A. Ahmad, M. Bača and S. Sultan, “Computing the metric dimension of Kayak Paddles graph and Cycles with chord”, Proyecciones, vol. 39, pp. 287-300, 2020.

[15] M. Mulyono and W Wulandari, “The Metric Dimension of Friendship Graph F_n, Lollipop Graph $L_{m,n}$ and Petersen Graph $P_{n, m}$”, Bulletin of Mathematics, Vol. 8, no. 2, pp. 117-124, 2016.

[16] I. J. L. Garces and J. B. Rosario, “Computing the metric dimension of truncated wheels”, 2015.

[17] H. M. A. Siddiqui, and M. Imran, “Computing the metric dimension of wheel related graphs”, Applied mathematics and computation, Vol. 242, pp. 624-632, 2014.

[18] G. Jäger and F. Drewes, “The metric dimension of $Z_n \times Z_n \times Z_n$ is $\lfloor 3n/2 \rfloor$”, Theoretical Computer Science, vol. 806, pp. 344-362, 2020.

[19] I. Tomescu and I. Javaid, “On the metric dimension of the Jahangir graph”, Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie, pp. 371-376, 2007.

[20] I. Tomescu and M. Imran, “R-sets and metric dimension of necklace graphs”, Applied Mathematics and Information Sciences, Vol. 9, no. 1, pp. 63-67, 2015.

[21] M. Imran, A. Q. Baig and M. K. Shafiq,“ Classes of convex polytopes with constant”, Utilitas mathematica, Vol. 90, pp. 85-99, 2013.

[22] M. Imran, A. AHMAD and A. SEMANIČOVÁ-FEŇOVČÍKOVÁ, “On classes of regular graphs with constant metric dimension”, Acta Mathematica Scientia, vol. 33, pp. 187-206, 2013.

[23] M. Imran, F. Bashir, A. Q. Baig, S. A. Bokhary, A. Riasat and I. Tomescu, “On metric dimension of flower graphs fnχm and convex polytopes”, Utilitas mathematica, Vol. 92, pp. 389–409, 2013.

[24] X. Zuo, A. Ali, G. Ali, M. K. Siddiqui, M. T. Rahim and A. Asare-Tuah, “On Constant Metric Dimension of Some Generalized Convex Polytopes”, Journal of Mathematics, vol. 2021, 2021.

[25] U. Vaghela and D. Parmar, “Difference perfect square cordial labeling of subdivision of snake graphs”, International Journal of Future Generation Communication and Networking, vol. 13, no. 2, pp. 275-297, 2020.