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Stochastic Stability and Optimal Control Analysis for a Tobacco Smoking Model

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Abstract: In this paper, a smoking model, which takes snuffing class and Brownian motion into consideration and is thus an extension of previously studied deterministic smoking models. We analytically show that this extended model system has one and only one positively bounded solution for any nonnegative initial values for the state variables. Interestingly, we find that the model system can exhibit sharp threshold characteristics whatever values of the basic reproductive number. By analyzing persistence, extinction and stationary distribution, we also find that the stochastic system is ergodic only when the coefficients of the noise terms are small. To eliminate gradually the infection out of the community, we introduce a stochastic system of two control variables and perform analysis, with results that can provide guidelines for tobacco control department. Results obtained by theoretical analysis are verified by numerical simulations.

Keywords: Stochastic Tobacco Smoking Model, *Itô* Formula, Extinction, Persistence, Stationary Distribution, Stochastic Optimal Control, Numerical Simulation

1. Introduction

Mathematical biology, particularly mathematical epidemiology, is a field of great concern for both mathematicians and biologists in the current century. In this field, many researchers are interested in describing the dynamics of infectious diseases and their control aspects in terms of mathematical language. Indeed, it was Brownlee [1] who (for the first) stand for the development of mathematical biology and provide a concrete base for the subject. The author used a probabilistic approach to study contagions within three years, he stated a law about the spread of infection [2]. A detailed mathematical touch to the subject was given in the work of Kermark and McKendrik [3]. Following the approach of Kermark-McKendrik, different types of infection models were proposed and analyzed in a more sophisticated way [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Similar to infectious diseases, smoking may be defined as the process by which an individual inhales the smoke of tobacco, or the process in which tobacco smoke is first taken by the mouth through cigars or pipes and then discharged from it. The habit of smoking was initially spread into Europe with the entrance of Columbus in the 16th century [15]. However, after and before this habit, other species of strange nature had adversely affected the human habitat and the whole ecosystem. It was Nicot who widely spread the use of tobacco in England as money yield and promoted it as a business (owing to this connection, nicotine was named after him). Since the cigarette manufacturing equipment was invented, the production speed for smoke was initially 200 units per minute in the late nineteen century and is currently 9000 units per minute. Smoking is a very dangerous social habit occurring almost in each part of the world. It not only can kill many persons but may also cause some severe diseases such as mouth, lung

and throat cancers. In addition, it would play an active role in spreading other relevant diseases dangerous for humanity [16, 17, 18].Studying the harm of smoking and its spread is significant.

From last three decades, researchers (almost from all scientific fields) are trying their level best to overcome the onset of various epidemics. Mathematicians have also been putting their efforts in curbing these diseases including other emerging infectious diseases by studying the spreading pattern and control alternatives related to the infection [19, 20, 21, 22]. In fact, studying the dynamics of various infections, mathematical modelling is a crucial tool as well as it can provide helps in the selection of best possible control strategies. In the case of deterministic modeling, numerous researchers have analyzed the stability and optimal control strategies of infectious disease model systems including those that used non-linear incidence functions [23, 24, 25, 26, 27]. For stochastic systems, many authors have performed the stability analysis for different epidemic model systems including those using nonlinear incidence functions [28, 29, 30, 31, 32, 33]. For stochastic epidemic models, however, there are rare literatures for optimal control theory [34, 35, 36]. In this work, we intend to elaborate the exponential stability of the proposed stochastic model and to present control strategies related to the quitting of smoking by using both stochastic and deterministic approaches. In the deterministic control formulation, the primary objective is to characterize the spreading rate that decreases the size of smoker population, by utilizing the minimum cost on the entire control program. In the stochastic control formulation, the main goal is to minimize the expected value. In order to achieve the desired goals, we will adopt the standard Pontryagin's principle for the deterministic model but the Hamilton Jacobi-Bellman equation for the stochastic model.

As a matter of fact in stability theory, one can improve the stability by incorporating the noise into a system of interest [37, 38, 39, 40]. Many authors studied stochastic epidemic models until the relevant stability theory became formally perfect [29, 30, 38, 39, 40]. Further, by comparing both deterministic and stochastic epidemic models, one can see that the latter can exhibit more degrees of freedom than the former, thus more approaching to the reality. Stochastic models with the Brownian motions have rich dynamical behaviors and the readers are suggested [37, 39]. By introducing a threshold parameter for a smoking model that considers standard Brownian effects, we show when the smoking behavior will persist and extinct across a population. In this analysis process, it is extremely important to look into the relationship of this threshold parameter with other known stochastic perturbations, including that of its variation with Brownian motions [34, 35]. In recent years, the tobacco use appears in the snuffing form. Thus, this factor of snuffing should be considered in smoking analysis when one models the smoking habit [41, 42]. Considering a snuffing population, we will stratify the entire community into five disjoint groups, namely, susceptible, snuffing population, casual smokers, chain and quit smokers.

The rest of this manuscript is arranged as follows. Sect. 2, is devoted to the formulation of a smoking model, including the biological feasibility of smoking problems. In Sect. 3, we prove the nonnegativity as well as the boundedness of the solutions to the model subject to any positive initial data. The extinction of the disease in terms of probabilities is presented and explained in Sect. 4. In Sect. 5, we study the persistence of the model system in mean of the epidemic. Section 6, discusses the necessary conditions for the existence of a stationary distribution. In Sect. 7, we investigated the existence of an optimal solution to the proposed control problem. Section 8, provides numerical examples to explain the empirical findings. The discussion is summarized in Sect. 9. Finally, in Sect. 10, we conclude the paper with a conclusion and give some simple discussion.

2. Model Formulation

Smoking is dangerous to health has been a universal truth, it doesn't only destroy the smokers health but also the passive smokers, and hence can be a threat to the entire society. World Health Organization (WHO) [42] indicates that 6 millions individual dies each year due to smoking and in a few decades. It is estimated that above 50 million deaths are due to the direct use of tobacco and second-hand smoking is responsible for nearly 0.6 million deaths. It is also worthy to mention that among one billion smokers around the globe, approximately 80% are from middle and low-income countries. In this work, a stochastic mathematical model is introduced to describes the smoking model dynamics. By considering snuffing individual we proposed a stochastic smoking model with random perturbation by parameter perturbation, we will stratify the entire community into five disjoint groups, namely; $V(t)$ (the susceptible), snuffing class $W(t)$, occasional smokers $X(t)$, chain smokers $Y(t)$ and quit smokers $Z(t)$ at time t. Assuming that the random effect is proportional to $(V(t), W(t), X(t), Y(t), Z(t))$, and total population can be expressed as

$$N = V + W + X + Y + Z. \tag{1}$$

It is revealed that the effects of environmental factors play a considerable role in the transmission dynamics of smoking dynamics. Therefore, the proposed smoking model taking into consideration the following assumptions:

A_1 : Homogeneous mixing which means that the population under consideration is well-mixed and each individual is equally-likely to mix with each other.

A_2 : The model's state variables and parameters are all non-negative.

A_3 : We assume that per unit time, the mean size of contacts is c.

A_4 : A portion of the current smokers try to quit smoking.

A_5 : The smokers quit smoking on temporal basis and thus could be susceptible again to smoking.

A_6 : In order to take the environmental noise into

consideration in the model, we set $B_i(t)$ for $i = 1, \cdots, 5$ (with $B_i(0) = 0 \; \forall i$) which will be the standard Brownian motion and $\alpha_i (i = 1, \cdots, 5)$ represent the Gauss noise intensity of the environment.

Assuming assumptions $(A_1 - A_6)$ and random effects are proportional to susceptible $V(t)$, snuffing population $W(t)$, casual smokers $X(t)$, chain $Y(t)$ and quit smokers $Z(t)$. Thus, we obtained a stochastic model like the one below which governing the dynamics of smoking habit

$$\begin{aligned}
dV(t) &= \left[\Pi - \frac{\beta V(t)W(t)}{N} - dV(t) + \lambda Z(t)\right]dt + \alpha_1 V(t)dB_1(t) \\
dW(t) &= \left[\frac{\beta V(t)W(t)}{N} - \frac{\delta W(t)X(t)}{N} - (\gamma + d)W(t)\right]dt + \alpha_2 W(t)dB_2(t) \\
dX(t) &= \left[\frac{\delta W(t)X(t)}{N} - (\mu + \omega + d)X(t)\right]dt + \alpha_3 X(t)dB_3(t) \\
dY(t) &= \left[\omega X(t) - (\kappa + d)Y(t)\right]dt + \alpha_4 Y(t)dB_4(t) \\
dZ(t) &= \left[\kappa Y(t) - (\lambda + d)Z(t)\right]dt + \alpha_5 Z(t)dB_5(t)
\end{aligned} \tag{2}$$

The biological interpretation of the model's parameters are mentioned in Table 1.

Where $B_i(t)$ for $i = 1 \cdots 5$ stand for the independent Brownian motions and α_1, $\alpha_2, \alpha_3, \alpha_4$, and α_5 are the intensities of the white noises. $\alpha_1 V dB_1(t), \alpha_2 W dB_1(t), \alpha_3 X dB_3(t), \alpha_4 Y dB_4(t)$ and $\alpha_5 Z dB_5(t)$ are used to the model for interaction between the indiyiduals and the environment.

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets.

Table 1. *Parameters description.*

Symbols	Description
Π	Inflow rate (either through migration or by birth)
β	The rate at which a susceptible person begins snuffing
λ	Relapse rate
γ	Tobacco related death rate in snuffing compartment
d	Natural death rate
ω	The proportion of occasional smokers who become chain smokers
μ	Death due to tobacco related diseases
δ	Rate through which snuffing population become casual smokers
κ	Quitting rate

3. Existence of Positive Solution and Its Uniqueness

In the view of biology, the solution of the epidemic model should be globally positive. So we first prove that the solution of system (2) satisfies this property.

Theorem 3.1. For initial value $(V(0), W(0), X(0), Y(0), Z(0)) \in R_+^5$ and $t \geq 0$, the solution of system (2) will not only be unique, but also absolutely remain in R_+^5. namely $(V(t), W(t), X(t), Y(t), Z(t)) \in R_+^5$ for every $t \geq 0$.

Proof Obviously, the coefficient of model (2) are continuous locally Lipschitz for any $(V(0), W, (0)X(0), Y(0), Z(0)) \in R_+^5$, which mean that the solution is unique and local $(V(t), W, (t)X(t), Y(t), Z(t))$ in $t \in [0, \tau_e)$, where the elapsed time of the explosion is τ_e.

If we can prove $\tau_e = \infty$ that actually indicates the global nature of the solution. Let k_0 be a non-negative constant, and the interval should be $[k_0, \frac{1}{k_0}]$. Here define $k \geq k_0$ and

$$\tau_k = inf\left\{t \in [0, \tau_e) : min\{V(t), W(t), X(t), Y(t), Z(t)\} \leq \frac{1}{k} \text{ or } max\{V(t), W(t), X(t), Y(t), Z(t)\} \geq k\right\}. \tag{3}$$

We set $inf\phi = \infty$, as ϕ is the null set. According to the definition, τ_k is increasing as k approaches ∞. Assume that τ_∞ is the limiting value i.e., $\lim_{k\to\infty} \tau_\infty = \tau_\infty$ a.s. In another word, we need to show that $\infty = \tau_\infty$ a.s. If this assertion is false, then there exist a pair of constants $T > 0$ and ϵ in the interval $(0, 1)$, such that

$$P\{T \geq \tau_\infty\} > \epsilon. \tag{4}$$

So an integer $k_1 \geq k_0$ will be there, such that

$$P\{T \geq \tau_k\} \geq \epsilon, \text{ for all } k \geq k_1. \tag{5}$$

Define a C^2- function $H : R_+^5 \to R_+$, where $R_+ = \{x \in R : x \leq 0\}$, then

$$\begin{aligned} H(V, W, X, Y, Z) = (V - 1 - \log V) + (W - 1 - \log W) + (X - 1 - \log X) + (Y - 1 - \log Y) \\ + (Z - 1 - \log Z). \end{aligned} \tag{6}$$

Making use of the $It\hat{o}$ formula, we obtain

$$\begin{aligned} dH(V, W, X, Y, Z) = LH(V, W, X, Y, Z)dt + \alpha_1(V - 1)dB_1(t) + \alpha_2(W - 1)dB_2(t) \\ + \alpha_3(X - 1)dB_3(t) + \alpha_3(Y - 1)dB_4(t) + \alpha_3(Z - 1)dB_5(t), \end{aligned} \tag{7}$$

where

$$\begin{aligned} LH &= \left(1 - \frac{1}{V}\right)\left(\Pi - \frac{\beta VW}{N} - dV + \lambda Z\right) + \frac{1}{2}{\alpha_1}^2 + \left(1 - \frac{1}{W}\right)\left(\frac{\beta VW}{N} - \frac{\delta WX}{N} - (\gamma + d)W\right) \\ &+ \frac{1}{2}{\alpha_2}^2 + \left(1 - \frac{1}{X}\right)\left(\frac{\delta WX}{N} - (\mu + \omega + d)X\right) + \frac{1}{2}{\alpha_3}^2 + \left(1 - \frac{1}{Y}\right)\left(\omega X - (\kappa + d)Y\right) + \frac{1}{2}{\alpha_4}^2 \\ &+ \left(1 - \frac{1}{Z}\right)\left(\kappa Y - (d + \lambda)Z\right) + \frac{1}{2}{\alpha_5}^2 \\ &= \Pi - \frac{\beta VW}{N} - dV + \lambda Z - \frac{\Pi}{V} + \frac{\beta W}{N} + d - \frac{\lambda Z}{V} + \frac{\beta VW}{N} - \frac{\delta WX}{N} - (\gamma + d)W - \frac{\beta V}{N} + \frac{\delta X}{N} \\ &+ (\gamma + d) + \frac{\delta WX}{N} - (\mu + \omega + d)X - \frac{\delta W}{N} + (\mu + \omega + d) + \omega X - (\kappa + d)Y - \frac{\omega X}{Y} + (\kappa + d) \\ &+ \kappa Y - (\lambda + d)Z - \frac{\kappa Y}{Z} + (d + \lambda) + \frac{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2}{2} \\ &= \Pi - dV - \frac{\Pi}{V} + d + \frac{\beta W}{N} - \frac{\lambda Z}{V} - (\gamma + d)W - \frac{\beta V}{N} + \frac{\delta X}{N} + (\gamma + d) - (\mu + d)X - \frac{\delta W}{N} \\ &+ (\mu + \omega + d) - dY - \frac{\omega X}{Y} + (\kappa + d) - dZ - \frac{\kappa Y}{Z} + (d + \lambda) + \frac{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2}{2} \\ &\leq \Pi + \beta + d + \delta + (\gamma + d) + (\mu + \omega + d) + (\kappa + d) + (d + \lambda) + \frac{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2}{2} \\ &\leq \Pi + \beta + 5d + \delta + \gamma + \mu + \omega + \kappa + \lambda + \frac{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2}{2} := K. \end{aligned} \tag{8}$$

Since K is positive constant which is independent of (V, W, X, Y, Z) and t, we can get

$$\begin{aligned} dH = Kdt + \alpha_1(V - 1)dB_1(t) + \alpha_2(W - 1)dB_2(t) \\ + \alpha_3(X - 1)dB_3(t) + \alpha_3(Y - 1)dB_4(t) + \alpha_3(Z - 1)dB_5(t), \end{aligned} \tag{9}$$

Integrating both sides Eq. (9) from 0 to $T \wedge \tau_k$ and taking expectations, then we can obtain

$$\begin{aligned} EH\left[\left(V(\tau_k \wedge T), W(\tau_k \wedge T), X(\tau_k \wedge T), Y(\tau_k \wedge T), Z(\tau_k \wedge T)\right)\right] &\leq H(V(0), W(0), X(0), Y(0), Z(0)) + TK \\ &< \infty \end{aligned} \tag{10}$$

Putting $\Omega_k = \tau_k \leq T$, $k \geq k_1$. Also from Eq. (5), it could be noted that $P(\Omega_k) \geq \epsilon$. Let $\omega \in \Omega_k$, at least one $V(\tau_k, \omega)$, $W(\tau_k, \omega)$, $X(\tau_k, \omega)$, $Y(\tau_k, \omega)$, $Z(\tau_k, \omega)$, exists, which is equal to k or $\frac{1}{k}$.
So $H(V(\tau_k), W(\tau_k), X(\tau_k), Y(\tau_k), Z(\tau_k))$, more then $-\log k + k - 1$ or $-1 + \frac{1}{k} + \log k$. Moreover

$$H(V(\tau_k,\omega), W(\tau_k,\omega), X(\tau_k,\omega), Y(\tau_k,\omega), Z(\tau_k,\omega)) \geq E\big(k - 1 - \log\big) \wedge \left(\frac{1}{k} - 1 + \log k\right). \tag{11}$$

Following Eqs. (10) and (11), we obtain

$$\begin{aligned} H(V(0), W(0), X(0), Y(0), Z(0)) + TK &\geq E\left[1_{\Omega(\omega)} H\big(V(\tau_k), W(\tau_k), X(\tau_k), Y(\tau_k), Z(\tau_k)\big)\right] \\ &\geq \epsilon\left[\left(\frac{1}{k} - 1 + \log k\right) \wedge (k - 1 - \log k)\right], \end{aligned} \tag{12}$$

where the function $1_{\Omega(\omega)}$ is called the indicator function of Ω. Assuming $k \to \infty$ will leads to the contradiction

$$\infty > H(V(0), W(0), X(0), Y(0), Z(0)) + TK = \infty \tag{13}$$

Therefore, we must have $\tau_\infty = \infty$ a.s. The proof is complete.

4. Extinction of the Disease

This section is dedicated to the analysis of the extinction of the proposed system. For simplification, we define

$$\big\langle S(r) \big\rangle = \frac{1}{t} \int_0^t S(r) ds. \tag{14}$$

Lemma 4.1. (Strong Law)[29, 33] Let $\{M\}_{0 \leq t} = M$ be a continuous and real-valued along with Local martingale that vanishes with the limit t approaches 0, then

$$\begin{aligned} &\lim_{t\to\infty} \big\langle M, M \big\rangle_t = \infty, \text{ a.s., } \Rightarrow \lim_{t\to\infty} \frac{M_t}{\big\langle M, M \big\rangle_t} = 0, \text{ a.s.} \\ &\lim_{t\to\infty} \sup \frac{\big\langle M, M \big\rangle_t}{t} < 0, \text{ a.s., } \Rightarrow \lim_{t\to\infty} \frac{M_t}{t} = 0, \text{ a.s.} \end{aligned} \tag{15}$$

Lemma 4.2. Let $(V(t) + W(t) + X(t) + Y(t) + Z(t))$ (being functions of t) be the solution(s) of model (2) with $(V(0), W(0), X(0), Y(0), Z(0)) \in \mathbb{R}^5_+$, then $\limsup_{t\to\infty}(V(t) + W(t) + X(t) + Y(t) + Z(t)) < \infty$, a.s
Moreover

$$\lim_{t\to\infty} \frac{V(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{W(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{X(t)}{t} = 0 \text{ a.s}, \quad \lim_{t\to\infty} \frac{Y(t)}{t} = 0 \text{ a.s}, \quad \lim_{t\to\infty} \frac{Z(t)}{t} = 0 \text{ a.s}, \tag{16}$$

$$\lim_{t\to\infty} \frac{\ln V(t)}{t} = 0, \lim_{t\to\infty} \frac{\ln W(t)}{t} = 0, \lim_{t\to\infty} \frac{\ln X(t)}{t} = 0 \text{ a.s}, \lim_{t\to\infty} \frac{\ln Y(t)}{t} = 0 \text{ a.s}, \lim_{t\to\infty} \frac{\ln Z(t)}{t} = 0 \text{ a.s}, \tag{17}$$

and

$$\begin{aligned} &\lim_{t\to\infty} \frac{\int_0^t V(r) dB_1(r)}{t} = 0, \quad \lim_{t\to\infty} \frac{\int_0^t W(r) dB_2(r)}{t} = 0, \quad \lim_{t\to\infty} \frac{\int_0^t X(r) dB_3(r)}{t} = 0, \\ &\lim_{t\to\infty} \frac{\int_0^t Y(r) dB_4(r)}{t} = 0, \quad \lim_{t\to\infty} \frac{\int_0^t Z(r) dB_5(r)}{t} = 0, \quad \text{a.s} \end{aligned} \tag{18}$$

Proof From the system (2) we can have

$$\begin{aligned} d(V(t) + W(t) + X(t) + Y(t) + Z(t)) &= \Pi - d[V + W + X + Y + Z] - \gamma W(t) - \mu X(t) \\ &+ \alpha_1 V(t) dB_1(t) + \alpha_2 W(t) dB_2(t) + \alpha_3 X(t) dB_3(t) \\ &+ \alpha_4 Y(t) dB_4(t) + \alpha_5 Z(t) dB_5(t). \end{aligned} \tag{19}$$

Solving the Eq. (19), we can get

$$
\begin{aligned}
V(t)+W(t)+X(t)+Y(t)+Z(t) &= \frac{\Pi}{d}+\left(V(0)+W(0)+X(0)+Y(0)+Z(0)-\frac{\Pi}{d}\right)e^{-dt}\\
&\quad -\gamma\int_0^t W(r)e^{-d(t-u)}dt-\mu\int_0^t X(r)e^{-d(t-r)}dt\\
&\quad +\alpha_1\int_0^t V(r)e^{-d(t-r)}dB_1(r)+\alpha_2\int_0^t W(r)e^{-d(t-r)}dB_2(r)\\
&\quad +\alpha_3\int_0^t X(r)e^{-d(t-r)}dB_3(r)+\alpha_4\int_0^t Y(r)e^{-d(t-r)}dB_4(r)\\
&\quad +\alpha_5\int_0^t Z(r)e^{-d(t-r)}dB_5(r), \qquad (20)\\
&\leq \frac{\Pi}{d}+\left(V(0)+W(0)+X(0)+Y(0)+Z(0)-\frac{\Pi}{d}\right)e^{-dt}\\
&\quad +\alpha_1\int_0^t V(r)e^{-d(t-r)}dB_1(r)+\alpha_2\int_0^t W(r)e^{-d(t-r)}dB_2(r)\\
&\quad +\alpha_3\int_0^t X(r)e^{-d(t-r)}dB_3(r)+\alpha_4\int_0^t Y(r)e^{-d(t-r)}dB_4(r)\\
&\quad +\alpha_5\int_0^t Z(r)e^{-d(t-r)}dB_5(r).
\end{aligned}
$$

Now we define

$$S(t)=X(0)+B(t)+M(t)-G(t), \qquad (21)$$

where

$$
\begin{aligned}
S(0) &= V(0)+W(0)+X(0)+Y(0)+Z(0),\\
B(t) &= \frac{\Pi}{d}(1-e^{-dt}),\\
G(t) &= (V(0)+W(0)+X(0)+Y(0)+Z(0))(1-e^{-dt}),\\
M(t) &= \alpha_1\int_0^t V(r)e^{-d(t-r)}dB_1(r)+\alpha_2\int_0^t W(r)e^{-d(t-r)}dB_2(r) \qquad (22)\\
&\quad +\alpha_3\int_0^t X(r)e^{-d(t-r)}dB_3(r)+\alpha_4\int_0^t Y(r)e^{-d(t-r)}dB_4(r)\\
&\quad +\alpha_5\int_0^t Z(r)e^{-d(t-r)}dB_5(r).
\end{aligned}
$$

Obviously, $M(t)$ is a continuous local martingale with $M(0)=0$. And from relation Eq. (20), we have $V(t)+W(t)+X(t)+Y(t)+Z(t)\leq S(t)$ a.s. for all positive t. One can see that $W(t)$ and $X(t)$ are continuous adapted increasing processes on $t\geq 0$ with $W(0)=X(0)$, we get $\lim_{t\to\infty}X(t)\leq\infty$ a.s. Thus

$$\lim_{t\to\infty}\sup(V(t)+W(t)+X(t)+Y(t)+Z(t))<\infty \text{ a.s.} \qquad (23)$$

Thus, Eq. (16) holds. Keeping in view relation (23), it is handy to show that

$$
\begin{aligned}
&\lim_{t\to\infty}\frac{V(t)}{t}=0,\quad \lim_{t\to\infty}\frac{W(t)}{t}=0,\quad \lim_{t\to\infty}\frac{X(t)}{t}=0\quad \lim_{t\to\infty}\frac{Y(t)}{t}=0,\ \lim_{t\to\infty}\frac{Z(t)}{t}=0, \text{a.s,}\\
&\lim_{t\to\infty}\frac{\ln V(t)}{t}=0,\quad \lim_{t\to\infty}\frac{\ln W(t)}{t}=0,\quad \lim_{t\to\infty}\frac{\ln X(t)}{t}=0\quad \lim_{t\to\infty}\frac{\ln Y(t)}{t}=0,\ \lim_{t\to\infty}\frac{\ln Z(t)}{t}=0\text{a.s.}
\end{aligned}
$$

Set

$$M_1(t) = \int_0^t V(r)dB_1(r), \quad M_2(t) = \int_0^t W(r)dB_2(r), \quad M_3(t) = \int_0^t X(r)dB_3(r),$$
$$M_4(t) = \int_0^t Y(r)dB_4(r), \quad M_5(t) = \int_0^t Z(r)dB_5(r).$$

Because of the quadratic variation, we can write

$$\langle M_1(t), M_2(t)\rangle = \int_0^t V^2(r)dr \leq \Big(sup_{t\geq 0}V^2(t)\Big)t. \tag{24}$$

By using Lemma 4.1 and Eq. (23), we get

$$\lim_{t\to\infty} \frac{\int_0^t V(r)dB_1(r)}{t} = 0, \text{ a.s.}$$

Similarly, we also get

$$\lim_{t\to\infty} \frac{\int_0^t W(r)dB_2(r)}{t} = 0, \lim_{t\to\infty} \frac{\int_0^t X(u)dB_3(r)}{t} = 0, \frac{\int_0^t Y(u)dB_4(r)}{t} = 0, \frac{\int_0^t Z(r)dB_5(r)}{t} = 0 \text{ a.s,}$$

which proves Eq. (18) and hence the Lemma 4.2.

Based on this, we give the definition of the reproductive parameter denoted by $\mathbf{R_s}$ for stochastic model (2) as follows:

$$\mathbf{R_s} = \frac{\beta}{(\delta+\gamma+d+\frac{\alpha_2^2}{2})}$$

Theorem 4.1. Let $(V(t), W(t), X(t), Y(t), Z(t))$ be the solution of system (2) with $(V(0), W(0), X(0), Y(0), Z(0)) \in R_+^5$, if $\mathbf{R_s} < 1$, then the solution of stochastic model (2) obeys

$$\begin{aligned}
&\lim_{t\to\infty} \langle V(t)\rangle = \frac{\Pi}{d}, \text{ a.s,}\\
&\lim_{t\to\infty} \langle W(t)\rangle = 0 \text{ a.s,}\\
&\lim_{t\to\infty} \langle X(t)\rangle = 0 \text{ a.s,}\\
&\lim_{t\to\infty} \langle Y(t)\rangle = 0 \text{ a.s,}\\
&\lim_{t\to\infty} \langle Z(t)\rangle = 0 \text{ a.s,}
\end{aligned} \tag{25}$$

namely the disease extinct with probability one.

Proof By employing direct integration to model (2), we may obtain the following results easily

$$\begin{aligned}
\frac{V(t)-V(0)}{t} &= \Pi - \frac{\beta\langle VW\rangle}{\langle N\rangle} - d\langle V\rangle + \lambda\langle Z\rangle + \frac{\alpha_1\int_0^t V(r)dB_1(r)}{t},\\
\frac{W(t)-W(0)}{t} &= \frac{\beta\langle VW\rangle}{\langle N\rangle} - \frac{\delta\langle WX\rangle}{\langle N\rangle} - (\gamma+d)\langle W\rangle + \frac{\alpha_2\int_0^t W(r)dB_2(r)}{t},\\
\frac{X(t)-X(0)}{t} &= \frac{\delta\langle WX\rangle}{\langle N\rangle} - (\mu+\omega+d)\langle X\rangle + \frac{\alpha_3\int_0^t X(r)dB_3(r)}{t}.\\
\frac{Y(t)-Y(0)}{t} &= \omega\langle X\rangle - (\kappa+d)\langle Y\rangle + \frac{\alpha_4\int_0^t Y(r)dB_4(r)}{t}.\\
\frac{Z(t)-Z(0)}{t} &= \kappa\langle Y\rangle - (d+\lambda)\langle Z\rangle + \frac{\alpha_5\int_0^t Z(r)dB_5(r)}{t}.
\end{aligned} \tag{26}$$

Apply $Itô$ formula on the second Eq. of system (2), then we can get

$$d\log W(t) = \left[\frac{\beta V}{N} - \frac{\delta X}{N} - (\gamma + d) - \frac{\alpha_2^2}{2}\right]dt + \alpha_2 dB_2(t). \tag{27}$$

By utilizing the tools of integration in Eq. (27) keeping in view the limits $[0, t]$ and multiplying the resultant expression by $\frac{1}{t}$, we have

$$\begin{aligned}\frac{\log W(t) - \log W(0)}{t} &= \beta\frac{\langle V\rangle}{\langle N\rangle} - \frac{\delta\langle X\rangle}{\langle N\rangle} - (\gamma + d) - \frac{\alpha_2^2}{2} + \frac{\alpha_2}{t}\int_0^t dB_2(r),\\ &\leq \beta - (\delta + \gamma + d + \frac{\alpha_2^2}{2}) + \frac{\alpha_2}{t}\int_0^t dB_2(r).\\ &= \beta - (\delta + \gamma + d + \frac{\alpha_2^2}{2}) + \frac{\alpha_2}{t}\int_0^t dB_2(r).\\ &= (\delta + \gamma + d + \frac{\alpha_2^2}{2})(\mathbf{R_s} - 1) + \frac{\alpha_2}{t}\int_0^t dB_2(r).\end{aligned} \tag{28}$$

Moreover $M(t) = \frac{\alpha_2}{t}\int_0^t dB_2(r)$ with $M(0) = 0$ and clearly, this is local martingale and continuous function. Here, by using the Lemma (4.1) and $t \to \infty$, we obtain

$$\lim_{t\to\infty} \sup \frac{M(t)}{t} = 0 \tag{29}$$

If $\mathbf{R_0} < 1$ is satisfied, then Eq. (28) become

$$\lim_{t\to\infty} \sup \frac{\log W(t)}{t} \leq \left(\delta + \gamma + d + \frac{\alpha_2^2}{2}\right)\left(\mathbf{R_s} - 1\right) < 0 \text{ a.s.} \tag{30}$$

Clearly, Eq. (30) indicates that

$$\lim_{t\to\infty} \langle W(t)\rangle = 0 \text{ a.s.} \tag{31}$$

Further, by considering equation number 3^{rd} in system (26), integrating the equation over $[0, t]$ and then dividing the obtain result by t as well as by using Eq. (31), we have

$$\begin{aligned}\frac{X(t) - X(0)}{t} &= \frac{\delta\langle WX\rangle}{\langle N\rangle} - (\mu + \omega + d)\langle X\rangle + \frac{\alpha_3\int_0^t X(r)dB_3(r)}{t}.\\ \langle X\rangle &= \frac{1}{(\mu + \omega + d)}\left[\frac{\delta\langle WX\rangle}{\langle N\rangle} + \frac{X(0) - X(t)}{t} + \frac{\alpha_3\int_0^t X(r)dB_3(r)}{t}\right]\end{aligned} \tag{32}$$

which simply implies that

$$\lim_{t\to\infty} \langle X(t)\rangle = 0 \text{ a.s.} \tag{33}$$

In the similar way, we have

$$\lim_{t\to\infty} \langle Y(t)\rangle = 0 \text{ a.s,} \tag{34}$$

and

$$\lim_{t\to\infty} \langle Z(t)\rangle = 0 \text{ a.s.} \tag{35}$$

Finally, from integrating the first equation of model (26) over period $[0, t]$, as well as dividing the result by t and using Eq. (31) and Eq. (35), we have

$$\begin{aligned}\frac{V(t)-V(0)}{t} &= \Pi - \frac{\beta\langle VW\rangle}{\langle N\rangle} - d\langle V\rangle + \lambda\langle Z\rangle + \frac{\alpha_1\int_0^t V(r)dB_1(r)}{t},\\ \langle V\rangle &= \frac{1}{d}\left[\Pi - \frac{\beta\langle VW\rangle}{\langle N\rangle} + \frac{V(0)-V(T)}{t} + \lambda\langle Z\rangle + \frac{\alpha_1\int_0^t V(r)dB_1(r)}{t},\right]\end{aligned} \tag{36}$$

which implies that

$$\lim_{t\to\infty}\langle V(t)\rangle = \frac{\Pi}{d} \text{ a.s.} \tag{37}$$

and it proves the complete result.

5. Persistence of the Disease

This portion of the study will gives a condition under which smoking will persist for model (2). The main result are presented by the following lemmas and theorem.

Definition 5.1. [32] The system (2) under discussion is called persistent only if

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t (W+X)(r)dr > 0 \text{ a.s.} \tag{38}$$

Theorem 5.1. If $\mathbf{R_0^s} = \frac{(\mu+\omega+d+\frac{\alpha_1^2}{2})}{(\gamma+d+\frac{\alpha_1^2}{2})}$ then subject to an initial data $(V(0),W(0),X(0),Y(0),Z(0)) \in R_+^5$, the disease $W(t)$ and $I(t)$ has the axiom

$$\liminf_{t\to\infty}\langle (W+X)\rangle \geq \frac{(\mathbf{R_0^s}-1)}{\delta} \text{ a.s.} \tag{39}$$

If we have the case of $\mathbf{R_0^s} > 1$, then we can assume that the disease will persist in the community.

Proof Set

$$G_1 = -lnW + lnX \tag{40}$$

Applying $Itô$ formula, so we have

$$dG_1 = \mathcal{L}G_1 - \alpha_1 dB_1(t) + \alpha_2 dB_2(t) \tag{41}$$

$$\begin{aligned}\mathcal{L}G_1 &= \mathcal{L}(-lnW) + \mathcal{L}(lnX)\\ &= -\frac{\beta V}{N} + \frac{\delta X}{N} + (\gamma+d) + \frac{\alpha_1^2}{2} + \frac{\delta W}{N} - (\mu+\omega+d) - \frac{\alpha_1^2}{2}\\ &\leq -\frac{\beta V}{N} + \delta(W+X) + (\gamma+d) + \frac{\alpha_1^2}{2} - (\mu+\omega+d) - \frac{\alpha_1^2}{2}\\ &\leq \delta(W+X) + (\gamma+d+\frac{\alpha_1^2}{2}) - (\mu+\omega+d+\frac{\alpha_1^2}{2})\\ &\leq -\left[\frac{(\mu+\omega+d+\frac{\alpha_1^2}{2})}{(\gamma+d+\frac{\alpha_1^2}{2})} - 1\right] + \delta(W+X)\\ &\leq -\left[(\mathbf{R_0^s}-1)\right] + \delta(W+X)\end{aligned} \tag{42}$$

Substituting Eq. (42) into Eq. (40), then integrating both side of the random smoking model (2)

$$\begin{aligned}\frac{G_1(W(t),X(t)) - G_1(W(0)),X(0))}{t} &\leq -\left[(\mathbf{R_0^s}-1)\right] + \delta\langle (W+X)\rangle - \frac{\alpha_1 B_1(t)}{t} + \frac{\alpha_2 B_2(t)}{t}\\ &\leq -(\mathbf{R_0^s}-1) + \delta\langle (W+x)\rangle + \Psi(t),\end{aligned} \tag{43}$$

where $\Psi(t) = -\frac{\alpha_1 B_1(t)}{t} + \frac{\alpha_2 B_2(t)}{t}$. From strong law as stated in Lemma (4.1), we arrive

$$\lim_{t\to\infty}\Psi(t) = 0, \tag{44}$$

From Eq. (43), we have

$$\left\langle (W+X) \right\rangle \geq \frac{(\mathbf{R_0^s}-1)}{\delta} - \frac{\Psi(t)}{\delta} + \frac{1}{\delta}\left(\frac{G_1(W(t),X(t)) - G_1(W(0),X(0))}{t} \right). \tag{45}$$

According to Lemma (4.2) and Eq. (44), the limit superior of Eq. (5), we have

$$\liminf_{t\to\infty} \left\langle (W+X) \right\rangle \geq \frac{(\mathbf{R_0^s}-1)}{\delta} \text{ a.s,} \tag{46}$$

which completes the proof of the theorem (5.1).

6. Stationary Distribution and Ergodicity of the Disease

In the stochastic version of the system, the endemic equilibrium does not exist. As a result, we can't analyze the disease's persistence by looking at the endemic equilibrium's stability; instead, we need to look into the presence and uniqueness of the stationary distribution for the system (2). We follow Hasminskii et.al. [43] and the method of [44] and will prove the ergodicity and stationary distribution for the model (2) solution.

Let $X(t)$ be Markov process and regular time and homogeneous over R^d

$$dG(t) = bGdt + \sum_{r=1}^{k} \sigma_r(G) dB_r(t). \tag{47}$$

The diffusion array is define as follow

$$A(x) = (a_{ij}(x)), \qquad a_{ij}(t) = \sum_{r=1}^{k} \sigma_r^i(t)\sigma_r^j(x).$$

Lemma 6.1. ([43]). The Markov process $G(t)$ has ergodic stationary distribution $\Pi(\cdot)$ and unique a bounded open domain $U \in R^d$ with regular boundary such that its closure $\bar{U} \in R^d$ having the following properties:

1. In the open domain U and some some neighborhood of there, the smallest eigenvalue of the diffusion matrix $A(t)$ is bounded away from zero
2. If $x \in R^d \backslash U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x\in K} E^x \tau < \infty$ for every compact subset $K \subset R^n$. Moreover, if $f(\cdot)$ is a function integrable with respect to the measure Π, then

$$P\left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T f(G^X(t))dt = \int_{R^d} f(x)\Pi(dx) \right\} = 1.$$

We present the theorem stated below for the required stationary process as well as ergodicity.

Theorem 6.1. For the proposed system (2) with $(V,W,X,Y,Z)(0) \in R_+^5$, if $\mathbf{R_s} > 1$, then the stationary distribution $\pi(\cdot)$ exists, and is ergodic.

Proof It is to be noted that system (2) can be also written as

$$d\begin{bmatrix} V \\ W \\ X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \Pi - \frac{\beta VW}{N} - dV(t) + \lambda Z \\ \frac{\beta VW}{N} - \frac{\delta WX}{N} - (\gamma + d)W \\ \frac{\delta WX}{N} - (\mu + \omega + d)X \\ \omega X - (\kappa + d)Y \\ \kappa Y - (d+\lambda)Z \end{bmatrix} dt + \begin{bmatrix} \alpha_1 V(t) dB_1(t) \\ \alpha_2 W(t) dB_2(t) \\ \alpha_3 X(t) dB_3(t) \\ \alpha_4 Y(t) dB_4(t) \\ \alpha_5 Z(t) dB_{53}(t) \end{bmatrix}.$$

Consequently the diffusion matrix for the associated problem (2) is as

$$B = \begin{bmatrix} \alpha_1^2 V^2 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2^2 W^2 & 0 & 0 & 0 \\ 0 & 0 & \alpha_3^2 X^2 & 0 & 0 \\ 0 & 0 & 0 & \alpha_4^2 Y^2 & 0 \\ 0 & 0 & 0 & 0 & \alpha_5^2 Z^2 \end{bmatrix}$$

Since $U \subset R_+^5$ and $\xi \in R_+^5 \setminus \{(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \in R_+^5 : \xi_1 = \xi_3 = \xi_2 = \xi_4 = \xi_5\}$, then a positive number C exists as

$$\begin{aligned}\sum_{i,j=1}^{5} a_{ij}(V,W,X,Y,Z)\xi_i\xi_j &= \alpha_1^2 V^2\xi_1^2 + \alpha_2^2 W^2\xi_2^2 + \alpha_3^2 X^2\xi_3^2 + \alpha_4^2 Y^2\xi_4^2 + \alpha_5^2 Z^2\xi_5^2 \\ &\geq C.\end{aligned} \tag{48}$$

Then condition (1) of Lemma (6.1) holds. Let $\mathcal{V}_1 W$, $\mathcal{V}_2(V,W)$ and $\mathcal{V}_3(W,X,Y,Z)$ be the function for $(V,W,X,Y,Z) \in R_+^5$ and defined by

$$\begin{aligned}\mathcal{V}_1 W &= \frac{1}{q} W^{-q}, \\ \mathcal{V}_2(V,W) &= \frac{1}{q} W^{-q}(\frac{\Pi}{d} - V), \\ \mathcal{V}_3(V,X,Y,Z) &= \frac{1}{V} + \frac{1}{Z} + Y + X,\end{aligned} \tag{49}$$

where q is positive constant and we will determine it later. Defining a function like

$$\begin{aligned}V(V,W,X,Y,Z) &= \mathcal{V}_1 W + \mathcal{V}_2(V,W) + \mathcal{V}_3(V,X,Y,Z), \\ V(V,W,X,Y,Z) &= \frac{1}{q} W^{-q} + \frac{1}{q} W^{-q}(-V + \frac{\Pi}{d}) + \frac{1}{V} + \frac{1}{Z} + Y + X.\end{aligned} \tag{50}$$

By $Itô$ formula and the application of system (2) we obtain

$$\begin{aligned}\mathcal{V}_1 W &= -W^{-q}(\frac{\beta V}{N} - \frac{\delta X}{N} - (\gamma + d)) + \frac{1}{2}(q+1)\alpha_2^2 W^{-q} \\ &\leq W^{-q}\bigg(- (\delta + \gamma + d + \frac{\alpha_1^2}{2})(\mathbf{R_s} - 1) + \frac{q}{2}\alpha_2^2\bigg)\end{aligned} \tag{51}$$

Then we compute $LV_2(V,W)$

$$\begin{aligned}LV_2(V,W) &= \frac{1}{q} W^{-q}(\frac{\Pi}{d} - V)\bigg(\frac{\beta V}{N} - \frac{\delta X}{N} - (\gamma + d) + \frac{1}{2}(q+1)\alpha_2^2\bigg) \\ &\quad - \frac{1}{q} W^{-q}(\Pi - \frac{\beta V W}{N} - dV(t) + \lambda Z) \\ &\leq W^{-q}(\frac{\Pi}{d} - V)((\delta + \gamma + d) - \frac{d}{q} + \frac{1}{2}(q+1)\alpha_2^2) + \frac{\beta V W}{qN} - \frac{\lambda Z}{q}\end{aligned} \tag{52}$$

For $V_3(S,R,Z)$, it implies that

$$\begin{aligned}V_3(S,R,Z) &= -\frac{\Pi}{V^2} + \frac{\beta W}{VN} + \frac{d}{V} - \frac{\lambda Y}{V^2} + \frac{\delta W}{N} - (\mu + \omega + d) \\ &\quad + \frac{\omega X}{Y} - (\lambda + \kappa + d) - \frac{\kappa Y}{Z^2} + \frac{(d+\lambda)}{Z}.\end{aligned} \tag{53}$$

Combining (51), (52), and (53), we have

$$\begin{aligned}LV &\leq W^{-q}\bigg(- (\delta + \gamma + d + \frac{\alpha_1^2}{2})(\mathbf{R_s} - 1) + \frac{q}{2}\alpha_2^2\bigg) \\ &\quad + W^{-q}(\frac{\Pi}{d} - V)((\delta + \gamma + d) - \frac{d}{q} + \frac{1}{2}(q+1)\alpha_2^2) + \frac{\beta V W}{qN} - \frac{\lambda Z}{q} \\ &\quad - \frac{\Pi}{V^2} + \frac{\beta W}{VN} + \frac{d}{V} - \frac{\lambda Y}{V^2} + \frac{\delta W}{N} - (\mu + \omega + d) \\ &\quad + \frac{\omega X}{Y} - (\kappa + d) - \frac{\kappa Y}{Z^2} + \frac{(d+\lambda)}{Z}.\end{aligned} \tag{54}$$

Since $\mathbf{R_s} \geq 0$, and choose q sufficiently small such that

$$-(\delta+\gamma+d+\frac{\alpha_1^2}{2})(\mathbf{R_s}-1)+\frac{q}{2}\alpha_2^2<0$$
$$(\delta+\gamma+d)-\frac{d}{q}+\frac{1}{2}(q+1)\alpha_2^2<0.$$

It is simple from Eq. (54), sufficiently large

$$LV \leq -1 \quad for \quad (V,W,X,Y,Z).$$

So Lemma (6.1) holds and the model (2) admits an ergodic invariant distribution $\Pi(\cdot)$, which is unique. Hence $(V_t, W_t, X_t, Y_t, Z_t)$ is ergodic and

$$P\left\{\lim_{t\to\infty}\frac{1}{t}\int_0^t \chi(V_s,W_s,X_s,Y_s,Z_s)\in\Gamma ds=\int_{R^5}\chi\Gamma^{\pi}(dx)\right\}=1 \tag{55}$$

where $\chi\Gamma$ is the characteristic function of Γ.

7. Investigation of the Stochastic Optimal Control

optimal control strategies have been a hot topic in recent years. These techniques are used to create the best management strategies for a variety of diseases. The optimal control theory is a useful tool for determining the best information circulation strategy [34, 35]. It has numerous applications in the fields of dynamical systems, economics and physics [36]. We refer the readers to [34, 36, 37] for more details on the required conditions of optimality and other similar issues related to state-constrained control systems. The main target of this disease-control plan is one that has the greatest chance of reducing the number of infectious people at the lowest cost. The basic steps in modeling is to connect a model(s) with some natural phenomenon. This move is not an essay assignment owing to a physical lack of knowledge of the phenomenon. In certain instances, we only have data from experiments and some literature outcomes at our hands. To find a well-suited mathematical model, extensive referenced research and numerous examinations with various mathematical model(s) are essential to use. The mathematical formalism might need to be adjusted to fit the experimental data, so a first representation is rarely convenient. Some recent studies have yielded impressive results. In [45], the authors improved and extended the established techniques for studying the interaction of $n-$species by using the random competitive delay model. Aside from the above, this paper formulates the optimum harvesting strategy for the system under consideration. Liu and Meng also looked at optimum harvesting problem and used stochastic delay model for presenting the dynamical aspects of the system [46]. The readers are advised to see [34, 35] and the references therein for recent developments in the stochastic optimal control methods.

To reduce the habit smoking in a community, we intend to include two control measures in model (2) and will follow the same techniques as presented in [34]. The two control variables are $u_1(t)$ and $u_2(t)$ which respectively represent the education campaign and anti-nicotine gum/medicine.

This part of the study is dedicated to the analysis of stochastic control problem. Based on some hypothesis, we updated model (2) to a control problem keeping in view two control measures with the same meaning as mentioned above. Thus, the associated stochastic controlled version of model (2) gives as the following form:

$$\begin{aligned}
dV &= \left[\Pi-\frac{\beta VW}{N}-dV+\lambda Z\right]dt+\alpha_1 V dB_1(t)\\
dW &= \left[\frac{\beta VW}{N}-\frac{\delta WX}{N}-(\gamma+d)W\right]dt+\alpha_2 W dB_2(t)\\
dX &= \left[\frac{\delta WX}{N}-(\mu+\omega+d)X-u_1X\right]dt+\alpha_3 X dB_3(t)\\
dY &= \left[\omega X-(\kappa+d)Y-u_2Y\right]dt+\alpha_4 Y dB_4(t)\\
dZ &= \left[\kappa Y-(d+\lambda)Z+u_1X+u_2Y\right]dt+\alpha_5 Z dB_5(t)
\end{aligned} \tag{56}$$

with the initial condition

$$V(0) > 0, W(0) \geq 0, X(0) \geq 0, Y(0) > 0, Z(0) \geq 0 \tag{57}$$

The vectors are described as follows for ease of use:

$$\begin{aligned} y(t) &= [y_1(t), y_2(t), y_3(t), y_4(t), y_5(t)]', \\ &= [V(t), W(t), X(t), Y(t), Z(t)]', \\ u(t) &= [u_1(t), u_2(t)]', \end{aligned} \tag{58}$$

Furthermore, Eq. (57) can be re-write

$$dy(t) = g(y(t))dw(t) + f(y(t), u(t))dt. \tag{59}$$

Where

$$\begin{aligned} f(y(t), u(t)) &= [f_1(y(t), u(t)), f_2(y(t), u(t)), f_3(y(t), u(t)), f_4(y(t), u(t)), f_5(y(t), u(t))]', \\ g(x) &= [g_1(x), g_2(x), g_3(x), g_4(x), g_5(x)]', \end{aligned} \tag{60}$$

and where

$$\begin{aligned} f_1 &= \left[\Pi - \frac{\beta VW}{N} - dV + \lambda Z\right]dt + \alpha_1 V dB_1(t), \\ f_2 &= \left[\frac{\beta VW}{N} - \frac{\delta WX}{N} - (\gamma + d)W\right]dt + \alpha_2 W dB_2(t), \\ f_3 &= \left[\frac{\delta WX}{N} - (\mu + \omega + d)X - u_1 X\right]dt + \alpha_3 X dB_3(t), \\ f_4 &= \left[\omega X - (\kappa + d)Y - u_2 Y\right]dt + \alpha_4 Y dB_4(t), \\ f_5 &= \left[\kappa Y - (d + \lambda)Z + u_1 X + u_2 Y\right]dt + \alpha_5 Z dB_5(t). \end{aligned} \tag{61}$$

$g_1 = \alpha_1 V, g_2 = \alpha_2 W, g_3 = \alpha_3 X, g_4 = \alpha_4 Y, g_5 = \alpha_5 Z.$

We considered a cost function with the form of quadratic in nature as follows:

$$G(u) = \frac{1}{2}E\left\{\int_0^{t_f}\left(A_1 W + A_2 X + A_3 Y - A_4 Z + \sum_{i=1}^{2}\frac{B_i}{2}u_i^2\right)dt + \frac{k_1}{2}V^2 + \frac{k_2}{2}W^2 + \frac{k_3}{2}X^2 + \frac{k_4}{2}Y^2 + \frac{k_5}{2}Z^2\right\}, \tag{62}$$

where $A_1, A_2, A_3, A_4, B_1, B_2$ and k_j for $j = 1 \cdots, 5$ are positive constants.

Our aim is to present an effective and efficient control set $u^* = (u_1^*(t), u_2^*(t))$ such that

$$J(u) \geq J(u^*), \text{ for all } u \in U, \tag{63}$$

here U denotes the feasible control set with the following definition:

$$U = \left\{u_i(t): \quad u_i(t) \in [0, u_i^{\max}], \quad \forall u_i \in L^2[0, t_f] \quad t \in (0, t_f], \quad i = 1, 2\right\} \tag{64}$$

Where $u_i^{\max} \in R_+$ for $i = 1, 2$. First of all, we must define the Hamiltonian $H_m(x, u, p, q)$ (in order to apply the stochastic optimality techniques) in such a manner that

$$H(y, u, p, q) = \langle g(y), q\rangle - l(y, u) + \langle f(y, u), p\rangle, \tag{65}$$

where the notion $\langle\cdot,\cdot\rangle$ stand for the inner product in Euclidean sense, whereas, $p = [p_1, p_2, p_3, p_4, p_5]'$ and $q = [q_1, q_2, q_3, q_4, q_5]'$ are the adjoint vectors. Referring to the optimality principle, we have

$$dy^*(t) = g(y^*(t))dW(t) + \frac{\partial H(y^*, p, u^*, q)}{\partial p}dt. \tag{66}$$

$$dp^* = q(t)dW(t) - \frac{\partial H(y^*, p, u^*, q)}{\partial y}dt. \tag{67}$$

$$H_m\left(y^*, p, u^*, q\right) = \max_{u \in U} H\left(y^*, p, u^*, q\right). \tag{68}$$

where $y^*(t)$ stand for the optimal curve of $y(t)$. The starting and ending criterion of Eq. (66) and (67) are

$$y^*(0) = y_0 \tag{69}$$

$$p\left(t_f\right) = -\frac{\partial h\left(y^*\left(t_f\right)\right)}{\partial y}, \tag{70}$$

respectively. As Eq. (68) implies that the optimal control $u^*(t)$ depends on $p(t), q(t)$ and $y^*(t)$, thus, we can write

$$u^*(t) = \phi\left(p, y^*, q\right) \tag{71}$$

and ϕ is functional relationship which needs to be calculated by Eq. (68). The related Hamiltonian function will takes the form

$$\begin{aligned} H = & \left(A_1 W + A_2 X + A_3 Y - A_4 Z + \frac{B_1}{2} u_1^2 + \frac{B_2}{2} u_2^2 + \frac{k_1}{2} V^2 + \frac{k_2}{2} W^2 + \frac{k_3}{2} X^2 + \frac{k_4}{2} Y^2 + \frac{k_5}{2} Z^2\right) \\ & + p_1\left(\Pi - \frac{\beta V W}{N} - dV + \lambda Z\right) + p_2\left(\frac{\beta V W}{N} - \frac{\delta W X}{N} - (\gamma + d) W\right) \\ & + p_3\left(\frac{\delta W X}{N} - (\mu + \omega + d) X - u_1 X\right) + p_4\left(\omega X - (\kappa + d) Y - u_2 Y\right) \\ & + p_5\left(\kappa Y - (d + \lambda) Z + u_1 X + u_2 Y\right) + \alpha_1 V q_1 + \alpha_2 W q_2 + \alpha_3 X q_3 + \alpha_4 Y q_4 + \alpha_5 Z q_5. \end{aligned} \tag{72}$$

Keeping in view relation (67) and taking the respective derivatives of H w.r.t V, W, X, Y and Z, we may write p_1', p_2', p_3', p_4', and p_5' in the following form

$$\begin{aligned} \frac{dp_1(t)}{dt} &= p_1 \frac{\beta W^*}{N} + p_1 d - p_2 \frac{\beta W^*}{N} + \alpha_1 q_1, \\ \frac{dp_2(t)}{dt} &= -A_1 + p_1 \frac{\beta V^*}{N} - p_2 \frac{\beta V^*}{N} + p_2 \frac{\delta X^*}{N} - p_3 \frac{\delta X^*}{N} - p_2(\gamma + d) + \alpha_2 q_2, \\ \frac{dp_3(t)}{dt} &= -A_2 - p_3 \frac{\delta W^*}{N} + p_2 \frac{\delta W *}{N} + p_3(\mu + \omega + d) + p_3 u_1^* - p_4 \omega - p_5 u_1^* + \alpha_3 q_3, \\ \frac{dp_4(t)}{dt} &= -A_3 + p_4(\kappa + d) + p_4 u_2^* - p_1 u_2^* - p_5 \kappa + \alpha_4 q_4, \\ \frac{dp_5(t)}{dt} &= -A_4 + p_5(\lambda + d) + p_1 \lambda + \alpha_5 q_5. \end{aligned} \tag{73}$$

supported by the subsidiary starting and final conditions of the form

$$(V^*, W^*, X^*, Y^*, Z^*)(0) = (\hat{V}, \hat{W}, \hat{X}, \hat{Y}, \hat{Z}), \;\; p\left(t_f\right) = -\frac{\partial h\left(x^*\left(t_f\right)\right)}{\partial x}, \tag{74}$$

and

$$h(V, W, X, Y, Z) = \frac{k_1}{2} V^2 + \frac{k_2}{2} W^2 + \frac{k_3}{2} X^2 + \frac{k_4}{2} Y^2 + \frac{k_5}{2} Z^2, \tag{75}$$

where $p_1\left(t_f\right) = -k_1 V, \;\; p_2\left(t_f\right) = -k_2 W, \;\; p_3\left(t_f\right) = -k_3 X, p_4\left(t_f\right) = -k_4 Y, p_4\left(t_f\right) = -k_4 Z$. Now differentiating Hamiltonian equation with respect to u_1 and u_2 we get the following optimal controls u_1^* and u_2^*

$$\begin{aligned} u_1^* &= \max\left\{\min\left\{1, \frac{1}{B_1}(p_3 - p_5) X^\star\right\}, 0\right\} \\ u_2^* &= \max\left\{\min\left\{1, \frac{1}{B_2}(p_4 - p_5) Y^\star\right\}, 0\right\} \end{aligned} \tag{76}$$

In order to deal with optimization systems, we are searching for control measures in a dynamical framework for achieving certain goals. To formulate and solve an optimal system, first of all, we need to use the tools of differential equations or stochastic differential equations in model formulation and then design and set the boundaries of the control measures as done in relation (61). The next step is the formulation of an objective functional and usually consist of states and control variables as we did in equation (62). To create an objective functional, it is important to balance the opposing variables. Since, the optimal control outcomes are highly dependent on the nature and form of the objective functional, therefore, it is consider to be a critical task which should be approached with caution. In case of more than two factors in the objective functional, weights constants should be assigned to such factors based on their relative significance. Prior to applying the maximum principle of Pontryagin's [47], it is necessary to demonstrate the existence of an

optimal control through the compactness claim. The control measure(s) used in such arguments are usually bounded Lebesgue measurable or piecewise continuous functions. Subsequently, to show that all possible control yields a bounded value of the objective functional, we must develop a minimizing/maximizing of control/state(s) which converges in the desired control/state spaces. The problem of getting control measures which optimize the cost function given a state system and initial condition is transformed into the case of maximizing/minimizing the Hamiltonian point-wise with the help of this principle. Following [47], we can easily write the Hamiltonian for the problem via the following formula

$$Hamiltonian(H) = (integrand\, of\, the\, objective\, functional) + (adjoint)(RHS\, of\, Differential\, equations).$$

Sufficient theory may be developed by optimizing the function H considering its derivation w.r.t the control at the point u^* and basically, this principle is known as optimality condition. To derive the adjoint system, one have to consider the partial differentiation of H w.r.t y, and finally by using Eq. (74), we have the transversility conditions.

8. Numerical Simulations

In order to verify the above theoretical predictions, we use a standard numerical method to solve system (2). Specifically, we construct, based on the stochastic Runge-Kutta method of order 4, a scheme with details below:

$$\begin{aligned}
V_{i+1} &= V_i + \left[\Pi - \frac{\beta V_i W_i}{N} - dV_i + \lambda Z_i\right] \triangle t + \alpha_1 V_i \sqrt{\triangle t}\zeta_{1,i} + \frac{\alpha_1^2}{2} V_i(\zeta_{1,i}^2 - 1) \triangle t,\\
W_{i+1} &= W_i + \left[\frac{\beta V_i W_i}{N} - \frac{\delta W_i X_i}{N} - (\gamma + d)W_i\right] \triangle t + \alpha_2 W_i \sqrt{\triangle t}\zeta_{2,i} + \frac{\alpha_2^2}{2} W_i(\zeta_{2,i}^2 - 1) \triangle t,\\
X_{i+1} &= X_i + \left[\frac{\delta W_i X_i}{N} - (\mu + \omega + d)X_i\right] \triangle t + \alpha_3 X_i \sqrt{\triangle t}\zeta_{3,i} + \frac{\alpha_3^2}{2} X_i(\zeta_{3,i}^2 - 1) \triangle t,\\
X_{i+1} &= Y_i + \left[\omega X_i - (\kappa + d)Y_i\right] \triangle t + \alpha_4 Y_i \sqrt{\triangle t}\zeta_{4,i} + \frac{\alpha_4^2}{2} Y_i(\zeta_{4,i}^2 - 1) \triangle t,\\
Z_{i+1} &= Z_i + \left[\kappa Y_i - (\lambda + d)Z_i\right] \triangle t + \alpha_5 Z_i \sqrt{\triangle t}\zeta_{5,i} + \frac{\alpha_5^2}{2} Z_i(\zeta_{5,i}^2 - 1) \triangle t.
\end{aligned} \tag{77}$$

In this scheme, $\zeta_{i,j}(i = 1, 2, 3, 4, 5)$, represent the Gaussian free stochastic variables with each following the $N(0, 1)$ distributions, Δt is the step size, and $\alpha_i > 0, (i = 1, 2, 3, 4, 5)$ stand for the white noise intensities.

By making use of the above numerical scheme, we will separately simulate stochastic stability and stochastic optimal control, with the aim to verify the above theoretical predictions.Since we are interested in qualitative behaviors of the stochastic system, i.e., model (2), as well as in qualitative effects of the proposed stochastic optimal control, we will make technical treatments for the data generated by the scheme.

In simulations, we adopt biologically reasonable parameter values including noise intensities, which are listed in Table (2). The initial sizes of each compartmental population are also listed in Table 2 and the time period is set as the [0, 140] interval.

8.1. Simulations of Qualitative Behavior

By making use of the stochastic stability theory, we have derived a set of sufficient conditions for the extinction of smoking from a population, seeing the above Theorem 4.1. Numerical simulations verify that the conclusion of this theorem holds only when the basic reproduction rate is less than one i.e., $\mathbf{R}_0 < 1$. In reality, this case implies smoking elimination with probability one. From Figure 1 that clearly shows the extinction of the disease under the theorem's conditions, we observe that the solution curves of the stochastic and deterministic models all converge to the unique smoking-free fixed point.

The above Theorem 5.1 describes the behavior of smoking persistence. To verify the conclusion of this theorem, we perform numerical simulations for the stochastic model described by Eq. (2). Using the parameter values listed in Table 2 (S_1), we calculate $\mathbb{R}_0^s$ and find that its value is more than one, implying that all the conditions of Theorem 3 are satisfied. Numerical results are demonstrated in Figure 2. From this figure, we observe that smoking tends to persist in the population since the noise is weak in this case. This indicates that the conclusion of Theorem 5.1 (smoking persistence) is indeed correct or is numerically verified. If simulating model (2) for more than 1000 times, we can obtain the mean extinction time for the smoking population. Note that the extinction time would be different for a distinct noise intensity. Therefore, to decrease the mean extinction time, one would need to increase the noise intensity. The above assumptions for the obtained results are due to Theorem 6.1. In addition, we use Figure 3 to show the probability distribution histograms of random variables V, W, X, Y, and Z, which

further verify our theoretical predictions.

For clarity, we list conditions for numerical results in the following two scenarios.

Example 8.1. (Stochastic disease-free stability) Given the parameter values listed in Table 2 (S_1). we can easily calculate the basic reproduction number $\mathbf{R_0} < 1$. In order to guarantee $\mathbf{R_0} < 1$ the solution of model (2) must satisfy

$$\lim_{t\to\infty} \sup \frac{\log W(t)}{t} \leq 0, \quad a.s.$$

and

$$\lim_{t\to\infty} \sup \frac{\log X(t)}{t} \leq 0, \quad a.s.$$

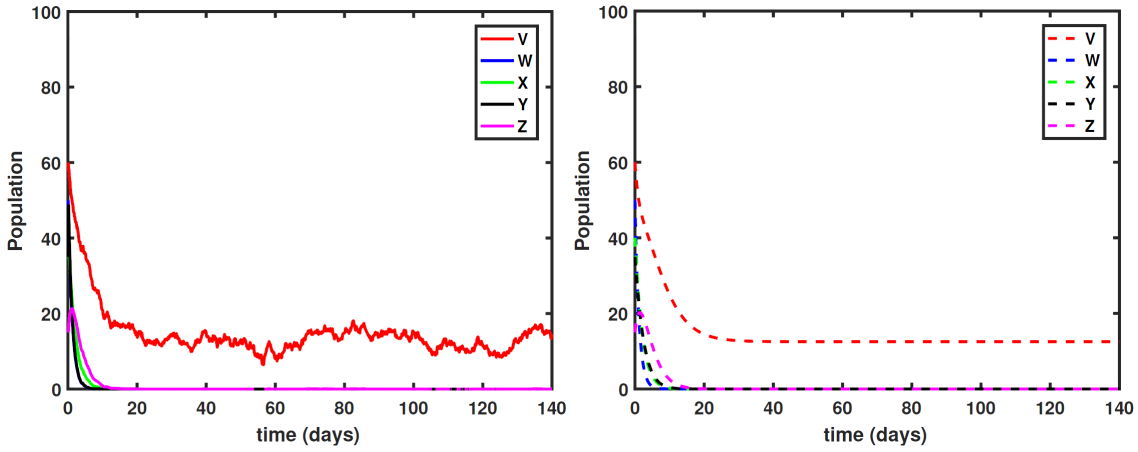
according to Theorem 4.1

In this scenario, the smoking habit can be eradicated from a population. In fact, numerical results shown in Figure 1 verifies the theoretical prediction.

Example 8.2. (Stochastic endemic stability) Given the parameter values listed in Table 2 (S_2). we can verify $\mathbb{R}_0^s > 1$. In this case, the smoking will persist by Theorem 5.1,and the simulation(s) of Figure 2 also support this theoretical result. Theorem 5.1 implies that model (2) has the persistence supported by Figure 2, and there exists an ergodic stationary distribution in the proposed model (2), which is confirmed by Figure 3.

Table 2. *Parameters values used in simulating model (2).*

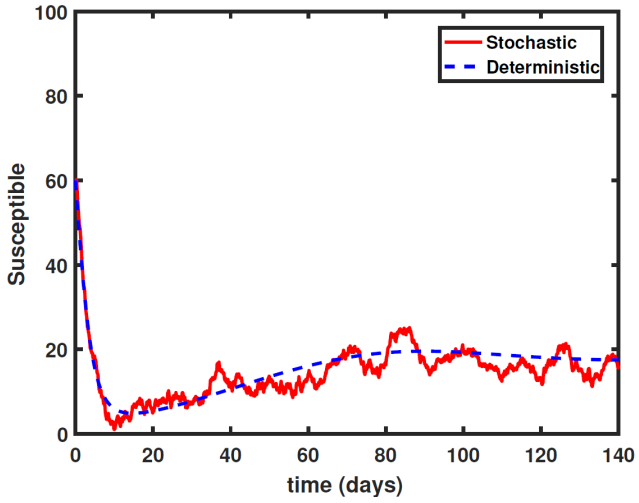
Parameters	**Values (S_1)**	**Values (S_2)**	**Values (S_3)**
Π	2.50 Per month	0.80 Per month	0.90 Per month
μ	0.10 Per month	0.01 Per month	0.00 Per month
β	0.45 Per month	0.90 Per month	0.45 Per month
γ	0.70 Per month	0.07 Per month	0.00 Per month
ω	0.20 Per month	0.001 Per month	0.002 Per month
κ	0.40 Per month	0.001 Per month	0.075 Per month
δ	0.30 Per month	0.010 Per month	0.02 Per month
λ	2.50 Per month	0.002 Per month	0.0025 Per month
d	0.20 Per month	0.01 Per month	0.08
α_1	0.170	0.205	0.243
α_2	0.120	0.315	0.305
α_3	0.125	0.320	0.245
α_4	0.140	0.210	0.15
α_5	0.135	0.220	0.35
$V(0)$	60	60	430
$W(0)$	50	50	10
$X(0)$	40	40	30
$Y(0)$	35	35	20
$Z(0)$	15	15	10



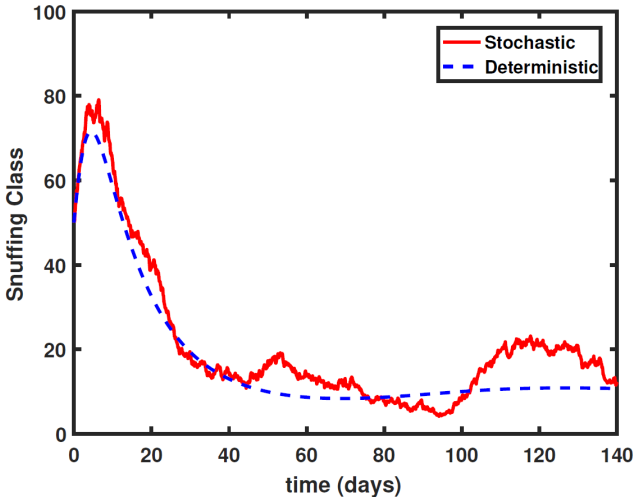
(a) Numerical results of stochastic model

(b) Numerical results of deterministic model

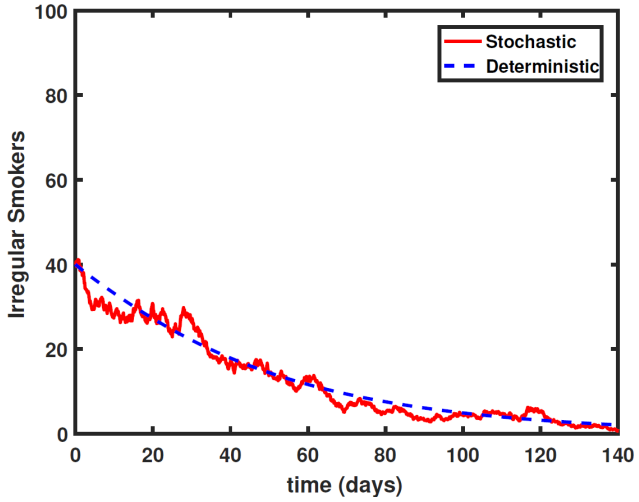
Figure 1. *Simulation results of susceptible, snuffing individuals, casual smokers, chain and quit smokers for the stochastic and deterministic models of system (2).*



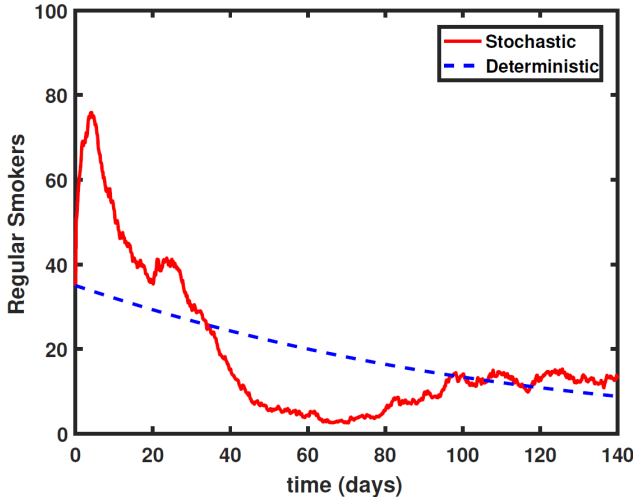
(a) $V(t)-$Susceptible smokers



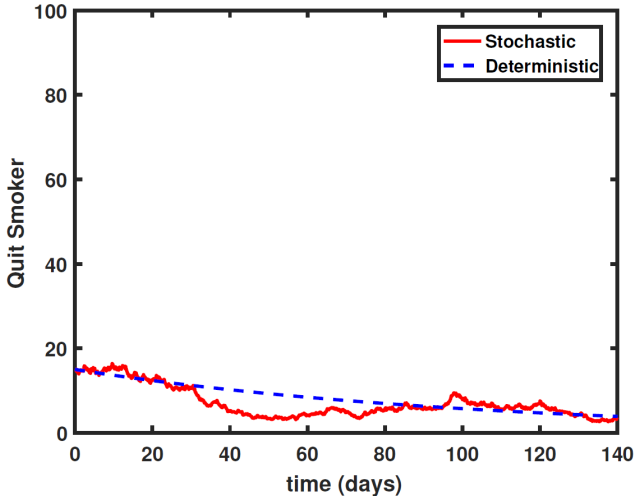
(b) $W(t)-$Snuffing Population



(c) $X(t)-$Irregular smokers

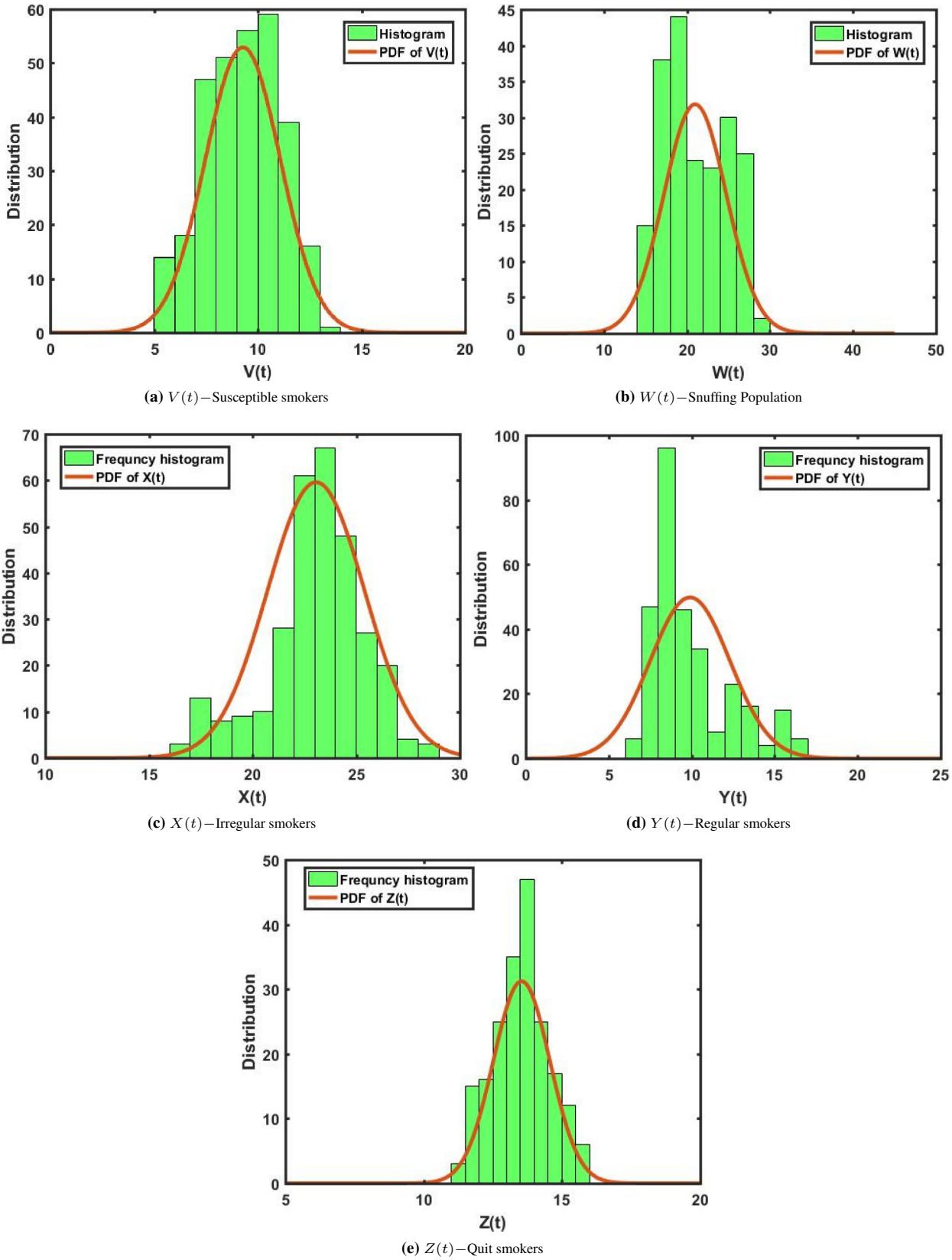


(d) $Y(t)-$Regular smokers



(e) $Z(t)-$Quit smokers

Figure 2. *Simulations of susceptible, snuffing individuals, casual smokers, chain and quit smokers for the stochastic models (2) with its corresponding deterministic version.*



(a) $V(t)-$Susceptible smokers

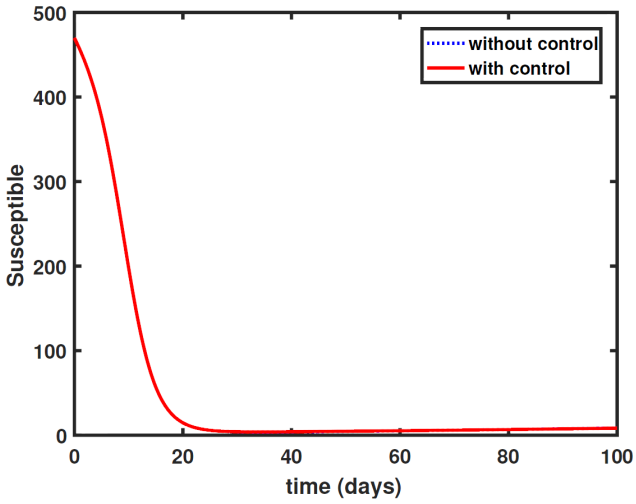
(b) $W(t)-$Snuffing Population

(c) $X(t)-$Irregular smokers

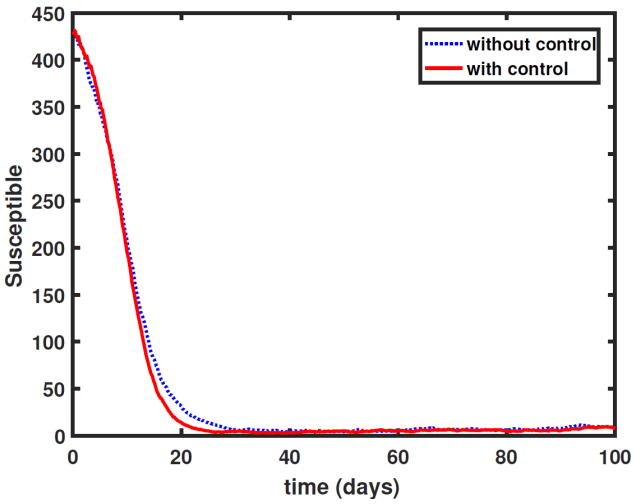
(d) $Y(t)-$Regular smokers

(e) $Z(t)-$Quit smokers

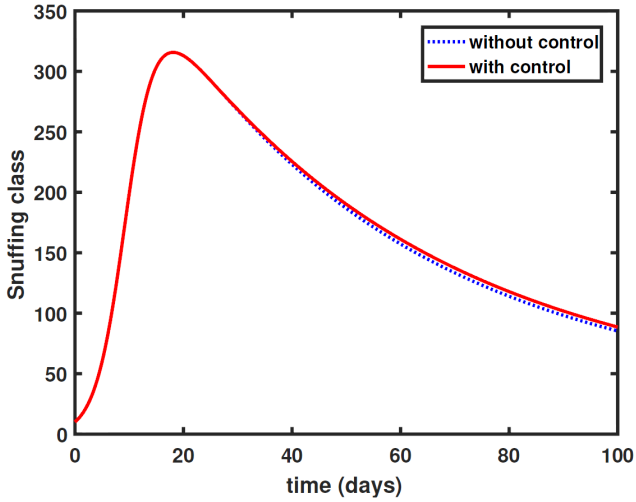
Figure 3. *The histogram and the probability of susceptible, snuffing individuals, casual smokers, chain and quit smokers for the stochastic model (2).*



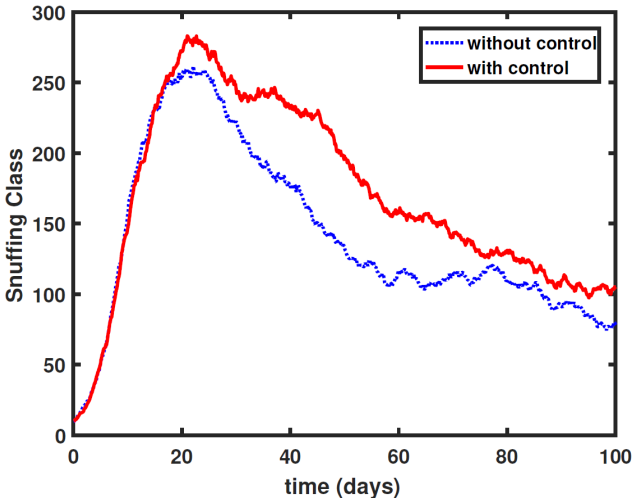
(a) The variation of susceptible smokers of deterministic model with and without control



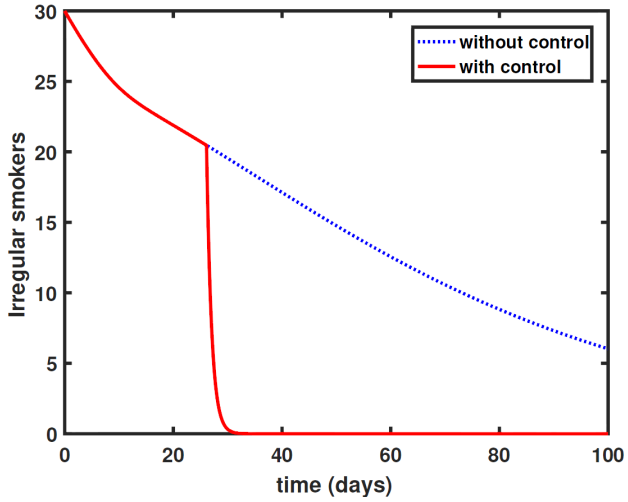
(b) The variation of susceptible smokers of stochastic model with and without control



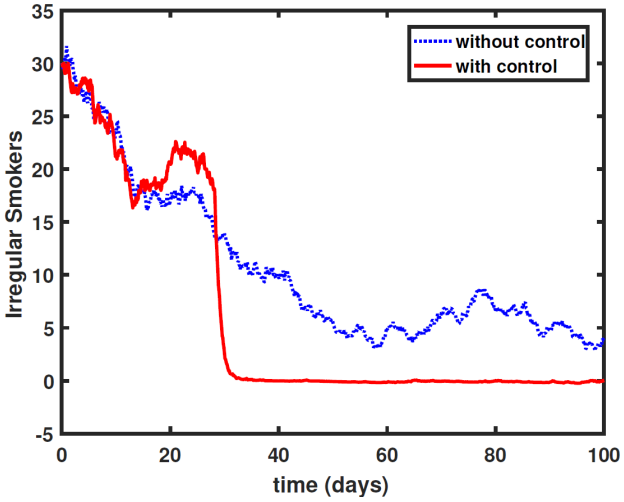
(c) The variation of Snuffing population of deterministic model with and without control



(d) The variation of Snuffing population of stochastic model with and without control

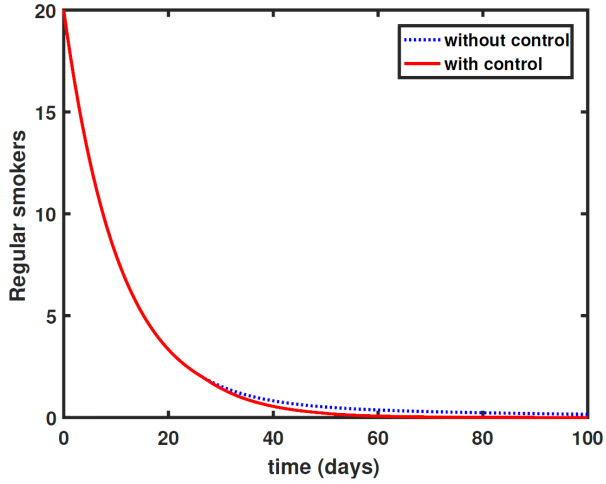


(e) The variation of irregular smokers of deterministic model with and without control

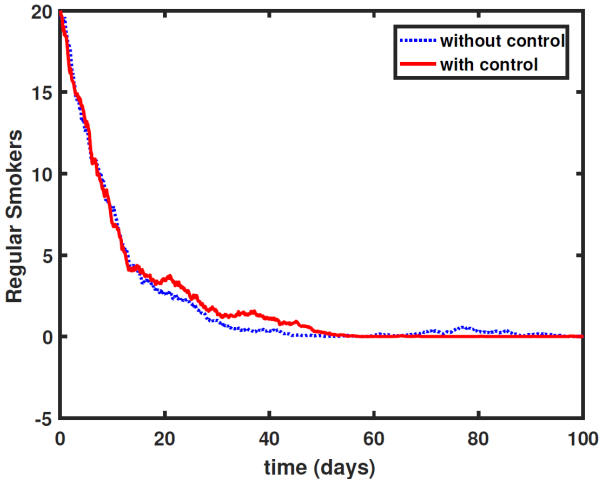


(f) The variation of irregular smokers of stochastic model with and without control

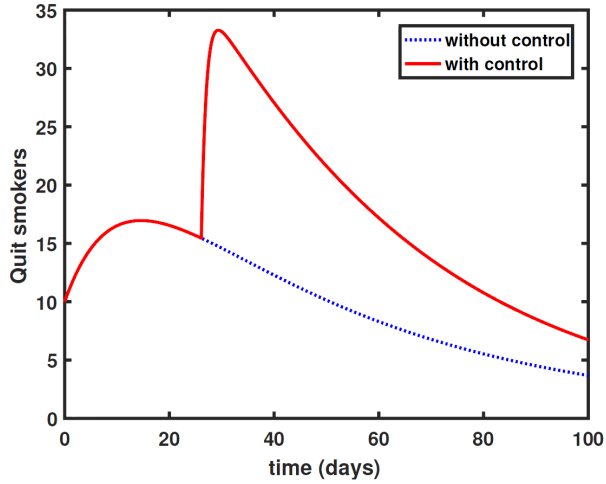
Figure 4. *Numerical results of susceptible smokers, snuffing class and irregular smokers for the stochastic models (2) with its corresponding deterministic version, under the regulation with and without control.*



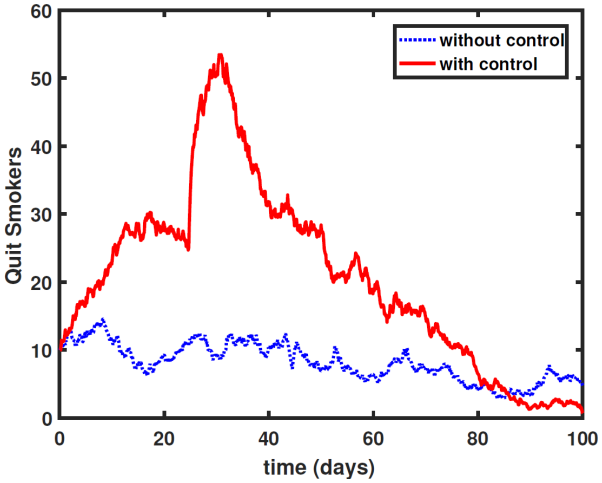
(a) Regular smokers in case of deterministic model both with and without control



(b) Regular smokers in case of stochastic model both with and without control

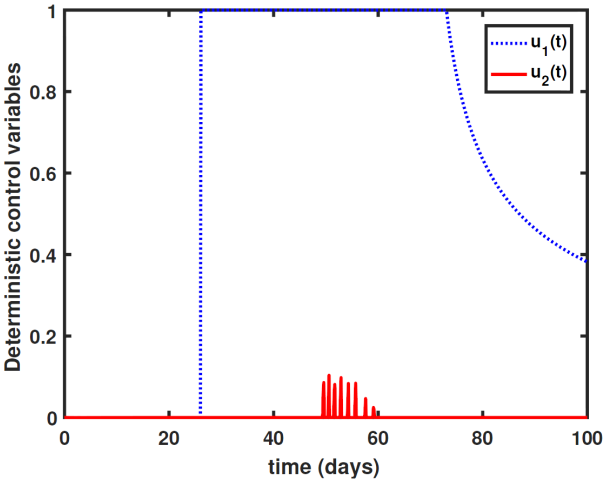


(c) Quit smokers in case of deterministic model both with and without control

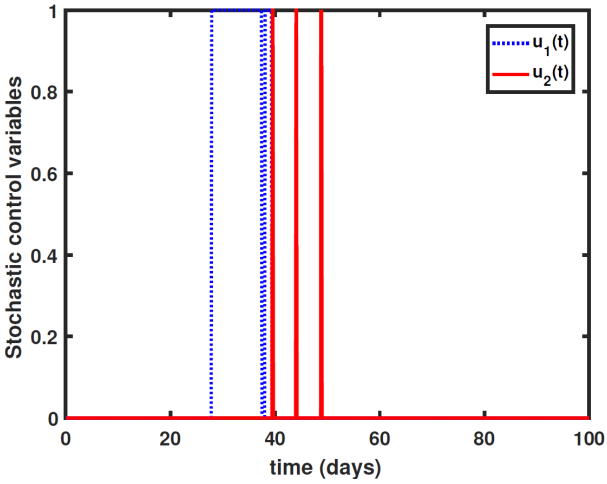


(d) Quit smokers in case of stochastic model both with and without control

Figure 5. *Numerical results of regular smokers and quit smokers for the stochastic models (2) with its corresponding deterministic version, both with and without controls.*



(a) Trajectories of optimal control variables $u_1(t)$ and $u_2(t)$ of deterministic system.



(b) Trajectories of optimal control variables $u_1(t)$ and $u_2(t)$ of stochastic system.

Figure 6. *Simulations of control variables $u_1(t)$ and $u_2(t)$ for the stochastic model (2) and its corresponding deterministic version.*

8.2. Numerical Simulations for Stochastic Optimal Control

In this part, we will numerically verify the effectiveness of the proposed optimal control strategy for system (2). Numerical simulations are carried out also by using the above fourth-order Runge-Kutta scheme. Numerical results are demonstrated in Figure 4 and Figure 5, where Figures 4a, 4c, 4e, respectively represent the dynamic curves of susceptible smokers, snuffing class and irregular smokers with and without the parameter values for the optimal control of the corresponding deterministic version of model (2). We can see that the simulations well explain the results by the above theoretical predictions: The infected and susceptible tend to decrease whereas the recovered and vaccinated individuals increases. Figure 5 shows the results for regular smokers and quit smokers in the cases with and without controls. Figures 5a and 5c, demonstrate results for regular smokers and quit smokers in the case with and without (optimal) control of the corresponding deterministic version of model (2). In addition, Figure 6a shows the dynamic process of the optimal control variables also in the deterministic version. From these numerical results, we can observe the clear differences between the two cases with and without controls, indicating the effects of our proposed control strategies. The optimal strategy can be achieved by approximately solving state and adjoint equations. In this solving process, one must keep in mind the transversality conditions. The method is simply stated as follows. First, the state system(73) is solved using the above numerical scheme. After that, according to the current iterations of the states, the corresponding adjoint system (73) is solved by backward techniques along with transversality relations (74). Then, the control measures are modified with the help of convex combinations of the preceding control(s) and with the parameter values from the new characterization (76). The algorithm is repeated until we find the values of unknown parameters that are sufficiently close to the one obtained in the previous iteration. The parameters as well as the noise intensities are shown in Table 2(S_3), the initial state variables value are also presented in Table 2 (S_3). Figure 4, Figure 5, Figure 6 illustrate that the decrease of the disease is proportional to the increase of the treatment in the infected compartment. The simulations carried out for the optimal control under strategies 1 and 2 graphically reflected in Figures 4b, 4d, 4f. One can see respective sharp decreases in susceptible smokers, snuffing class and irregular smokers. We point out that this decrease is due to the optimal control strategy. Figures 5b and 5d suggest that the regular smokers and quit smokers in model (2) are increasing. In addition, Figure 6b demonstrates the dynamic process of the optimal control variables in model (2), which clearly show the differences between control and no control cases.

9. Discussion

It is a universal fact that smoking is harmful to health. It not only damages the health of smokers, but also destroys the health of passive smokers, and hence the hazard even waves across society. In this manuscript, we deeply investigate the impact of smoking cessation and smoking cessation treatment on the control of smoking, and give a clear comparison between the controlled and uncontrolled model. By carrying out the numerical experiments, it is obvious that under the high contact rate of transmittable contacts between snuffing class and susceptible the number of new smokers rise at a femtoliter speed. So, with the human interventions (with control) the number of smokers decreased dramatically, even it can approach a situation without smoking. In this situation, the solution of our system oscillates more and more closely around the smoking-free equilibrium state. In another aspect, by controlling for some parameters, the number of smokers could be stabilized in another stable state of the model (non-zero state). Namely, in this case, the solution of the system will oscillate more and more closely around the smoking-present equilibrium state. The control of relevant parameters can be specifically implemented by applying some of the laws established by public authorities. For example, the explicit setting of some smoking free places could effectively reduce the transmission risk of smoking. In addition, in the future work, the authors intend to extend the theory to age-structured stochastic epidemic models where both controlled and without control problems.

10. Conclusion

This paper presented a mathematical formulation for theoretical analysis of stochastic smoking models. The necessary conditions for extinction and persistence were derived. Although the proposed stochastic model's threshold is debated when the noise is weak or strong, we showed that smokers will tend to extinct if $\mathbf{R_0} < 1$. With the help of stochastic Lyapunov tools, we also derived sufficient conditions for the existence and uniqueness of an ergodic stationary distribution for the positive solution, and showed that the smoking will persist in a population if $\mathbb{R}_0 > 1$. To control the spread of smoking habit in the population via some external measures, we used the optimal control theory to analyze the stochastic model with and without control. We carried out numerical simulations by using the stochastic Runge-Kutta method of order 4, to support the obtained theoretical results. We showed that the proposed control strategy can prevent and even control the smoking in an effective and low-cost way. In the future, the authors intend to extend the theory established here and the results obtained here to age-structured stochastic epidemic models with or without controls.

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Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Authors Contribution

All authors carried out the proof. All authors conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

Competing Interests

All authors declare that they have no competing interests.

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