

# Discrete Probability of Random Möbius Groups: Random Subgroups by Two Generators

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**Abstract:** Let  $GL(2, \mathbb{R})$  be real Möbius groups. For any  $2 \times 2$  matrix  $A$  in  $GL(2, \mathbb{R})$  induces real Möbius transformations  $g$  by the formula  $A \rightarrow g_A = g$  where  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, g = \frac{ax+b}{cx+d}$ . The collection of all real Möbius transformations for which  $ad-bc=1$  takes the values 1 forms a group which can be identified with  $PSL(2, \mathbb{R})$ . We write  $f \in {}^*PSL(2, \mathbb{R})$  to mean that  $f$  is a random variable in  $PSL(2, \mathbb{R})$ . In this paper, we study the random Möbius subgroup  ${}^*PSL(2, \mathbb{R})$ . We can get some new results: (1) If  $f, g \in {}^*PSL(2, \mathbb{R})$ , the probability  $|tr^2(f) - 4| + |tr(fgf^{-1}g^{-1}) - 2| \leq 1$  is greater than 0.027282. (2) If  $f$  is hyperbolic transformation and  $g$  is parabolic transformation, then the probability  $\langle f, g \rangle$  is discrete is greater than  $7/12$ . (3) If  $f$  is elliptic of order  $n$  and  $g$  is elliptic of order 2, then the probability  $\langle f, g \rangle$  is discrete is greater than  $2/n$ . (4) The probability that random chosen  $f, g \in {}^*PSL(2, \mathbb{R})$  generate an elementary or non-discrete group  $\langle f, g \rangle$  is greater than 0.0302049.

**Keywords:** Random Discrete Möbius Group,  ${}^*PSL(2, \mathbb{R})$ , Jørgensen's Inequality

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## 1. Introduction

Throughout this paper, we will adopt the same notations and definitions as [2, 13, 14, 15, 16] such as, discrete group, Fuchsian group, elementary group,  ${}^*PSL(2, \mathbb{R})$  and so on. For example, if  $G$  is discrete and if  $X, A_1, A_2, \dots$  are in  $G$  with  $A_n \rightarrow X$  then  $A_n = X$  for all sufficiently large  $n$ . A group  $G$  of Möbius transformation is a Fuchsian group if and only if there is some  $G$ -invariant disc in which  $G$  acts discontinuously. A subgroup  $G$  of  $PSL(2, \mathbb{C})$  is said to be elementary if and only if there exists a finite  $G$ -orbit in  $\mathbb{R}^3$ . See [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 18, 19, 20] etc. for more details.

It is well known that Random group, introduced in the 1943 by Auerbach [1], has been extensively studied over the group of random symmetric group [4] and random finite group [11, 12]. G. J. Martin and G. O'Brien [13, 14]

introduce a geometrically natural probability measure on the group  $PSL(2, \mathbb{R})$ . They studied the probability that random Möbius subgroup  $\langle f, g \rangle$  is discrete and given a lower bound for the probability a group generated by two random hyperbolic elements or parabolic elements of  ${}^*PSL(2, \mathbb{R})$  is discrete.

Theorem MO1 [13, Theorem 11.6] The probability that two randomly chosen hyperbolic transformation  $f, g \in {}^*PSL(2, \mathbb{R})$  generate a discrete group  $\langle f, g \rangle$  is at least  $2/5$ .

Theorem MO2 [13, Theorem 10.3] The probability that two randomly chosen parabolic transformation  $f, g \in {}^*PSL(2, \mathbb{R})$  generate a discrete group  $\langle f, g \rangle$  is at least  $1/6$ .

We further study the probability a group generated by a hyperbolic element and a parabolic element of  ${}^*PSL(2, \mathbb{R})$  is

discrete and obtain the following theorems.

**Theorem 1.1** Let  $f, g$  be random chosen hyperbolic element and parabolic element in  $*PSL(2, \mathbb{R})$ . Then the probability  $\langle f, g \rangle$  is discrete is at least  $7/12$ .

G. J. Martin and G. O'Brier studied the probability a group generated by a elliptic element of order  $n$  and a hyperbolic element of  $*PSL(2, \mathbb{R})$  is discrete.

**Theorem MO3** [13, Theorem 12.5] Let  $f = \xi z$  and  $\xi^n = 1$ . Let  $g \in *PSL(2, \mathbb{R})$  be a randomly chosen hyperbolic. Then the probability  $\langle f, g \rangle$  is discrete is  $2/n^2$ .

We further study the probability a group generated by two elliptic elements of  $*PSL(2, \mathbb{R})$  is discrete and obtain the following theorems.

**Theorem 1.2** Let  $f, g$  be random chosen elements in  $*PSL(2, \mathbb{R})$ . If  $f$  is elliptic of order  $n$  and  $g$  is elliptic of order 2. Then the probability  $\langle f, g \rangle$  is discrete is  $2/n$ .

G. J. Martin [15, 16] had results such as the p.d.f for trace, Commutators, Cross ratios. We will obtain the p.d.f for  $m(f)$  and have the following theorem.

**Theorem 1.3** Let  $f$  be randomly chosen element of  $PSL(2, \mathbb{R})$ . Then the random variable  $m^2(f)$  has the p.d.f

$$F(s) = \begin{cases} F_1(s) & 0 < s \leq 8 \\ F_2(s) & s \geq 8 \end{cases}$$

Where,

$$F_1(s) = \frac{1}{8\pi^2} \frac{1}{\frac{s}{8} + 8} \int_1^{\frac{s}{8}+1} \sqrt{\frac{y}{\left(\frac{s}{8} + 1 - y\right)(2-y)(y-1)}} dy$$

$$F_2(s) = \frac{1}{8\pi^2} \frac{1}{\frac{s}{8} + 1} \int_1^2 \sqrt{\frac{y}{\left(\frac{s}{8} + 1 - y\right)(2-y)(y-1)}} dy$$

Finally, by using Theorem 1.3, we will obtain the following theorems.

**Theorem 1.4** Let  $f, g$  be random chosen elements in  $*PSL(2, \mathbb{R})$ . Then the probability  $|tr^2(f) - 4| + |tr(fgf^{-1}g^{-1}) - 2| \leq 1$  is greater than 0.0272821.

**Theorem 1.5** Let  $f, g$  be random chosen elements in  $*PSL(2, \mathbb{R})$ . Then the probability  $\langle f, g \rangle$  is non-discrete or elementary is greater than 0.0302049.

## 2. Preliminaries

### 2.1. Random Möbius Groups

If  $A \in PSL(2, \mathbb{C})$  has the form

$$A = \pm \begin{pmatrix} a & c \\ \bar{c} & \bar{a} \end{pmatrix}, |a|^2 - |c|^2 = 1. \quad (1)$$

Then the group of all matrices satisfying (1) will be denoted  $\mathcal{F}$ . It is not difficult to construct an algebraic isomorphism  $\mathcal{F} \equiv PSL(2, \mathbb{R}) \equiv Isom^+(\mathbb{H}^2)$ .

*The probability distribution.* The probability space is  $(\mathcal{F}, \mu_{\mathcal{F}})$ , the space of matrices with the following imposed distribution of the entries of an element of  $\mathcal{F}$ .

- (i)  $\xi = a/|a|$  and  $\eta = c/|c|$  are chosen uniformly in the circle  $\mathbb{S}$ , with arclength measure.
- (ii)  $t = |a| \geq 1$  is chosen so that

$$2 \arcsin(1/t) \in [0, \pi]$$

is uniformly distributed.

**Lemma 2.1.1** [13, lemma 2.3] The random variable  $|a| \in [1, \infty)$  has the p.d.f

$$F_{|a|}(x) = \frac{2}{\pi} \frac{1}{x\sqrt{x^2 - 1}}.$$

we write  $\arg(a) \in_{\mu} [0, 2\pi]_{\mathbb{R}}$  if  $\arg(a)$  uniformly distributed on  $\mathbb{R} \bmod 2\pi$ . The following lemma will be crucial.

**Lemma 2.1.2** [13, lemma 2.4] If  $\arg(a), \arg(b) \in_{\mu} [0, 2\pi]_{\mathbb{R}}$ , then  $\arg(ab), \arg(a/b) \in_{\mu} [0, 2\pi]_{\mathbb{R}}$ . Hence

$$\arg(a^k) = k \arg(a) \in_{\mu} [0, 2\pi]_{\mathbb{R}} \text{ for } k \in \mathbb{Z}.$$

For any

$$A = \begin{pmatrix} a & c \\ \bar{c} & \bar{a} \end{pmatrix} \in \mathcal{F} \quad (2)$$

$$|tr(A)| = |2Re(a)| = |2a \cdot \cos(\theta)|, \theta = \arg(a). \quad (3)$$

Using the obvious symmetries, by lemma 2.1, we may calculate

$$Pr[\{|tr(A)| > 2\}] = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \left(1 - \frac{2}{\pi} \theta\right) d\theta = \frac{1}{2} \quad (4)$$

**Corollary 2.1** [13, corollary 2.4] Let  $f \in \mathcal{F}$  be a Möbius transformation chosen random from the distribution described in (i) and (ii). Then the probability that  $f$  is hyperbolic is equal to  $1/2$ ,  $f$  is elliptic is equal to  $1/2$  and  $f$  is parabolic is equal to 0.

See [13, Section 2] for more details.

### 2.2. Jørgensen's Inequality

In 1976, Jørgensen proved a necessary condition for a non-elementary two generator subgroup  $PSL(2, \mathbb{C})$  to be discrete, which is called Jørgensen's inequality.

**Theorem J** [21, lemma 1] If  $f, g \in PSL(2, \mathbb{C})$  generate a discrete and nonelementary group  $\langle f, g \rangle$ . Then

$$|tr^2(f) - 4| + |tr(fgf^{-1}g^{-1}) - 2| \geq 1 \quad (5)$$

For each  $f \in \mathcal{F}$ , we let  $m^2(f)$  denote the norm

$$m^2(f) = \|A - A^{-1}\|^2 = 2|a - \bar{a}|^2 + 8|c|^2 \quad (6)$$

We have the following theorem from [9, Theorem 2.7].

**Theorem  $FG_1$**  If  $f$  and  $g$  in  $PSL(2, \mathbb{R})$ . Then

$$|tr^2(f) - 4| \leq \frac{1}{2}m(f)^2 \quad (7)$$

$$|tr(fgf^{-1}g^{-1}) - 2| \leq \frac{1}{16}m(f)^2m(g)^2. \quad (8)$$

Both of these inequalities are sharp.

According to (7) and (8), we have

$$|tr^2(f) - 4| + |tr(fgf^{-1}g^{-1}) - 2| \leq m(f)^2 \left( \frac{1}{2} + \frac{1}{16}m(g)^2 \right) \quad (9)$$

### 2.3. An experiment on $m^2(f)m^2(g)$

If we select two random Möbius transformations, say,

$$f = \begin{pmatrix} e^{i\phi_1} \csc(\theta_1) & e^{i\alpha_1} \cot(\theta_1) \\ e^{-i\alpha_1} \cot(\theta_1) & e^{-i\phi_1} \csc(\theta_1) \end{pmatrix}, \quad (10)$$

$$g = \begin{pmatrix} e^{i\phi_2} \csc(\theta_2) & e^{i\alpha_2} \cot(\theta_2) \\ e^{-i\alpha_2} \cot(\theta_2) & e^{-i\phi_2} \csc(\theta_2) \end{pmatrix}, \quad (11)$$

where  $\theta_1, \theta_2 \in_\mu [0, \pi/2]_{\mathbb{R}}$  and  $\alpha_1, \alpha_2, \phi_1, \phi_2 \in_\mu [0, 2\pi]_{\mathbb{R}}$ .

Then

$$m^2(f)m^2(g) = 64 \left( \frac{\sin^2(\phi_1)}{\sin^2(\theta_1)} + \cot^2(\theta_1) \right) \left( \frac{\sin^2(\phi_2)}{\sin^2(\theta_2)} + \cot^2(\theta_2) \right) \quad (12)$$

We made several independent runs through about  $10^7$  random matrix pairs of elements to generate the histogram in Figure 1. We found the probability that

$$m^2(f)m^2(g) \geq 16 \left( 2 \cos\left(\frac{2\pi}{7}\right) - 1 \right) \quad (13)$$

to be about 0.0308797.

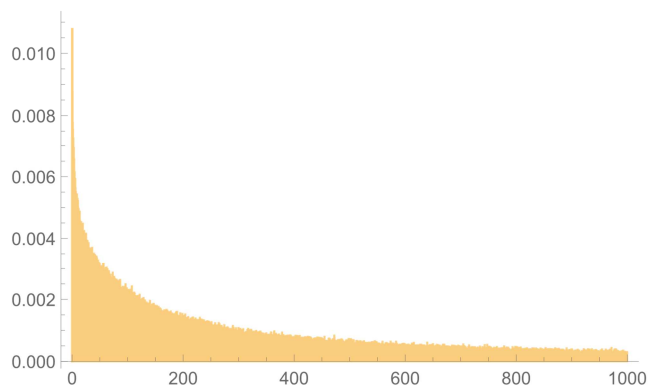


Figure 1. Histogram of  $m^2(f)m^2(g)$  values.

We also found

$$\Pr \left\{ m(f)^2 \left( \frac{1}{2} + \frac{1}{16}m(g)^2 \right) \leq 1 \right\} \approx 0.027282. \quad (14)$$

According to Theorem J and Lemma 2.3, we have

$$\Pr \left\{ |tr^2(f) - 4| + |tr(fgf^{-1}g^{-1}) - 2| \leq 1 \right\} \geq 0.027282 \quad (15)$$

We have the following theorem from [9, Theorem 4.19].

**Theorem  $FG_2$**  Suppose that  $\langle f, g \rangle$  is nonelementary discrete Fuchsian group subgroup of  $\mathbb{M}$ . Then

$$m^2(f)m^2(g) \geq 16(2 \cos(2\pi/7) - 1). \quad (16)$$

In addition, if  $f$  or  $g$  is parabolic, then

$$m^2(f)m^2(g) \geq 16. \quad (17)$$

If  $f$  and  $g$  are both elliptic, then

$$m^2(f)m^2(g) \geq 16 \cos^2(\pi/7). \quad (18)$$

If  $f$  and  $g$  are both hyperbolic, then

$$m^2(f)m^2(g) \geq 16(\cos(2\pi/7) + \cos(\pi/7) - 1)^2 \quad (19)$$

Finally, if  $f$  is hyperbolic and  $g$  is elliptic, then

$$m^2(f)m^2(g) \geq 16(\cos(2\pi/7) - 1) \quad (20)$$

Each of these inequalities is sharp.

Another discrete criterion can be found in [9, 16].

**Lemma 2.3.1** [10, Lemma 4.2] Suppose that  $\langle f, g \rangle$  has parameters  $(\gamma, \beta, -4)$  with  $\gamma \geq \beta$  and that  $f$  is a primitive elliptic of order  $q \geq 3$ . Then  $\langle f, g \rangle$  is a discrete group if and only if  $\gamma \geq \beta + 4$ .

**Lemma 2.3.2** [15, Theorem VII. A.13.] Let  $f_i, i = 1, 2, \dots, n$ , be hyperbolic transformations of the disk whose isometric disks are all disjoint. Then the group generated by these

hyperbolic transformations  $\langle f_1, f_2, \dots, f_n \rangle$  is discrete.

### 3. Proofs of the Main Results

*Proof of Theorem 1.1* Suppose that the arc  $\alpha_i = \alpha(m_{\alpha_i}, l_{\alpha_i})$   $i = 1, 2$ , of  $f$  and the arc  $\beta = \beta(m_{\beta_i}, l_{\beta_i})$  of  $g$ . Since  $f$  is hyperbolic, we have

$$\alpha_1 \cap \alpha_2 = \emptyset. \quad (21)$$

Thus

$$\alpha_1 \cap \beta = \emptyset, \alpha_2 \cap \beta = \emptyset \quad (22)$$

are independent. A little trigonometry reveals that

$$\alpha_i \cap \beta = \emptyset \Leftrightarrow \arg(m_{\alpha_i}, \bar{m}_{\beta}) \geq \frac{l_{\alpha} + l_{\beta}}{2} \quad (23)$$

Now the variables  $\theta_i = \arg(m_{\alpha_i}, \bar{m}_{\beta})$ ,  $i = 1, 2$ , are uniformly distributed in  $[0, \pi]$  and independent.

According lemma 2.1.2, we have

$$\begin{aligned} \Pr\left\{\min \theta_i \geq \frac{l_{\alpha} + l_{\beta}}{2}\right\} &= \Pr\left\{\min \theta_i - \frac{l_{\beta}}{2} \geq \frac{l_{\alpha}}{2}\right\} \\ &= \left(\frac{1}{\pi} \int_{\frac{l_{\alpha}}{2}}^{\pi} d\theta\right)^2 = \left(1 - \frac{1}{\pi} \frac{l_{\alpha}}{2}\right)^2. \end{aligned}$$

Let  $\frac{l_{\alpha}}{2} = \varphi$ , then  $\varphi \in \left[0, \frac{\pi}{2}\right]$ , thus

$$\Pr\left\{\min \theta_i \geq \frac{l_{\alpha} + l_{\beta}}{2}\right\} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \left(1 - \frac{\varphi}{\pi}\right)^2 d\varphi = \frac{7}{12}.$$

Using lemma 2.3.2, the probability  $\langle f, g \rangle$  is discrete is at least  $7/12$ .

*Proof of Theorem 1.2* With loss of generality,  $f$  and  $g$  can be represented that

$$f = \begin{pmatrix} e^{i\frac{\pi}{n}} & 0 \\ 0 & e^{-i\frac{\pi}{n}} \end{pmatrix}, g = \begin{pmatrix} i \csc(\theta) & e^{i\varphi} \cot(\theta) \\ e^{-i\varphi} \cot(\theta) & -i \csc(\theta) \end{pmatrix} \quad (24)$$

By calculation

$$\beta = -4 \sin^2\left(\frac{\pi}{n}\right), \gamma = 4 \sin^2\left(\frac{\pi}{n}\right) \cot^2(\theta)$$

Using lemma 2.3.1,  $\langle f, g \rangle$  is a discrete group if and only if  $\gamma \geq \beta + 4$ .

$$\begin{aligned} \Pr\{\gamma \geq \beta + 4\} &= \Pr\left\{4 \sin^2\left(\frac{\pi}{n}\right) \cot^2(\theta) \geq 4 \cos^2\left(\frac{\pi}{n}\right)\right\} \\ &= \Pr\left\{\theta \leq \frac{\pi}{n}\right\} = \frac{2}{\pi} \int_0^{\frac{\pi}{n}} d\theta = \frac{2}{n}. \end{aligned}$$

Then the probability  $\langle f, g \rangle$  is discrete is  $\frac{2}{n}$ .

*Proof of Theorem 1.3* We select a random Möbius transformation

$$f = \begin{pmatrix} e^{i\phi} \csc(\theta) & e^{i\varphi} \cot(\theta) \\ e^{-i\varphi} \cot(\theta) & e^{-i\phi} \csc(\theta) \end{pmatrix}, \theta \in_{\mu} [0, \frac{\pi}{2}], \phi, \varphi \in_{\mu} [0, 2\pi]$$

Then

$$m^2(f) = 8 \frac{\sin^2(\phi)}{\sin^2(\theta)} + 8 \cot^2(\theta)$$

The probability distribution functions of  $x = \sin^2(\theta)$  and  $y = \sin^2(\phi)$  are independent and identically distributed  $\Psi(x)$  and  $\Psi(y)$ ,

$$\Psi(x) = \frac{1}{\pi \sqrt{x(1-x)}}$$

The random variable has the c.d.f

$$P(S \leq s) = \frac{1}{\pi^2} \iint_{\left\{s, \frac{x+1-y}{y} \leq s\right\}} \frac{1}{\sqrt{x(1-x)}} \frac{1}{\sqrt{y(1-y)}} dx dy$$

We begin this calculation under the assumption  $0 \leq s \leq 8$ .

$$\begin{aligned} P_1(s) &= \frac{1}{\pi^2} \int_{\frac{s}{8}+1}^1 \int_0^{(\frac{s}{8}+1)y-1} \frac{1}{\sqrt{x(1-x)}} \frac{1}{\sqrt{y(1-y)}} dx dy \\ &= \frac{2}{\pi^2} \int_{\frac{s}{8}+1}^1 \frac{1}{\sqrt{y(1-y)}} \cdot \arcsin\left[\sqrt{\left(\frac{s}{8}+1\right)y-1}\right] dy \quad (25) \\ &= \frac{2}{\pi^2} \int_1^{\frac{s}{8}+1} \frac{\arcsin\sqrt{y-1}}{\sqrt{y(\frac{s}{8}+1-y)}} dy \end{aligned}$$

We can now differentiate (26) of  $s$  under the integral.

$$\begin{aligned} F_1(s) &= \frac{1}{8\pi^2} \int_{\frac{s}{8}+1}^1 \sqrt{\frac{y}{(1-y)[2-(\frac{s}{8}+1)y][(\frac{s}{8}+1)y-1]}} dy \\ &= \frac{1}{8\pi^2} \frac{1}{\frac{s}{8}+1} \int_1^{\frac{s}{8}+1} \sqrt{\frac{y}{(\frac{s}{8}+1-y)(2-y)(y-1)}} dy \quad (26) \end{aligned}$$

Similarity, suppose that  $s \geq 8$ , we can calculate

$$P_2(s) = \frac{2}{\pi^2} \int_1^2 \frac{\arcsin \sqrt{y-1}}{\sqrt{y(\frac{s}{8}+1-y)}} dy + 1 - \frac{2}{\pi} \cdot \arcsin \left[ \sqrt{\frac{2}{\frac{s}{8}+1}} \right]$$

and

$$F_2(s) = \frac{1}{8\pi^2} \frac{1}{\frac{s}{8}+1} \int_1^2 \sqrt{\frac{y}{(\frac{s}{8}+1-y)(2-y)(y-1)}} dy$$

The proof is completed.

A discussion similar to the proof of Theorem 1.3, we obtain some corollaries.

**Corollary 3.1** Let  $f$  be randomly chosen parabolic element of  ${}^*PSL(2, \mathbb{R})$ , then the random variable  $m^2(f)$  has the p.d.f

$$H(s) = \frac{4}{\pi} \frac{1}{\sqrt{s(16+s)}}$$

**Corollary 3.2** Let  $f$  be randomly chosen hyperbolic element of  ${}^*PSL(2, \mathbb{R})$ , then the random variable  $m^2(f)$  has the p.d.f

$$G(s) = \frac{1}{4\pi^2} \frac{1}{\frac{s}{8}+1} \int_1^{\frac{\frac{s}{4}+2}{\frac{s}{8}+1}} \sqrt{\frac{y}{(\frac{s}{8}+1-y)(2-y)(y-1)}} dy$$

**Corollary 3.3** Let  $f$  be randomly chosen elliptic element of  ${}^*PSL(2, \mathbb{R})$ , then the random variable  $m^2(f)$  has the p.d.f

$$T(s) = \begin{cases} T_1(s) & 0 < s \leq 8 \\ T_2(s) & s \geq 8 \end{cases}$$

Where,

$$T_1(s) = \frac{1}{4\pi^2} \frac{1}{\frac{s}{8}+1} \int_1^{\frac{\frac{s}{8}+1}{\frac{s}{8}+1}} \sqrt{\frac{y}{(\frac{s}{8}+1-y)(2-y)(y-1)}} dy$$

$$T_2(s) = \frac{1}{4\pi^2} \frac{1}{\frac{s}{8}+1} \int_1^2 \sqrt{\frac{y}{(\frac{s}{8}+1-y)(2-y)(y-1)}} dy$$

**Corollary 3.4** Let  $f$  be randomly chosen element of  ${}^*PSL(2, \mathbb{R})$ . Then the random variable  $\frac{1}{2} + \frac{1}{16} m^2(f)$  has the p.d.f

$$F(t) = \begin{cases} F_1(t) & \frac{1}{2} < t \leq 1 \\ F_2(t) & t \geq 1 \end{cases}$$

Where,

$$F_1(t) = \frac{1}{\pi^2 t} \int_1^{2t} \sqrt{\frac{y}{(2t-y)(2-y)(y-1)}} dy,$$

$$F_2(t) = \frac{1}{\pi^2 t} \int_1^2 \sqrt{\frac{y}{(2t-y)(2-y)(y-1)}} dy$$

**Corollary 3.5** Let  $f$  be randomly chosen parabolic element of  ${}^*PSL(2, \mathbb{R})$ . Then the random variable  $\frac{1}{2} + \frac{1}{16} m^2(f)$  has the p.d.f

$$H(t) = \frac{1}{\pi} \cdot \frac{1}{(t + \frac{1}{2}) \sqrt{t - \frac{1}{2}}}$$

**Corollary 3.6** Let  $f$  be randomly chosen hyperbolic element of  ${}^*PSL(2, \mathbb{R})$ . Then the random variable  $\frac{1}{2} + \frac{1}{16} m^2(f)$  has the p.d.f

$$G(t) = \frac{2}{\pi^2 t} \int_1^{\frac{4t}{2t+1}} \sqrt{\frac{y}{(2t-y)(2-y)(y-1)}} dy.$$

**Corollary 3.7** Let  $f$  be randomly chosen elliptic element of  ${}^*PSL(2, \mathbb{R})$ . Then the random variable  $\frac{1}{2} + \frac{1}{16} m^2(f)$  has the p.d.f

$$T(t) = \begin{cases} T_1(t) & \frac{1}{2} < t \leq 1 \\ T_2(t) & t \geq 1 \end{cases}$$

Where,

$$T_1(t) = \frac{1}{2\pi^2 t} \int_1^{\frac{4t}{2t+1}} \sqrt{\frac{y}{(2t-y)(2-y)(y-1)}} dy,$$

$$T_2(t) = \frac{1}{2\pi^2 t} \int_1^2 \sqrt{\frac{y}{(2t-y)(2-y)(y-1)}} dy.$$

**Proof of Theorem 1.4** Suppose that  $f, g$  be randomly chosen in  ${}^*PSL(2, \mathbb{R})$ , we need calculate the probability of  $m(f)^2(\frac{1}{2} + \frac{1}{16} m(g)^2) \leq 1$ . According to Theorem 1.3 and Corollary 3.5, we have

$$\begin{aligned}
Pr\left\{m(f)^2\left(\frac{1}{2} + \frac{1}{16}m(g)^2\right) \leq 1\right\} &= \iint_{\{s,t \leq 1\}} F(t) \cdot F(s) ds dt \\
&= \int_1^1 \int_1^{2t} \int_1^{\frac{1}{8t}+1} \frac{2}{\pi^4 t} \frac{\arcsin \sqrt{y-1}}{\sqrt{y(\frac{1}{8t}+1-y)}} \sqrt{\frac{x}{(2t-x)(2-x)(x-1)}} dy dx dt \\
&\quad + \int_1^\infty \int_1^2 \int_1^{\frac{1}{8t}+1} \frac{2}{\pi^4 t} \frac{\arcsin \sqrt{y-1}}{\sqrt{y(\frac{1}{8t}+1-y)}} \sqrt{\frac{x}{(2t-x)(2-x)(x-1)}} dy dx dt \\
&= 0.0161113 + 0.0111708 = 0.0272821.
\end{aligned}$$

Using (27), we have

$$Pr\left\{m(f)^2\left(\frac{1}{2} + \frac{1}{16}m(g)^2\right) \leq 1\right\} \leq Pr\{|tr^2(f) - 4| + |tr(fgf^{-1}g^{-1}) - 2| \leq 1\}$$

Then

$$Pr\{|tr^2(f) - 4| + |tr(fgf^{-1}g^{-1}) - 2| \leq 1\} \geq 0.0272821.$$

chosen in  ${}^*PSL(2, \mathbb{R})$ , we need calculate the probability of

$m(f)^2 m(g)^2 \leq 16(2 \cos(2\pi/7) - 1) = m$ . According to Theorem 1.3, we have

*Proof of Theorem 1.5* Suppose that  $f, g$  be randomly

$$\begin{aligned}
Pr\{m(f)^2 m(g)^2 \leq m\} &= \int_0^{\frac{m}{8}} \int_1^{\frac{t}{8}+1} \int_1^2 \frac{1}{4\pi^4} \frac{1}{\frac{t}{8}+1} \frac{\arcsin \sqrt{x-1}}{\sqrt{x(\frac{m}{8t}+1-x)}} \sqrt{\frac{y}{(\frac{t}{8}+1-y)(2-y)(y-1)}} dx dy dt \\
&\quad + \int_0^{\frac{m}{8}} \int_1^{\frac{t}{8}+1} \frac{1}{8\pi^2} \frac{1}{\frac{t}{8}+1} \left(1 - \frac{2}{\pi} \arcsin \sqrt{\frac{2}{\frac{m}{8t}+1}}\right) \sqrt{\frac{x}{(\frac{t}{8}+1-x)(2-x)(x-1)}} dx dt \\
&\quad + \int_{\frac{m}{8}}^8 \int_1^{\frac{t}{8}+1} \int_1^{\frac{m}{8t}+1} \frac{1}{4\pi^4} \frac{1}{\frac{t}{8}+1} \frac{\arcsin \sqrt{x-1}}{\sqrt{x(\frac{m}{8t}+1-x)}} \sqrt{\frac{y}{(\frac{t}{8}+1-y)(2-y)(y-1)}} dx dy dt \\
&\quad + \int_8^\infty \int_1^{\frac{m}{8t}+1} \int_1^2 \frac{1}{4\pi^4} \frac{1}{\frac{t}{8}+1} \frac{\arcsin \sqrt{y-1}}{\sqrt{y(\frac{1}{8t}+1-y)}} \sqrt{\frac{x}{(\frac{t}{8}+1-x)(2-x)(x-1)}} dx dy dt \\
&= 0.00222416 + 0.00805606 + 0.0155805 + 0.00434425 = 0.0302049.
\end{aligned}$$

Using (28), we have the probability  $\langle f, g \rangle$  is non-discrete or elementary is greater than 0.0302049.

According theorem 1.5, we further have the following corollaries.

*Corollary 3.8* Let  $f, g$  be random chosen elements in  ${}^*PSL(2, \mathbb{R})$  and  $f$  or  $g$  is parabolic. Then the probability  $\langle f, g \rangle$  is non-discrete or elementary is greater than 0.0881294.

*Corollary 3.9* Let  $f, g$  be random chosen elements in  ${}^*PSL(2, \mathbb{R})$  and  $f, g$  are both elliptic. Then the probability  $\langle f, g \rangle$  is non-discrete or elementary is greater than 0.0756016.

*Corollary 3.10* Let  $f, g$  be random chosen elements in  ${}^*PSL(2, \mathbb{R})$  and  $f, g$  are both hyperbolic. Then the probability  $\langle f, g \rangle$  is non-discrete or elementary is greater than

0.0329061.

*Corollary 3.11* Let  $f, g$  be random chosen elements in  ${}^*PSL(2, \mathbb{R})$  and  $f$  is hyperbolic and  $g$  is elliptic,. Then the probability  $\langle f, g \rangle$  is non-discrete or elementary is greater than 0.0302049.

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