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The Cordiality of the Join and Union of the Second Power of Fans

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Abstract: A graph is called cordial if it has a 0-1 labeling that satisfies certain conditions. A second power of a fan F_n^2 is the join of the null graph N_1 and the second power of path P_n^2, i.e. $F_n^2 = N_1 + P_n^2$. In this paper, we study the cordiality of the join and union of pairs of the second power of fans. and give the necessary and sufficient conditions that the join of two second powers of fans is cordial. we extend these results to investigate the cordiality of the join and the union of pairs of the second power of fans. Similar study is given for the union of such second power of fans. AMS Classification: 05C78.

Keywords: Join Graph, Second Power Graph, Cordial Graph

1. Introduction

Labeling problem is one of the most important ones in graph theory, and an excellent reference on this subject is the survey by Gallian [5]. Two well known labelings are graceful and harmonious. Graceful labelings were introduced independently by Rosa [8] and Golomb [6], while harmonious labelings were first studied by Graham and Sloane [7]. The concept of cordial labeling which contains aspects of both was introduced by Cahit [1].

Let $G = (V,E)$ be a graph. A binary vertex labeling is a function f: $V \rightarrow 0,1$ and a binary edge labeling is a function f^*: $E \rightarrow 0,1$ such that $f^*(vw) = (f(v) + f(w))$ (mod2)

Where $e = uv \in E$. Thus, $f^*(e) = 0$ if its two end vertices have the same label and $f^*(e) = 1$ if they have different labels. Let v_0 and v_1 be the numbers of vertices of G labeled 0 and 1 under f, respectively, and let e_0 and e_1 be the corresponding numbers of edge of G under f^*. Such a labeling is called cordial if both $|v_0 - v_1| \leq 1$ and $|e_0 - e_1| \leq 1$ hold. A graph is called cordial if it has a cordial labeling. Given two disjoint graphs G and H, their union $G \cup H$ is simply the unions of their sets of vertices and edges, while their join $G + H$ is obtained from $G \cup H$ togethers with all edges joining vertices of G and vertices of H. If G is undirected graph G^2 is a different graph that has the same vertices of G, but in which two vertices are adjacent if their distance in G is at most 2. Here, we consider only simple, connected, and undirected graph. A second power of a fan F_n^2 is the graph obtained from the join of the null graph N_1 and the second power of a path P_n^2, i.e. $F_n^2 = N_1 + P_n^2$. So the order of F_n^2 is $n + 1$ and its size is $3n - 3$. Obviously, $F_1^2 = P_2$, $F_2^2 = C_3$ and. $F_3^2 = K_4$ Many researches have been interested with cordial labeling of different kind graphs[1-4].

In this paper we investigate the cordiality of the join and union of pairs of second power of fans. In section 3, we show that F_n^2 is cordial if and only if $n \neq 3$. In section 4, we show that the join $F_n^2 + F_m^2$ is cordial for n and m if and only if $(n,m) \neq (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (2, 4), (3,1), (3, 2), (3, 3), (3, 4), (4, 3), (4, 4)$.

Moreover, we show that the union $F_n^2 \cup F_m^2$ is cordial for n and all m (or vice versa).

If and only if $(n, m) \neq (1, 1), (2, 2), (2, 4), (4, 2), (4, 4)$.

2. Terminology and Notations

Let us first introduce some convenient notations for graphs with $2r$ vertices; we let M_{2r} denote the labeling 01...01 (repeated r times), let M'_{2r} denote the labeling 10...10 (repeated r times). Sometimes we modify this by adding symbols at one end or the other (or both). Thus $0M_{2r}$ denoted the labeling 0 01 ... 01. When we label a fan or second power of a fan, we always start labelling the center, i.e N_1. For specific labeling L and M of the graphs G and H, respectively, we let $[L;M]$ denote the labeling of the join $G + H$ or the union $G \cup H$.

Additional notation that we use is the following.

For a given labeling of the join $G+H$ or union $G \cup H$, we let vi and ei for $i = 0,1$ be the numbers of labels that are i as before, we let x_i and a_i be the corresponding quantities for G, and we let y_i and b_i be those for H.

It follows that $v_0 = x_0 + y_0$, $v_1 = x_1 + y_1$, $e_0 = a_0 + b_0 + x_0y_0 + x_1y_1$ and $e_1 = a_1 + b_1 + x_0y_1 + x_1y_0$,thus, $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$. These equations will be used throughout to verify the conditions $|v_0 - v_1| \leq 1$ and $|e_0 - e_1| \leq 1$ in order to verify the cordiality of $G + H$ or $G \cup H$, we use the last two equations and show that $|v_0 - v_1| \leq 1$ and $|e_0 - e_1| \leq 1$.

3. The Cordiality of Second Power of Fans

In this section we study the cordiality of the second power of Fans F_n^2.

Theorem 3.1. The second power of fan F_n^2 is cordial if and only if $n \neq 3$.

Proof. Let us proof the necessary condition by transform its implication to its contrapositive. If $n = 3$, then $F_3^2 \equiv K_4$ which is not cordial [1]. To demonstrate the converse, two things immediately clear $F_1^2 \equiv P_2$ (n=1) and $F_2^2 \equiv C_3$ (n=2) both are cordial [2]. What is not clear is the case at $n \geq 4$. We claim that F_n^2 is cordial $\forall n \geq 4$. For this task, we study the following four cases.

Case(1) $n \equiv 0(mod\ 4)$.

Let $n = 4r$ and $r \geq 1$, then one can chose the label $1M_{4r}$ for F_{4r}^2. It is easy to verify that $x_o = 2r$, $x_1 = 2r{+}1$, $a_0 = 6r{-}2$ and $a_1 = 6r{-}1$. Therefore $x_o - x_1 = -1$ and $a_o - a_1 = -1$.

As an example, "Figure 1" illustrates F_8^2.

Case(2) $n \equiv 1(mod\ 4)$.

Let $n = 4r + 1$ and $r \geq 1$, then we label the vertices of F^2_{4r+1} by $10_31_2M'_{4r-4}$. It is easy to verify that $x_o = 2r + 1$, $x_1 = 2r + 1$, $a_0 = a_1 = 6r$. Therefore $x_o - x_1 = 0$ and $a_o - a_1 = 0$. As an example, "Figure 2" illustrates F_9^2.

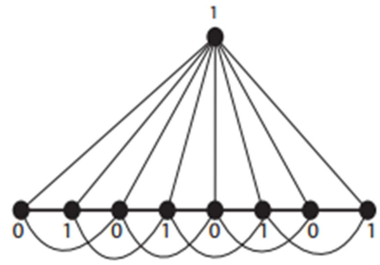
Case(3) $n \equiv 2(mod\ 4)$.

Let $n = 4r + 2$ and $r \geq 1$, then we label the vertices F^2_{4r+2} by $1M_{4r+2}$. It is easy to verify that $x_o = 2r + 1$, $x_1 = 2r + 2$,$a_0 = 6r + 1$ and $a_1 = 6r + 2$. Therefore $x_o - x_1 = -1$ and $a_o - a_1 = -1$. As an example, "Figure 3" illustrates F_{10}^2.

Case(4) $n \equiv 3(mod\ 4)$.

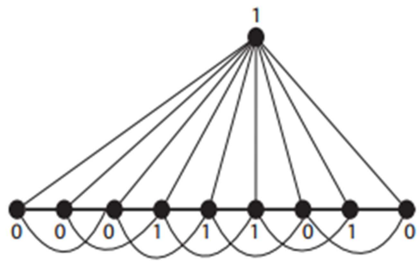
Let $n = 4r + 3$ and $r \geq 1$, then we label the vertices F^2_{4r+3} by $10_31_2M'_{4r-2}$. It is easy to verify that $x_o = x_1 = 2r{+}2$, $a_0 = a_1 = 6r{+}3$. Therefore $x_o - x_1 = 0$ and $a_o - a_1 = 0$.

As an example, "Figure 4" illustrates F_{11}^2. Thus F_n^2 is cordial for all n except at $n = 3$, and the theorem follows.



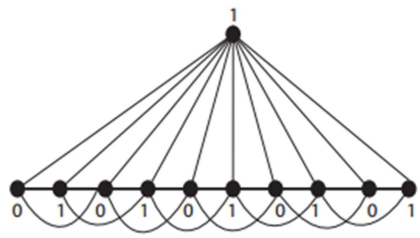
$x_0 = 4$, $x_1 = 5$, $a_0 = 10$,$a_1 = 11$, $v_0 - v_1 = -1$, $e_0 - e_1 = -1$

Figure 1. F_8^2.



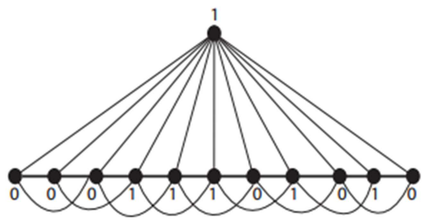
$x_0 = 5$, $x_1 = 5$, $a_0 = 12$,$a_1 = 12$, $v_0 - v_1 = 0$, $e_0 - e_1 = 0$

Figure 2. F_9^2.



$x_0 = 5$, $x_1 = 6$, $a_0 = 13$,$a_1 = 14$, $v_0 - v_1 = -1$, $e_0 - e_1 = -1$

Figure 3. F_{10}^2.



$x_0 = x_1 = 6$, $a_0 = a_1 = 15$, $v_0 - v_1 = 0$, $e_0 - e_1 = 0$

Figure 4. F_{11}^2.

4. Joins of a Pairs of Second Power of Fans

In this section, we show that $F_n^{2} + F_m^2$ is cordial for n and m if and only if $(n,m) \neq (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 3), (4, 4)$.

We reach our task through the following series of lemmas.

Lemma 4.1. The join $F_n^2{+}F_m^2$ is cordial for all $n > 4$ and $m > 4$.

Proof. For given values of i and j with $0 \leq i \leq 3$ and $0 \leq j \leq 3$, we use the labeljng A_i or A'_i or A''_i.

For $F_n^2$and B_j or B or B''_j for F_m^2 as given in "Table 1". Using this table and the fact that $v_0 - v_1 = (x_0{-}x_1){+}(y_0{-}y_1)$ and also,

$e_0-e_1 = (a_0-a_1)+(b_0-b_1)+(x_0-x_1)(y_0-y_1)$, we can compute the values shown in the last two columns of "Table 2"Since these are all either 0, 1, or −1, the lemma follows.

Table 1. *Labeling of second power of Fans.*

n=4r+i, i=0, 1, 2, 3	Labeling of F_n^2	x_0	x_1	a_0	a_1
	$A_0 = 1M_{4r}$, r > 1	2r	2r+1	6r-2	6r-1
i = 0	or $A'_0 = 0_4 1_3 M_{4r-6}$, r > 1	2r+1	2r	6r-1	6r-2
	or $A''_0 = 0M_{4r}$, r > 1	2r+1	2r	6r-2	6r-1
i = 1	$A_1 = 10_3 1_2 M'_{4r-4}$, r ≥ 1	2r+1	2r+1	6r	6r
	$A_2 = 1M_{4r+2}$, r ≥ 1	2r+1	2r+2	6r+1	6r+2
i = 2	or $A'_2 = 0_4 1_3 M_{4r-4}$, r > 1	2r+2	2r+1	6r+2	6r+1
	or $A''_2 = 0M_{4r+2}$, r ≥ 1	2r+2	2r+1	6r+1	6r+2
i = 3	$A_3 = 10_3 1_2 M'_{4r-2}$, r ≥ 1	2r+2	2r+2	6r+3	6r+3

Table 2. *Continued.*

m=4s+j, j=0,1,2,3	Labeling ofF_m^2	y_0	y_1	b_0	b_1
j = 0	$B_0 = 0_4 1_3 M_{4s-6}$, s > 1	2s+1	2s	6s-1	6s-2
	or $B'_0 = 1M_{4s}$, s > 1	2s	2s+1	6s-2	6s-1
j = 1	$B_1 = 10_3 1_2 M'_{4s-4}$, s ≥ 1	2s+1	2s+1	6s	6s
	$B_2 = 0_4 1_3 M_{4s-4}$, s > 1	2s+2	2s+1	6s+2	6s+1
j = 2	or $B'_2 = 0M_{4s+2}$, s ≥ 1	2s+2	2s+1	6s+1	6s+2
	or $B''_2 = 1M_{4s+2}$, s ≥ 1	2s+1	2s+2	6s+3	6s+3

Table 2. *Combinations of labeling.*

n = 4r+i, i = 0, 1, 2, 3	m = 4s+j, j = 0, 1, 2, 3	F_n^2	F_m^2	$v_0 - v_1$	$e_0 - e_1$
0	0	A_0	B_0	0	-1
0	1	A_0	B_1	-1	0
0	2	A_0	B_2	0	-1
0	3	A_0	B_3	-1	-1
1	0	A_1	B'_0	-1	-1
1	1	A_1	B_1	0	0
1	2	A_1	B'_2	1	-1
1	3	A_1	B_3	0	0
2	0	A'_2	B'_0	0	-1
2	1	A''_2	B_1	1	-1
2	2	A_2	B_2	0	-1
2	3	A_2	B_3	-1	-1
3	0	A_3	B'_0	-1	-1
3	1	A_3	B_1	0	0
3	2	A_3	B''_2	-1	-1
3	3	A_3	B_3	0	0

Lemma 4.2. $F_4^2+F_m^2$ is cordial for all m > 4.

Proof. We select the labeling $1_3 0_2$ for F_4^2 i.e., $xo = 2$, $x1 = 3$, $a_0 = 4$ and $a_1 = 5$. Now let $m = 4s + j$, where $j = 0,1,2,3$, then for the label of the vertices of F_m^2,

Where $m > 4$, we need to study the following four cases.

Case(1) j = 0.

We choose the labeling $0_4 1_3 M_{4s-6}$ for F_{4s}^2,where s > 1. Then $y_o = 2s+1$, $y_1 = 2s$, $b_0 = 6s-1$.

And $b_1 = 6s-2$. and hence $v_o - v_1 = 0$ and $e_0 - e_1 = -1$. Thus $F_4^2+F_{4s}^2$ is cordial for all s > 1. As an example, Figure 5 illustrates $F_4^2+F_8^2$.

Case(2) j = 1.

We choose the labeling 0413 M′4s−4 for F24s+1 where s ≥ 1. It is easy to verify that yo = y1 = 2s + 1, b0 = b1 = 6s. Therefore $v_o - v_1 = -1$ and $e_o - e_1 = -1$.Thus $F_4^2+F^2_{4s+1}$ is cordial for all s ≥ 1. As an example, Figure 6 illustrates $F_{4s}^2+F_9^2$.

Case(3) j = 2.

We choose the labeling $0_4 1_3 M_{4s-4}$ for F^2_{4s+2},where $s > 1$. Then $y_o = 2s + 2$, $y_1 = 2s + 1$,

$b_0 = 6s + 2$ and $b_1 = 6s + 1$. Therefore $v_o - v_1 = 0$ and $e_o - e_1 = -1$. At $s = 1$, we label the vertices of F_6^2 by 1000011, i.e., $y_o = 4$, $y_1 = 3$, $b_0 = 8$ and $b_1 = 7$. Therefore $v_o - v_1 = 0$ and $e_o - e_1 = -1$. Thus $F_4^2+F^2_{4s+2}$ is cordial for all $s \geq 1$. As an example, Figure 7 illustrates $F_4^2+F_6^2$.

Case(4) j = 3.

We choose the labeling $10_4 1_2 M'_{4s-2}$ for F^2_{4s+3} where $s \geq 1$. It is easy to verify that $y_o = y_1 = 2s + 2$, $b_0 = b_1 = 6s + 3$ and consequently, $v_o - v_1 = -1$ and $e_o - e_1 = -1$.

Thus $F_4^2+F^2_{4s+3}$ is cordial for all $s \geq 1$. As an example, Figure 8 illustrates $F_4^2 + F_{11}^2$.

Thus $F_4^2 + F_m^2$ is cordial for all $m > 4$ and the lemma follows.

The graph $F_4^2 + F_1^2$ is cordial since the labelling $[0_4 1, 1_2]$ is sufficient. In a straightforward way one can see that.

$F_3^2 + F_3^2 \equiv K_4 + K_4 \equiv K_8$, $F_2^2 + F_3^2 \equiv F_3^2 + F_2^2 \equiv C_3 + K_4 \equiv K_7$

$F_3^2 + F_1^2 \equiv F_1^2 + F_3^2 \equiv P_2 + K_4 \equiv K_2^2 + K_2^2 \equiv C_3 + C_3 \equiv K_6$

$F_2^2 + F_1^2 \equiv F_1^2 + F_2^2 \equiv P_2 + C_3 \equiv K_5$ and $F_1^2 + F_1^2 \equiv P_2 + P_2 \equiv K_4$

Because of all of these facts and since the complete graph, K_n, is cordial if and only if

$n \leq 3$ [1], we conclude that:

Lemma 4.3.The graphs $F_1^2 + F_1^2$, $F_1^2 + F_2^2$, $F_1^2 + F_3^2$, $F_2^2 + F_2^2$, $F_2^2 + F_3^2$ and $F_3^2 + F_3^2$ do not have cordial labeling.

Lemma 4.4.The three graphs $F_2^2 + F_4^2$, $F_3^2 + F_4^2$ and $F_4^2 + F_4^2$ do not have cordial labeling.

Proof.We shall only prove the first claim, and the other two can be proved similarly with slight modifications.To satisfy the first condition of cordiality,we have to label four vertices of $F_4^2 + F_4^2$ by 0and the other four vertices by 1; otherwise $|v_0 - v_1| \geq 2$. If we do that, then to show whether or not the second condition of cordiality holds, we need to discuss all possibilities of choices of labeling. There are two independent ways of labeling. The first is to label four vertices of F_4^2 by 0(or 1) and therefore the remaining vertices of, $F_2^2 + F_4^2$ are labelled 1(or 0). In this case, it is easy to show that for any order of 0 and 1 you choose,$|e_0 - e_1| \geq 2$. The second way is to label three vertices of F_4^2 by 0(or 1) and therefore the remaining two vertices of F_4^2 are labelled 1(or 0). So, two vertices of F_2^2 must be labelled 1(or 0)and the last is labelled 0(or 0). Similarly, it is a simple matter showing that $|e_0 - e_1| \geq 2$. Hence $F_2^2 + F_4^2$ is not cordial as we wanted to prove.

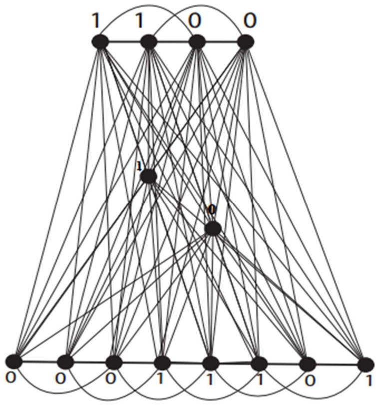


Figure 5. $F_4^2 + F_8^2$.

$x_0 = 2$, $x_1 = 3$, $a_0 = 4$, $a_1 = 5$, $y_0 = 5$, $y_1 = 4$, $b_0 = 11$, $b_1 = 10$, $v_0 - v_1 = 0$, $e_0 - e_1 = -1$.

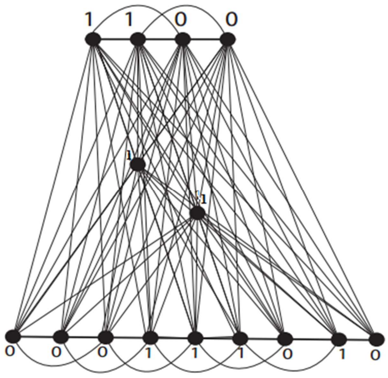


Figure 6. $F_4^2 + F_9^2$.

$x_0 = 2$, $x_1 = 3$, $a_0 = 4$, $a_1 = 5$, $y_0 = 5$, $y_1 = 5$, $b_0 = 12$, $b_1 = 12$, $v_0 - v_1 = -1$, $e_0 - e_1 = -1$.

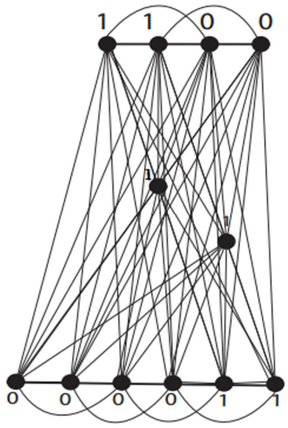


Figure 7. $F_4^2 + F_6^2$.

$x_0 = 2$, $x_1 = 3$, $a_0 = 4$, $a_1 = 5$, $y_0 = 4$, $y_1 = 3$, $b_0 = 8$, $b_1 = 7$, $v_0 - v_1 = 0$, $e_0 - e_1 = -1$.

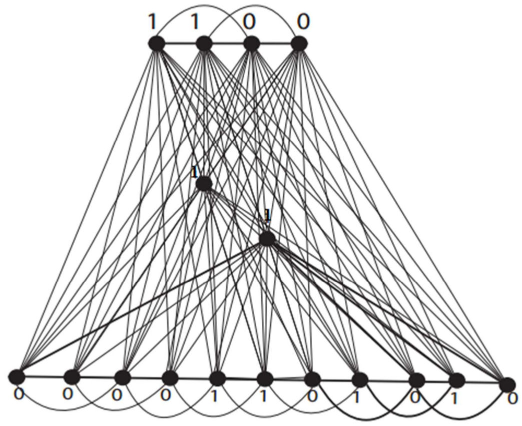


Figure 8. $F_4^2 + F_{11}^2$.

$x_0 = 2$, $x_1 = 3$, $a_0 = 4$,$a_1 = 5$,$y_0 = 6$, $y_1 = 6$, $b_0 = 15$, $b_1 = 15$, $v_0 - v_1 = -1$, $e_0 - e_1 = -1$.

As a direct consequence of all previous lemmas, we have.

Theorem 4.1. The join $F_n^2 + F_m^2$ is cordial for all n and m except at $(n,m) \neq (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 3), (3, 4)$, and $(4, 4)$.

5. Union of a Pairs of Second Power of Fans

In this section we prove that the union $F_n^2 \cup F_m^2$ is cordial for all n and all m except at $(n,m) \neq (1, 1), (2, 2), (2, 4), (4, 2), (4, 4)$. Let us start our study by investigating the cordiality of the union if $n > 4$ and $m > 4$.

Lemma 5.1. The union $F_n^2 \cup F_m^2$ of second power of Fans F_n^2 and F_m^2 is cordial for all $n > 4$ and $m > 4$.

Proof. For given values of i and j with $0 \leq i \leq 3$ and $0 \leq j \leq 3$, we use the labeling A_i or A'_i or A''_i for F_n^2 and B_j or B_j' or B_j for F_m^2 as indicated in Table 4.1. Using this table and formulae $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$, we can then compute the values shown in the last two columns of "Table 3". Since these values are all 0,1, or −1,the lemma follows.

Table 3. *Combinations of labeling.*

n = 4r + i, i = 0,1,2,3	m = 4s + j, j = 0,1,2,3	F_n^2	F_m^2	$v_0 - v_1$	$e_0 - e_1$
0	0	A_0	B_0	0	0
0	1	A_0	B_1	-1	-1
0	2	A_0	B_2	0	0
0	3	A_0	B_3	-1	-1
1	0	A_1	B'_0	-1	-1
1	1	A_1	B_1	0	0
1	2	A_1	B'_2	1	-1
1	3	A_1	B_3	0	0
2	0	A'_2	B'_0	0	0
2	1	A''_2	B_1	1	-1
2	2	A_2	B_2	0	0
2	3	A_2	B_3	-1	-1
3	0	A_3	B'_0	-1	-1
3	1	A_3	B_1	0	0
3	2	A_3	B''_2	-1	-1
3	3	A_3	B_3	0	0

Lemma 5.2. The union $F_4^2 \cup F_m^2$ is cordial for all m > 4.

Proof. Let us label the vertices of F_4^2 by 11100; then $x_o = 2$, $x_1 = 3$, $a_0 = 4$ and $a_1 = 5$. Now, let $m = 4s + j$, where j = o,1,2,3. For the label of the vertices of F_m^2, where $m > 4$, we need to study the following four cases.

Case(1) j = 0.

Choose the labeling $0_4 1_3 M_{4s-6}$ where s > 1 for the vertices of Fan

Then $y_o = 2s+1$, $y_1 = 2s$, $b_0 = 6s-1$ and $b_1 = 6s-2$. Therefore $v_o - v_1 = 0$ and $e_o - e_1 = 0$. An illustrated example is shown in Figure 9 ($F_4^2 \cup F_8^2$).

Case(2) j = 1.

We choose the labeling $0_3 1_2 M'_{4s-4}$, where s ≥ 1 for the vertices of F^2_{4s+1}. Then $y_o = y_1 = 2s + 1$, $b_0 = b_1 = 6s$. Hence $v_o\ v_1 = -1$ and $e_o - e_1 = -1$. An illustrated example is shown in Figure 10 ($F_4^2 \cup F_9^2$).

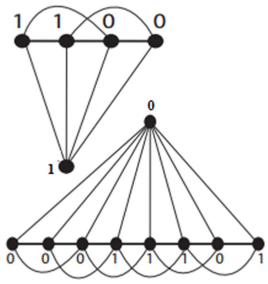
Case(3) j = 2.

We choose the labeling $0_4 1_3 M_{4s-4}$, where s > 1 for F^2_{4s+2}. Then $y_o = 2s+2$, $y_1 = 2s+1$, $b_0 = 6s+2$ and $b_1 = 6s+1$. Hence $v_o - v_1 = 0$ and $e_o - e_1 = 0$. An illustrated example is shown in Figure 11 ($F_4^2 \cup F_{10}^2$).

At s = 1, we label the vertices of F_6^2 by 1000011, so $y_o = 4$, $y_1 = 3$, $b_0 = 8$ and $b_1 = 7$. Therefore $v_o - v_1 = 0$ and $e_o - e_1 = 0$.

Case (4) j = 3.

We choose the labeling $1 0_3 1_2 M'_{4s-2}$ for F^2_{4s+3}. Then $y_o = y_1 = 2s+2$, $b_0 = b_1 = 6s+3$. Hence $v_o - v_1 = -1$ and $e_o - e_1 = -1$. An illustrated example is shown in Figure 12 ($F_4^2 \cup F_{11}^2$). Thus $F_4^2 \cup F_m^2$ is cordial for all $m > 4$ and the lemma follows.



$x_0 = 2$, $x_1 = 3$, $a_0 = 4$, $a_1 = 5$, $y_0 = 5$, $y_1 = 4$, $b_0 = 11$, $b_1 = 10$, $v_0 - v_1 = 0$, $e_o - e_1 = 0$.

Figure 9. $F_4^2 \cup F_8^2$.

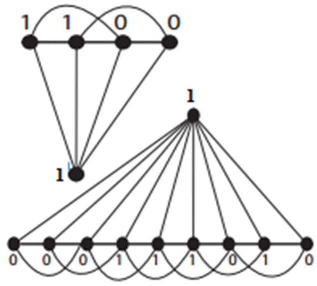


Figure 10. $F_4^2 \cup F_9^2$.

$x_0 = 2$, $x_1 = 3$, $a_0 = 4$, $a_1 = 5$, $y_0 = 5$, $y_1 = 5$, $b_0 = 12$, $b_1 = 12$, $v_0 - v_1 = -1$, $e_o - e_1 = -1$.

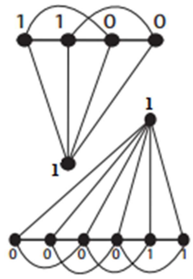


Figure 11. $F_4^2 \cup F_6^2$.

$x_0 = 2$, $x_1 = 3$, $a_0 = 4$, $a_1 = 5$, $y_0 = 4$, $y_1 = 3$, $b_0 = 8$, $b_1 = 7$, $v_0 - v_1 = 0$, $e_o - e_1 = 0$.

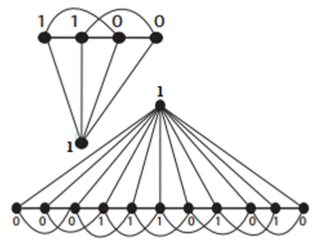


Figure 12. $F_4^2 \cup F_{11}^2$.

$x_0 = 2$, $x_1 = 3$, $a_0 = 4$, $a_1 = 5$, $y_0 = 6$, $y_1 = 6$, $b_0 = 15$, $b_1 = 15$, $v_0 - v_1 = -1$, $e_o - e_1 = -1$.

From [2, 3], $P_n \cup P_m$ is cordial for all n and m except the case $2P_2$.

And also $C_n \cup C_m$ if and only if $n+m$ is not congruent to $2(mod4)$.

As a consequence of these facts and since $F_1^2 \equiv P_2$ and $F_2^2 \equiv C_3$, the following is an obvious result.

Corollary 5.1. The graphs $F_1^2 \cup F_1^2$ and $F_2^2 \cup F_2^2$ are not cordial.

Lemma 5.3. The graphs $F_2^2 \cup F_4^2$ and $F_4^2 \cup F_4^2$ are not cordial.

Proof.We shall only prove the second claim, and the second is proved similarly.

To satisfy the first condition of cordiality,we have to label five vertices of $F_4^2 \cup F_4^2$ by 0 and the other five vertices by 1;otherwise $|v_0 - v_1| \geq 2$.If we do that,then to show whether or not the second condition of cordiality holds, we need to discuss all possibilities of choices of labeling. The graph $F_2^2 \cup F_4^2$ is the union of two similar pieces which is F_4^2. One way of labelling the first piece is to take three of its vertices with label 0(or 1) and the other two vertices are labelled 1(or0). Consequently, the labelling of the second piece must be exchanged,i.e,three of its vertices must be labelled 1(or 0) and the other two vertices are labelled 0(or1). In this case, it is easy to show that for any order of 0 and 1 you choose, $|e_0 - e_1| \leq 1$. By a routine work, one can show that, any other choice of labelling will not satisfy the second condition of cordiality.

Hence $F_4^2 \cup F_4^2$ is not cordial, and the lemma follows.

Now we come to the final lemma.

Lemma 5.4. If $n, m \leq 4$ and (n, m) is not one of $(1, 1)$,$(2, 2)$,$(2, 4)$,$(4, 2)$ and $(4, 4)$. Then $F_n^2 \cup F_m^2$, is cordial.

Proof. Looking at the last lemma and corollary, we need only.

To verify the following cases;

Case(1) n = 1.

$F_1^2 \cup F_m^2$ will be examined for all $m \leq 4$. Obviously, The graph $F_1^2 \cup F_2^2 \equiv P_2^2 \cup P_3^2$ is cordial since $P_n^2 \cup P_m^2$ is so for all n and m except at $2P_2$ and $2P_3$ [4]. The labelings [11;0001] and [11;00011] are sufficient for the graphs $F_1^2 \cup F_3^2$ and $F_1^2 \cup F_4^2$, respectively.

Case(2) n = 2.

The graph $F_2^2 \cup F_m^2$ will be examined for all $m \leq 4$. Firstly, $F_2^2 \cup F_1^2 \equiv F_1^2 \cup F_2^2$ is cordial from the above case. For the graph $F_2^2 \cup F_3^2$ the labeling: [011;0001] is sufficent.

Case(3) n = 3.

The graphs $F_3^2 \cup F_m^2$ will be examined for all $m \leq 4$. Since $F_3^2 \cup F_1^2 \equiv F_1^2 \cup F_3^2$ and $F_3^2 \cup F_2^2 \equiv F_2^2 \cup F_3^2$, $F_3^2 \cup F_1^2$ and $F_3^2 \cup F_2^2$ are cordial.

For the graphs $F_3^2 \cup F_3^2$ and $F_3^2 \cup F_4^2$ the labelings [0001;1110] and: [0001;01111], respectively,are sufficent.

Case(4) n = 4.

The graphs $F_4^2 \cup F_m^2$ for all $m \leq 4$ will be examined. Since $F_4^2 \cup F_1^2 \equiv F_1^2 \cup F_4^2$,

and $F_4^2 \cup F_3^2 \equiv F_3^2 \cup F_4^2$, $F_4^2 \cup F_1^2$ and $F_4^2 \cup F_3^2$ are both cordial as shown in the last two cases.

From the above results, we can establish the following theorem.

Theorem 5.1. The union $F_n^2 \cup F_m^2$ is cordial for all n and all m if and only if

$(n,m) \neq (1, 1), (2, 2), (2, 4), (4, 2), (4, 4)$.

6. Conclusion

We show that the second power of fans F_n^2 is cordial if and only if $n \neq 3$. and that the join $F_n^2 + F_m^2$ is cordial for all n *and m* if and only if

$(n,m) \neq (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 3), (4, 4)$.

Moreover, we show that the union $F_n^2 \cup F_m^2$ is cordial for all n and m if and only if

$(n,m) \neq (1, 1), (2, 2), (2, 4), (4, 2), (4, 4)$.

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