



Discrete Probability of Random Möbius Groups: Random Subgroups by Two Generators

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Abstract: Let $GL(2, \mathbb{R})$ be real Möbius groups. For any 2×2 matrix A in $GL(2, \mathbb{R})$ induces real Möbius transformations g by the formula $A \rightarrow g_A = g$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, g = \frac{ax+b}{cx+d}$. The collection of all real Möbius transformations for which $ad - bc = 1$ takes the values 1 forms a group which can be identified with $PSL(2, \mathbb{R})$. We write $f \in {}^*PSL(2, \mathbb{R})$ to mean that f is a random variable in $PSL(2, \mathbb{R})$. In this paper, we study the random Möbius subgroup ${}^*PSL(2, \mathbb{R})$. We can get some new results: (1) If $f, g \in {}^*PSL(2, \mathbb{R})$, the probability $|tr^2(f) - 4| + |tr(fgf^{-1}g^{-1}) - 2| \leq 1$ is greater than 0.027282. (2) If f is hyperbolic transformation and g is parabolic transformation, then the probability $\langle f, g \rangle$ is discrete is greater than $7/12$. (3) If f is elliptic of order n and g is elliptic of order 2, then the probability $\langle f, g \rangle$ is discrete is greater than $2/n$. (4) The probability that random chosen $f, g \in {}^*PSL(2, \mathbb{R})$ generate an elementary or non-discrete group $\langle f, g \rangle$ is greater than 0.0302049.

Keywords: Random Discrete Möbius Group, ${}^*PSL(2, \mathbb{R})$, Jørgensen's Inequality

1. Introduction

Throughout this paper, we will adopt the same notations and definitions as [2, 13, 14, 15, 16] such as, discrete group, Fuchsian group, elementary group, ${}^*PSL(2, \mathbb{R})$ and so on. For example, if G is discrete and if X, A_1, A_2, \dots are in G with $A_n \rightarrow X$ then $A_n = X$ for all sufficiently large n . A group G of Möbius transformation is a Fuchsian group if and only if there is some G -invariant disc in which G acts discontinuously. A subgroup G of $PSL(2, \mathbb{C})$ is said to be elementary if and only if there exists a finite G -orbit in \mathbb{R}^3 . See [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 18, 19, 20] etc. for more details.

It is well known that Random group, introduced in the 1943 by Auerbach [1], has been extensively studied over the group of random symmetric group [4] and random finite group [11, 12]. G. J. Martin and G. O'Brien [13, 14]

introduce a geometrically natural probability measure on the group $PSL(2, \mathbb{R})$. They studied the probability that random Möbius subgroup $\langle f, g \rangle$ is discrete and given a lower bound for the probability a group generated by two random hyperbolic elements or parabolic elements of ${}^*PSL(2, \mathbb{R})$ is discrete.

Theorem MO1 [13, Theorem 11.6] The probability that two randomly chosen hyperbolic transformation $f, g \in {}^*PSL(2, \mathbb{R})$ generate a discrete group $\langle f, g \rangle$ is at least $2/5$.

Theorem MO2 [13, Theorem 10.3] The probability that two randomly chosen parabolic transformation $f, g \in {}^*PSL(2, \mathbb{R})$ generate a discrete group $\langle f, g \rangle$ is at least $1/6$.

We further study the probability a group generated by a hyperbolic element and a parabolic element of ${}^*PSL(2, \mathbb{R})$ is

discrete and obtain the following theorems.

Theorem 1.1 Let f, g be random chosen hyperbolic element and parabolic element in $*PSL(2, \mathbb{R})$. Then the probability $\langle f, g \rangle$ is discrete is at least $7/12$.

G. J. Martin and G. O’Brier studied the probability a group generated by a elliptic element of order n and a hyperbolic element of $*PSL(2, \mathbb{R})$ is discrete.

Theorem MO3 [13, Theorem 12.5] Let $f = \xi z$ and $\xi^n = 1$. Let $g \in *PSL(2, \mathbb{R})$ be a randomly chosen hyperbolic. Then the probability $\langle f, g \rangle$ is discrete is $2/n^2$.

We further study the probability a group generated by two elliptic elements of $*PSL(2, \mathbb{R})$ is discrete and obtain the following theorems.

Theorem 1.2 Let f, g be random chosen elements in $*PSL(2, \mathbb{R})$. If f is elliptic of order n and g is elliptic of order 2. Then the probability $\langle f, g \rangle$ is discrete is $2/n$.

G. J. Martin [15, 16] had results such as the p.d.f for trace, Commutators, Cross ratios. We will obtain the p.d.f for $m(f)$ and have the following theorem.

Theorem 1.3 Let f be randomly chosen element of $PSL(2, \mathbb{R})$. Then the random variable $m^2(f)$ has the p.d.f

$$F(s) = \begin{cases} F_1(s) & 0 < s \leq 8 \\ F_2(s) & s \geq 8 \end{cases}$$

Where,

$$F_1(s) = \frac{1}{8\pi^2} \frac{1}{\frac{s}{8} + 8} \int_1^{\frac{s}{8} + 1} \sqrt{\frac{y}{\left(\frac{s}{8} + 1 - y\right)(2 - y)(y - 1)}} dy$$

$$F_2(s) = \frac{1}{8\pi^2} \frac{1}{\frac{s}{8} + 1} \int_1^2 \sqrt{\frac{y}{\left(\frac{s}{8} + 1 - y\right)(2 - y)(y - 1)}} dy$$

Finally, by using Theorem 1.3, we will obtain the following theorems.

Theorem 1.4 Let f, g be random chosen elements in $*PSL(2, \mathbb{R})$. Then the probability $|tr^2(f) - 4| + |tr(fgf^{-1}g^{-1}) - 2| \leq 1$ is greater than 0.0272821.

Theorem 1.5 Let f, g be random chosen elements in $*PSL(2, \mathbb{R})$. Then the probability $\langle f, g \rangle$ is non-discrete or elementary is greater than 0.0302049.

2. Preliminaries

2.1. Random Möbius Groups

If $A \in PSL(2, \mathbb{C})$ has the form

$$A = \pm \begin{pmatrix} a & c \\ \bar{c} & \bar{a} \end{pmatrix}, |a|^2 - |c|^2 = 1. \tag{1}$$

Then the group of all matrices satisfying (1) will be denoted \mathcal{F} . It is not difficult to construct an algebraic isomorphism $\mathcal{F} \cong PSL(2, \mathbb{R}) \cong Isom^+(\mathbb{H}^2)$.

The probability distribution. The probability space is $(\mathcal{F}, \mu_{\mathcal{F}})$, the space of matrices with the following imposed distribution of the entries of an element of \mathcal{F} .

- (i) $\xi = a/|a|$ and $\eta = c/|c|$ are chosen uniformly in the circle \mathbb{S} , with arclength measure.
- (ii) $t = |a| \geq 1$ is chosen so that

$$2 \arcsin(1/t) \in [0, \pi]$$

is uniformly distributed.

Lemma 2.1.1 [13, lemma 2.3] The random variable $|a| \in [1, \infty)$ has the p.d.f

$$F_{|a|}(x) = \frac{2}{\pi} \frac{1}{x\sqrt{x^2 - 1}}.$$

we write $arg(a) \in_{\mu} [0, 2\pi]_{\mathbb{R}}$ if $arg(a)$ uniformly distributed on $\mathbb{R} \bmod 2\pi$. The following lemma will be crucial.

Lemma 2.1.2 [13, lemma 2.4] If $arg(a), arg(b) \in_{\mu} [0, 2\pi]_{\mathbb{R}}$, then $arg(ab), arg(a/b) \in_{\mu} [0, 2\pi]_{\mathbb{R}}$. Hence $arg(a^k) = k arg(a) \in_{\mu} [0, 2\pi]_{\mathbb{R}}$ for $k \in \mathbb{Z}$.

For any

$$A = \begin{pmatrix} a & c \\ \bar{c} & \bar{a} \end{pmatrix} \in \mathcal{F} \tag{2}$$

$$|tr(A)| = |2Re(a)| = |2a \cdot \cos(\theta)|, \theta = arg(a). \tag{3}$$

Using the obvious symmetries, by lemma 2.1, we may calculate

$$Pr\{|tr(A)| > 2\} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \left(1 - \frac{2}{\pi} \theta\right) d\theta = \frac{1}{2} \tag{4}$$

Corollary 2.1 [13, corollary 2.4] Let $f \in \mathcal{F}$ be a Möbius transformation chosen random from the distribution described in (i) and (ii). Then the probability that f is hyperbolic is equal to $1/2$, f is elliptic is equal to $1/2$ and f is parabolic is equal to 0.

See [13, Section 2] for more details.

2.2. Jørgensen’s Inequality

In 1976, Jørgensen proved a necessary condition for a non-elementary two generator subgroup $PSL(2, \mathbb{C})$ to be discrete, which is called Jørgensen’s inequality.

Theorem J [21, lemma 1] If $f, g \in PSL(2, \mathbb{C})$ generate a discrete and nonelementary group $\langle f, g \rangle$. Then

$$|tr^2(f) - 4| + |tr(fgf^{-1}g^{-1}) - 2| \geq 1 \tag{5}$$

For each $f \in \mathcal{F}$, we let $m^2(f)$ denote the norm

$$m^2(f) = \|A - A^{-1}\|^2 = 2|a - \bar{a}|^2 + 8|c|^2 \tag{6}$$

We have the following theorem from [9, Theorem 2.7].

Theorem FG₁ If f and g in $PSL(2, \mathbb{R})$. Then

$$|tr^2(f) - 4| \leq \frac{1}{2}m(f)^2 \tag{7}$$

$$|tr(fgf^{-1}g^{-1}) - 2| \leq \frac{1}{16}m(f)^2m(g)^2. \tag{8}$$

Both of these inequalities are sharp.

According to (7) and (8), we have

$$|tr^2(f) - 4| + |tr(fgf^{-1}g^{-1}) - 2| \leq m(f)^2 \left(\frac{1}{2} + \frac{1}{16}m(g)^2 \right) \tag{9}$$

2.3. An experiment on $m^2(f)m^2(g)$

If we select two random Möbius transformations, say,

$$f = \begin{pmatrix} e^{i\phi_1} \csc(\theta_1) & e^{i\alpha_1} \cot(\theta_1) \\ e^{-i\alpha_1} \cot(\theta_1) & e^{-i\phi_1} \csc(\theta_1) \end{pmatrix}, \tag{10}$$

$$g = \begin{pmatrix} e^{i\phi_2} \csc(\theta_2) & e^{i\alpha_2} \cot(\theta_2) \\ e^{-i\alpha_2} \cot(\theta_2) & e^{-i\phi_2} \csc(\theta_2) \end{pmatrix}, \tag{11}$$

where $\theta_1, \theta_2 \in_{\mu} [0, \pi/2]_{\mathbb{R}}$ and $\alpha_1, \alpha_2, \phi_1, \phi_2 \in_{\mu} [0, 2\pi]_{\mathbb{R}}$.

Then

$$m^2(f)m^2(g) = 64 \left(\frac{\sin^2(\phi_1)}{\sin^2(\theta_1)} + \cot^2(\theta_1) \right) \left(\frac{\sin^2(\phi_2)}{\sin^2(\theta_2)} + \cot^2(\theta_2) \right) \tag{12}$$

We made several independent runs through about 10^7 random matrix pairs of elements to generate the histogram in Figure 1. We found the probability that

$$m^2(f)m^2(g) \geq 16 \left(2 \cos\left(\frac{2\pi}{7}\right) - 1 \right) \tag{13}$$

to be about 0.0308797.

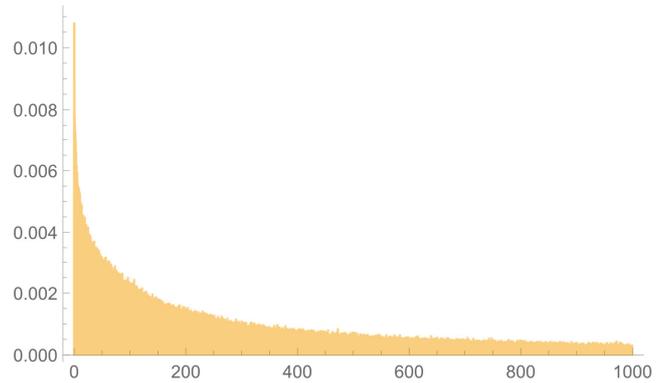


Figure 1. Histogram of $m^2(f)m^2(g)$ values.

We also found

$$\Pr \left\{ m(f)^2 \left(\frac{1}{2} + \frac{1}{16}m(g)^2 \right) \leq 1 \right\} \approx 0.027282. \tag{14}$$

According to Theorem J and Lemma 2.3, we have

$$\Pr \left\{ |tr^2(f) - 4| + |tr(fgf^{-1}g^{-1}) - 2| \leq 1 \right\} \geq 0.027282 \tag{15}$$

We have the following theorem from [9, Theorem 4.19].

Theorem FG₂ Suppose that $\langle f, g \rangle$ is nonelementary discrete Fuchsian group subgroup of \mathbb{M} . Then

$$m^2(f)m^2(g) \geq 16(2 \cos(2\pi/7) - 1). \tag{16}$$

In addition, if f or g is parabolic, then

$$m^2(f)m^2(g) \geq 16. \tag{17}$$

If f and g are both elliptic, then

$$m^2(f)m^2(g) \geq 16 \cos^2(\pi/7). \tag{18}$$

If f and g are both hyperbolic, then

$$m^2(f)m^2(g) \geq 16(\cos(2\pi/7) + \cos(\pi/7) - 1)^2 \tag{19}$$

Finally, if f is hyperbolic and g is elliptic, then

$$m^2(f)m^2(g) \geq 16(\cos(2\pi/7) - 1) \tag{20}$$

Each of these inequalities is sharp.

Another discrete criterion can be found in [9, 16].

Lemma 2.3.1 [10, Lemma 4.2] Suppose that $\langle f, g \rangle$ has parameters $(\gamma, \beta, -4)$ with $\gamma \geq \beta$ and that f is a primitive elliptic of order $q \geq 3$. Then $\langle f, g \rangle$ is a discrete group if and only if $\gamma \geq \beta + 4$.

Lemma 2.3.2 [15, Theorem VII. A.13.] Let $f_i, i = 1, 2, \dots, n$, be hyperbolic transformations of the disk whose isometric disks are all disjoint. Then the group generated by these

hyperbolic transformations $\langle f_1, f_2, \dots, f_n \rangle$ is discrete.

3. Proofs of the Main Results

Proof of Theorem 1.1 Suppose that the arc $\alpha_i = \alpha(m_{\alpha_i}, l_{\alpha_i})$ $i = 1, 2$, of f and the arc $\beta = \beta(m_{\beta_i}, l_{\beta_i})$ of g . Since f is hyperbolic, we have

$$\alpha_1 \cap \alpha_2 = \emptyset. \tag{21}$$

Thus

$$\alpha_1 \cap \beta = \emptyset, \alpha_2 \cap \beta = \emptyset \tag{22}$$

are independent. A little trigonometry reveals that

$$\alpha_i \cap \beta = \emptyset \Leftrightarrow \arg(m_{\alpha_i}, \bar{m}_{\beta}) \geq \frac{l_{\alpha_i} + l_{\beta}}{2} \tag{23}$$

Now the variables $\theta_i = \arg(m_{\alpha_i}, \bar{m}_{\beta}), i = 1, 2$, are uniformly distributed in $[0, \pi]$ and independent.

According lemma 2.1.2, we have

$$\begin{aligned} \Pr \left\{ \min \theta_i \geq \frac{l_{\alpha_i} + l_{\beta}}{2} \right\} &= \Pr \left\{ \min \theta_i - \frac{l_{\beta}}{2} \geq \frac{l_{\alpha_i}}{2} \right\} \\ &= \left(\frac{1}{\pi} \int_{\frac{l_{\alpha_i}}{2}}^{\pi} d\theta \right)^2 = \left(1 - \frac{1}{\pi} \frac{l_{\alpha_i}}{2} \right)^2. \end{aligned}$$

Let $\frac{l_{\alpha_i}}{2} = \varphi$, then $\varphi \in \left[0, \frac{\pi}{2} \right]$, thus

$$\Pr \left\{ \min \theta_i \geq \frac{l_{\alpha_i} + l_{\beta}}{2} \right\} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \left(1 - \frac{\varphi}{\pi} \right)^2 d\varphi = \frac{7}{12}.$$

Using lemma 2.3.2, the probability $\langle f, g \rangle$ is discrete is at least $7/12$.

Proof of Theorem 1.2 With loss of generality, f and g can be represented that

$$f = \begin{pmatrix} e^{i\frac{\pi}{n}} & 0 \\ 0 & e^{-i\frac{\pi}{n}} \end{pmatrix}, g = \begin{pmatrix} i \csc(\theta) & e^{i\varphi} \cot(\theta) \\ e^{-i\varphi} \cot(\theta) & -i \csc(\theta) \end{pmatrix} \tag{24}$$

By calculation

$$\beta = -4 \sin^2 \left(\frac{\pi}{n} \right), \gamma = 4 \sin^2 \left(\frac{\pi}{n} \right) \cot^2(\theta)$$

Using lemma 2.3.1, $\langle f, g \rangle$ is a discrete group if and only if $\gamma \geq \beta + 4$.

$$\begin{aligned} \Pr \{ \gamma \geq \beta + 4 \} &= \Pr \left\{ 4 \sin^2 \left(\frac{\pi}{n} \right) \cot^2(\theta) \geq 4 \cos^2 \left(\frac{\pi}{n} \right) \right\} \\ &= \Pr \left\{ \theta \leq \frac{\pi}{n} \right\} = \frac{2}{\pi} \int_0^{\frac{\pi}{n}} d\theta = \frac{2}{n}. \end{aligned}$$

Then the probability $\langle f, g \rangle$ is discrete is $\frac{2}{n}$.

Proof of Theorem 1.3 We select a random Möbius transformation

$$f = \begin{pmatrix} e^{i\phi} \csc(\theta) & e^{i\varphi} \cot(\theta) \\ e^{-i\varphi} \cot(\theta) & e^{-i\phi} \csc(\theta) \end{pmatrix}, \theta \in_{\mu} [0, \frac{\pi}{2}], \phi, \varphi \in_{\mu} [0, 2\pi]$$

Then

$$m^2(f) = 8 \frac{\sin^2(\phi)}{\sin^2(\theta)} + 8 \cot^2(\theta)$$

The probability distribution functions of $x = \sin^2(\theta)$ and $y = \sin^2(\phi)$ are independent and identically distributed $\Psi(x)$ and $\Psi(y)$,

$$\Psi(x) = \frac{1}{\pi \sqrt{x(1-x)}}$$

The random variable has the c.d.f

$$P(S \leq s) = \frac{1}{\pi^2} \iint_{\left\{ \frac{x+1-y}{y} \leq s \right\}} \frac{1}{\sqrt{x(1-x)}} \frac{1}{\sqrt{y(1-y)}} dx dy$$

We begin this calculation under the assumption $0 \leq s \leq 8$.

$$\begin{aligned} P_1(s) &= \frac{1}{\pi^2} \int_{\frac{s}{8}+1}^1 \int_0^{(\frac{s}{8}+1)y-1} \frac{1}{\sqrt{x(1-x)}} \frac{1}{\sqrt{y(1-y)}} dx dy \\ &= \frac{2}{\pi^2} \int_{\frac{s}{8}+1}^1 \frac{1}{\sqrt{y(1-y)}} \cdot \arcsin \left[\sqrt{\left(\frac{s}{8}+1 \right) y - 1} \right] dy \tag{25} \\ &= \frac{2}{\pi^2} \int_1^{\frac{s}{8}+1} \frac{\arcsin \sqrt{y-1}}{\sqrt{y \left(\frac{s}{8}+1-y \right)}} dy \end{aligned}$$

We can now differentiate (26) of s under the integral.

$$\begin{aligned} F_1(s) &= \frac{1}{8\pi^2} \int_{\frac{s}{8}+1}^1 \sqrt{\frac{y}{(1-y)[2 - (\frac{s}{8}+1)y][(\frac{s}{8}+1)y-1]}} dy \\ &= \frac{1}{8\pi^2} \frac{1}{\frac{s}{8}+1} \int_1^{\frac{s}{8}+1} \sqrt{\frac{y}{(\frac{s}{8}+1-y)(2-y)(y-1)}} dy \tag{26} \end{aligned}$$

Similarity, suppose that $s \geq 8$, we can calculate

$$P_2(s) = \frac{2}{\pi^2} \int_1^2 \frac{\arcsin \sqrt{y-1}}{\sqrt{y(\frac{s}{8}+1-y)}} dy + 1 - \frac{2}{\pi} \cdot \arcsin \left[\sqrt{\frac{2}{\frac{s}{8}+1}} \right]$$

and

$$F_2(s) = \frac{1}{8\pi^2} \frac{1}{\frac{s}{8}+1} \int_1^2 \sqrt{\frac{y}{(\frac{s}{8}+1-y)(2-y)(y-1)}} dy$$

The proof is completed.

A discussion similar to the proof of Theorem 1.3, we obtain some corollaries.

Corollary 3.1 Let f be randomly chosen parabolic element of ${}^*PSL(2, \mathbb{R})$, then the random variable $m^2(f)$ has the p.d.f

$$H(s) = \frac{4}{\pi} \frac{1}{\sqrt{s(16+s)}}$$

Corollary 3.2 Let f be randomly chosen hyperbolic element of ${}^*PSL(2, \mathbb{R})$, then the random variable $m^2(f)$ has the p.d.f

$$G(s) = \frac{1}{4\pi^2} \frac{1}{\frac{s}{8}+1} \int_1^{\frac{\frac{s}{4}+2}{\frac{s}{8}+1}} \sqrt{\frac{y}{(\frac{s}{8}+1-y)(2-y)(y-1)}} dy$$

Corollary 3.3 Let f be randomly chosen elliptic element of ${}^*PSL(2, \mathbb{R})$, then the random variable $m^2(f)$ has the p.d.f

$$T(s) = \begin{cases} T_1(s) & 0 < s \leq 8 \\ T_2(s) & s \geq 8 \end{cases}$$

Where,

$$T_1(s) = \frac{1}{4\pi^2} \frac{1}{\frac{s}{8}+1} \int_{\frac{\frac{s}{4}+2}{\frac{s}{8}+1}}^{\frac{s+1}{8}} \sqrt{\frac{y}{(\frac{s}{8}+1-y)(2-y)(y-1)}} dy$$

$$T_2(s) = \frac{1}{4\pi^2} \frac{1}{\frac{s}{8}+1} \int_{\frac{\frac{s}{4}+2}{\frac{s}{8}+1}}^2 \sqrt{\frac{y}{(\frac{s}{8}+1-y)(2-y)(y-1)}} dy$$

Corollary 3.4 Let f be randomly chosen element of ${}^*PSL(2, \mathbb{R})$. Then the random variable $\frac{1}{2} + \frac{1}{16} m^2(f)$ has the p.d.f

$$F(t) = \begin{cases} F_1(t) & \frac{1}{2} < t \leq 1 \\ F_2(t) & t \geq 1 \end{cases}$$

Where,

$$F_1(t) = \frac{1}{\pi^2 t} \int_1^{2t} \sqrt{\frac{y}{(2t-y)(2-y)(y-1)}} dy,$$

$$F_2(t) = \frac{1}{\pi^2 t} \int_1^2 \sqrt{\frac{y}{(2t-y)(2-y)(y-1)}} dy$$

Corollary 3.5 Let f be randomly chosen parabolic element of ${}^*PSL(2, \mathbb{R})$. Then the random variable $\frac{1}{2} + \frac{1}{16} m^2(f)$ has the p.d.f

$$H(t) = \frac{1}{\pi} \cdot \frac{1}{(t + \frac{1}{2}) \sqrt{t - \frac{1}{2}}}$$

Corollary 3.6 Let f be randomly chosen hyperbolic element of ${}^*PSL(2, \mathbb{R})$. Then the random variable $\frac{1}{2} + \frac{1}{16} m^2(f)$ has the p.d.f

$$G(t) = \frac{2}{\pi^2 t} \int_1^{\frac{4t}{2t+1}} \sqrt{\frac{y}{(2t-y)(2-y)(y-1)}} dy.$$

Corollary 3.7 Let f be randomly chosen elliptic element of ${}^*PSL(2, \mathbb{R})$. Then the random variable $\frac{1}{2} + \frac{1}{16} m^2(f)$ has the p.d.f

$$T(t) = \begin{cases} T_1(t) & \frac{1}{2} < t \leq 1 \\ T_2(t) & t \geq 1 \end{cases}$$

Where,

$$T_1(t) = \frac{1}{2\pi^2 t} \int_{\frac{4t}{2t+1}}^{2t} \sqrt{\frac{y}{(2t-y)(2-y)(y-1)}} dy,$$

$$T_2(t) = \frac{1}{2\pi^2 t} \int_{\frac{4t}{2t+1}}^2 \sqrt{\frac{y}{(2t-y)(2-y)(y-1)}} dy.$$

Proof of Theorem 1.4 Suppose that f, g be randomly chosen in ${}^*PSL(2, \mathbb{R})$, we need calculate the probability of $m(f)^2(\frac{1}{2} + \frac{1}{16} m(g)^2) \leq 1$. According to Theorem 1.3 and Corollary 3.5, we have

$$\begin{aligned}
 Pr\left\{m(f)^2\left(\frac{1}{2} + \frac{1}{16}m(g)^2\right) \leq 1\right\} &= \iint_{\{s,t \leq 1\}} F(t) \cdot F(s) ds dt \\
 &= \int_{\frac{1}{2}}^1 \int_1^{2t} \int_1^{\frac{1}{8t}+1} \frac{2}{\pi^4 t} \frac{\arcsin\sqrt{y-1}}{\sqrt{y\left(\frac{1}{8t}+1-y\right)}} \sqrt{\frac{x}{(2t-x)(2-x)(x-1)}} dy dx dt \\
 &\quad + \int_1^\infty \int_1^2 \int_1^{\frac{1}{8t}+1} \frac{2}{\pi^4 t} \frac{\arcsin\sqrt{y-1}}{\sqrt{y\left(\frac{1}{8t}+1-y\right)}} \sqrt{\frac{x}{(2t-x)(2-x)(x-1)}} dy dx dt \\
 &= 0.0161113 + 0.0111708 = 0.0272821.
 \end{aligned}$$

Using (27), we have

$$Pr\left\{m(f)^2\left(\frac{1}{2} + \frac{1}{16}m(g)^2\right) \leq 1\right\} \leq Pr\{|tr^2(f) - 4| + |tr(fgf^{-1}g^{-1}) - 2| \leq 1\}$$

Then

$$Pr\{|tr^2(f) - 4| + |tr(fgf^{-1}g^{-1}) - 2| \leq 1\} \geq 0.0272821.$$

chosen in ${}^*PSL(2, \mathbb{R})$, we need calculate the probability of

$m(f)^2 m(g)^2 \leq 16(2 \cos(2\pi/7) - 1) = m$. According to Theorem 1.3, we have

Proof of Theorem 1.5 Suppose that f, g be randomly

$$\begin{aligned}
 &Pr\{m(f)^2 m(g)^2 \leq m\} \\
 &= \int_0^{\frac{m}{8}} \int_1^{\frac{t}{8}+1} \int_1^2 \frac{1}{4\pi^4} \frac{1}{\frac{t}{8}+1} \frac{\arcsin\sqrt{x-1}}{\sqrt{x\left(\frac{m}{8t}+1-x\right)}} \sqrt{\frac{y}{\left(\frac{t}{8}+1-y\right)(2-y)(y-1)}} dx dy dt \\
 &\quad + \int_0^{\frac{m}{8}} \int_1^{\frac{t}{8}+1} \frac{1}{8\pi^2} \frac{1}{\frac{t}{8}+1} \left(1 - \frac{2}{\pi} \arcsin \sqrt{\frac{2}{\frac{m}{8t}+1}}\right) \sqrt{\frac{x}{\left(\frac{t}{8}+1-x\right)(2-x)(x-1)}} dx dt \\
 &\quad + \int_{\frac{m}{8}}^8 \int_1^{\frac{t}{8}+1} \int_1^{\frac{m}{8t}+1} \frac{1}{4\pi^4} \frac{1}{\frac{t}{8}+1} \frac{\arcsin\sqrt{x-1}}{\sqrt{x\left(\frac{m}{8t}+1-x\right)}} \sqrt{\frac{y}{\left(\frac{t}{8}+1-y\right)(2-y)(y-1)}} dx dy dt \\
 &\quad + \int_8^\infty \int_1^{\frac{m}{8t}+1} \int_1^2 \frac{1}{4\pi^4} \frac{1}{\frac{t}{8}+1} \frac{\arcsin\sqrt{y-1}}{\sqrt{y\left(\frac{1}{8t}+1-y\right)}} \sqrt{\frac{x}{\left(\frac{t}{8}+1-x\right)(2-x)(x-1)}} dx dy dt \\
 &= 0.00222416 + 0.00805606 + 0.0155805 + 0.00434425 = 0.0302049.
 \end{aligned}$$

Using (28), we have the probability $\langle f, g \rangle$ is non-discrete or elementary is greater than 0.0302049.

According theorem 1.5, we further have the following corollaries.

Corollary 3.8 Let f, g be random chosen elements in ${}^*PSL(2, \mathbb{R})$ and f or g is parabolic. Then the probability $\langle f, g \rangle$ is non-discrete or elementary is greater than 0.0881294.

Corollary 3.9 Let f, g be random chosen elements in ${}^*PSL(2, \mathbb{R})$ and f, g are both elliptic. Then the probability $\langle f, g \rangle$ is non-discrete or elementary is greater than 0.0756016.

Corollary 3.10 Let f, g be random chosen elements in ${}^*PSL(2, \mathbb{R})$ and f, g are both hyperbolic. Then the probability $\langle f, g \rangle$ is non-discrete or elementary is greater than

0.0329061.

Corollary 3.11 Let f, g be random chosen elements in ${}^*PSL(2, \mathbb{R})$ and f is hyperbolic and g is elliptic. Then the probability $\langle f, g \rangle$ is non-discrete or elementary is greater than 0.0302049.

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