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# General Distance Energies and General Distance Estrada Index of Random Graphs

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**Abstract:** In 2000s, Gutman and Güngör introduced the concept of distance energy and the distance Estrada index for a simple graph *G* respectively. Moreover, many researchers established a large number of upper and lower bounds for these two invariants. But there are only a few graphs attaining the equalities of those bounds. In this paper, however, the exact estimates to general distance energy are formulated for almost all graphs by probabilistic and algebraic approaches. The bounds to general distance Estrada index are also established for almost all graphs by probabilistic and algebraic approaches. The results of this paper generalize the results of the distance energy and distance Estrada of random graph.

**Keywords:** E-R Random Graph, General Distance Matrix, General Distance Energy, General Distance Estrada Index

## 1. Introduction

Let $G$ be a simple connected graph with vertex set $\{v_1, v_2, \cdots v_n\}$. Denote by $A = A(G) \in \mathbb{R}^{n\times n}$ the adjacency matrix of $G$, and $\lambda_1, \lambda_2, \cdots \lambda_n$ are eigenvalues of $A(G)$. The distance between vertices $v_i$ and $v_j$ of $G$, denoted by $d_{ij}$, is defined to be the length (i.e., the number of edges) of the shortest path from $v_i$ to $v_j$. The distance matrix of $G$, denoted by $D(G)$, is the $n\times n$ matrix whose $(i, j)$-entry is equal to $d_{ij}$ $(i, j = 1, 2, \ldots, n)$, (see [1], [2]). Note that $d_{ii} = 0,\ i = 1, 2, \ldots, n$. The eigenvalues of $D(G)$ are said to be the $D$-eigenvalues of $G$. Since $D(G)$ is a real symmetric matrix, the $D$-eigenvalues are real and can be ordered in non-increasing order, $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ The Harary matrix $H(G)$ of $G$ (see [3]), which can be regarded as a generalization of $D(G)$ and is initially called reciprocal distance matrix, is a $n\times n$ matrix whose $(i, j)$-entry is equal to $\frac{1}{d_{ij}}$ if $i \neq j$ and $0$ otherwise. The eigenvalues of the Harary matrix $H(G)$ are denoted by $\beta_1, \beta_2, \ldots, \beta_n$ and are said to be the $H$-eigenvalues of $G$. Since the Harary matrix is symmetric, its eigenvalues are real and can be ordered as

$$\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n.$$

For the Hückel molecular orbital approximation, the total π-electron energy in conjugated hydrocarbons is given by the sum of absolute values of the eigenvalues corresponding to the molecular graph $G$ in which the maximum degree is not more than three in general. In 1970s, Gutman [4] extended the concept of energy to all simple graphs $G$, and defined the energy of $G$ as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

In theoretical chemistry, the energy of a given molecular graph is related to the total π-electron energy [5] of the molecule represented by that graph. The graph energy has been studied extensively by many mathematicians and chemists, and many results have been obtained on this invariant of graphs (see [6]).

The distance energy, $D\mathcal{E}(G)$, of $G$ is defined as

$$D\mathcal{E}(G) = \sum_{i=1}^{n} |\rho_i| \tag{1}$$

The concept of distance energy, Eq. (1), was recently introduced [7]. This definition was motivated by the much older [8] and nowadays extensively studied [9, 10, 11, 12] graph distance energy.

In [13], Estrada introduced another graph-spectrum-based invariant of graphs, which was later called the Estrada index, defined as

$$EE(G) = \sum_{i=1}^{n} e^{\lambda_i} .$$

Since then, there were various applications of the Estrada index. Initially, it was used to quantify the degree of folding of long chain polymeric molecules, especially those of proteins [14, 15]. And later, a connection between $EE(G)$ and the concept of extended atomic branching was established [16], which was an attempt to apply $EE(G)$ in quantum chemistry. In addition, Estrada and Rodríguez-Velázquez showed that $EE(G)$ provides a measure of the centrality of complex networks [17, 18]. Recently, a information-theoretical application of $EE(G)$ was put forward. Carbó-Dorca endeavored to find connections between $EE(G)$ and the Shannon entropy [19].

The distance Estrada index of a connected graph $G$ was introduced in [20] as

$$DEE(G) = \sum_{i=1}^{n} e^{\rho_i} .$$

Some upper and lower bounds for $DEE(G)$ were established in [20] and [21].

In 2010, Güngör introduced the Harary energy and Harary Estrada index of a graph [22]:

$$H\mathcal{E}(G) = \sum_{i=1}^{n} |\beta_i| , \quad HEE(G) = \sum_{i=1}^{n} e^{\beta_i} .$$

The Erdős-Rényi random graph $\mathcal{G}_n(p)$, named after Erdős and Rényi [23], consists of all graphs on n vertices in which the edges are chosen independently with probability $p$, where $p$ is a constant and $0 < p < 1$. Recently, there have been a lot of work on chemical indices of random graphs. In [24], Du et al. have considered the energy of Erdős-Rényi model $\mathcal{G}_n(p)$. They obtained that almost every graph $G_n(p)$ in $\mathcal{G}_n(p)$ satisfies

$$\mathcal{E}(G_n(p)) = \left( \frac{8}{3\pi} \sqrt{p(1-p)} + o(1) \right) n^{3/2} \quad a.s. \tag{2}$$

In [25], Chen et al. have considered the Estrada index of Erdős-Rényi model $\mathcal{G}_n(p)$. They obtained that almost every graph $G_n(p)$ in $\mathcal{G}_n(p)$ satisfies

$$EE(G_n(p)) = e^{np} \left( e^{O(\sqrt{n})} + o(1) \right) \quad a.s. \tag{3}$$

Du et al. also presented exact estimates of Lapacian-energy like invariant, incidence energy, and distance energy, and a tight bound of signless Laplacian energy for almost all graphs (see [26]). They obtained that almost every graph $G_n(p)$ in $\mathcal{G}_n(p)$ satisfies

$$RD\mathcal{E}(G_n(p)) = \left( \frac{8}{3\pi} \sqrt{p(1-p)} + o(1) \right) n^{3/2} \quad a.s. \tag{4}$$

In [27], Shang established better lower and upper bounds to $DEE(G)$ for almost all graphs. He obtained that almost every graph $G_n(p)$ in $\mathcal{G}_n(p)$ satisfies

$$\begin{aligned} &\left[ e^{np}(e^{n2}+1)+n-2 \right] e^{-3O(\sqrt{n})} \\ &\le DEE(G_n(p)) \\ &\le \left[ e^{n2}(n-1)+1 \right] e^{-3O(\sqrt{n})} \quad a.s. \end{aligned} \tag{5}$$

Throughout this paper, following the term introduced in Bollobás's book [28]. Almost every (a.e. for short) graph $G_n(p)$ in $\mathcal{G}_n(p)$ has a certain property $Q$ if the probability that $G_n(p)$ has $Q$ converges to 1 as $n$ becomes infinite. Sometimes, "almost all" can replace "almost every". An event in a probability space holds asymptotically almost surely (a.s. for short) if its probability goes to one as $n$ tends to infinity. Evidently, almost every graph in $\mathcal{G}_n(p)$ has $Q$ if the probability of random graphs satisfying $Q$ converges almost surely.

Denote by $\Delta(G)$ the diameter of $G$. The diameter of a random graph $G_n(p)$ has the following properties.

Lemma 1 ([1]). Suppose that $p^2 n - 2\log n \to \infty$ and $n^2(1-p) \to \infty$ as $n \to \infty$. Then

$$\Delta G_n(p) = 2 \quad a.s.$$

Since $p$ is a constant with $0 < p < 1$ in this paper, it follows from Lemma 1 that a.e. graph $G_n(p)$ has diameter 2. The distance matrix of a random graph $G_n(p) \in \mathcal{G}_n(p)$ is denoted as

$$\mathbf{D} = \mathbf{D}(G_n(p)) .$$

Recall that the diameter of a graph $G$ is the greatest distance between two vertices of $G$. Let $\mathcal{G}'_n(p) \subseteq \mathcal{G}_n(p)$ be a subset containing all graphs with diameter 2 and $G_n(p) \in \mathcal{G}'_n(p)$. Evidently, the entries of $\mathbf{D}(G_n(p))$ satisfy the following [14]

$$\mathbf{D}(ij) = \begin{cases} 0, & \text{if } v_i = v_j; \\ 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 2, & \text{if } v_i \text{ and } v_j \text{ are nonadjacent.} \end{cases}$$

In order to study the more general form of distance energy of random graphs, we introduce the concept of general distance matrix of a random graph $RD_\alpha(G_n(p))$, where $G_n(p)\in \mathcal{G}'_n(p)$ and $\alpha\neq 0$ is a real number. The entries of $RD_\alpha(G_n(p))$ satisfy the following

$$RD_\alpha(ij)=\begin{cases}0, & \text{if } v_i=v_j;\\ 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent;}\\ 2^\alpha, & \text{if } v_i \text{ and } v_j \text{ are nonadjacent,}\end{cases} \tag{6}$$

Clearly, $RD_1(G_n(p))=\mathrm{D}(G_n(p))$. Similar to the definition of the Harary matrix $H(G)$ of $G$, we can give a definition of the Harary matrix $RD_{-1}(G_n(p))$ of a random graph $G_n(p)\in \mathcal{G}'_n(p)$.

The goal of this paper is to formulate the exact estimates to general distance energy and establish bounds to general distance Estrada index for almost all graphs constructed from Erdős-Rényi random graph model.

In Section 2, two exact estimates are formulated to $RD\mathcal{E}_\alpha(G_n(p))$ for almost all $G_n(p)\in\mathcal{G}_n(p)$, where $\mathcal{G}_n(p)$ is Erdös-Rényi model.

i If $\alpha>0$, then

$$RD\mathcal{E}_\alpha(G_n(p))=(2^\alpha-1)\left(\frac{8}{3\pi}\sqrt{p(1-p)}+o(1)\right)n^{3/2}\ a.s.$$

ii If $\alpha<0$, then

$$RD\mathcal{E}_\alpha(G_n(p))=(1-2^\alpha)\left(\frac{8}{3\pi}\sqrt{p(1-p)}+o(1)\right)n^{3/2}\ a.s.$$

In Section 3, In Section 2, two asymptotically inequation are formulated to $RDEE_\alpha(G_n(p))$ for almost all $G_n(p)\in\mathcal{G}_n(p)$.

i If $\alpha>0$, then

$$\begin{aligned}&\left[e^{np(2^\alpha-1)}(e^{n2^\alpha}+1)+n-2\right]e^{-2^\alpha-(2^\alpha-1)O(\sqrt{n})}\\ &\le RDEE_\alpha(G_n(p))\\ &\le\left[e^{n2^\alpha}(n-1)+1\right]e^{-2^\alpha-(2^\alpha-1)O(\sqrt{n})}\quad a.s.\end{aligned}$$

ii If $\alpha<0$, then

$$\begin{aligned}&\left[e^{n2^\alpha}+n-1\right]e^{-2^\alpha+(1-2^\alpha)O(\sqrt{n})}\\ &\le RDEE_\alpha(G_n(p))\\ &\le\left[e^{np(1-2^\alpha)}(e^{n2^\alpha}+1)+(n-2)e^{n2^\alpha}\right]e^{-2^\alpha+(1-2^\alpha)O(\sqrt{n})}\quad a.s.\end{aligned}$$

## 2. General Distance Energies of Random Graphs

This section starts by the definition of the adjacency matrix of the random graph and an important lemma which will play an important role in the proofs of our main results. General distance energies $RD\mathcal{E}_\alpha(G_n(p))$ for random graphs shall be estimated.

Let $A_n := A(G_n(p))$ denote the adjacency matrix of the random grap $G_n(p)\in\mathcal{G}_n(p)$. Apparently, $A(G_n(p))$ is a symmetric random matrix in which the diagonal entries are zeros while $a_{ij}$ $(i<j)$ is 1 or 0, with probability $p$ or $1-p$ respectively. Evidently, the matrix (6) can be rewritten as

$$RD_\alpha(G_n(p))=2^\alpha(J_n-I_n)-(2^\alpha-1)A_n, \tag{7}$$

where $J_n$ is $n\times n$ matrix in which all entries equal 1 and $I_n$ is the unit $n\times n$ matrix.

Lemma 2 (Fan Ky's inequality [29]). Let $X, Y, Z$ be real symmetric matrices of order $n$ such that $X+Y=Z$. Then

$$\mathcal{E}(X)+\mathcal{E}(Y)\ge\mathcal{E}(Z).$$

Theorem 1. Let $G_n(p)\in\mathcal{G}_n(p)$ and $\alpha\neq 0$ be a fix real number. Then general distance energies of $G_n(p)\in\mathcal{G}_n(p)$ can be given as follows.

i If $\alpha>0$, then

$$RD\mathcal{E}_\alpha(G_n(p))=(2^\alpha-1)\left(\frac{8}{3\pi}\sqrt{p(1-p)}+o(1)\right)n^{3/2}\ a.s.$$

ii If $\alpha<0$, then

$$RD\mathcal{E}_\alpha(G_n(p))=(1-2^\alpha)\left(\frac{8}{3\pi}\sqrt{p(1-p)}+o(1)\right)n^{3/2}\ a.s.$$

Proof: Let $\mathcal{G}'_n(p)\subseteq\mathcal{G}_n(p)$ be a subset containing all graphs with diameter 2. In view of Lemma 1, it show to show our result for the graph $G_n(p)\in\mathcal{G}'_n(p)$. This problem can be explained according to the following proof. i If $\alpha>0$, (7) is equivalent to

$$\underline{A_n}=\frac{2^\alpha}{2^\alpha-1}(J_n-I_n)-\frac{RD_\alpha}{2^\alpha-1}. \tag{8}$$

The following results can be obtained from (7), (8) and Lemma 2.

$$\mathcal{E}(RD_\alpha)\le 2^\alpha\mathcal{E}(J_n-I_n)+(2^\alpha-1)\mathcal{E}(-A_n)$$

and

$$\mathcal{E}(-A_n)\le\frac{2^\alpha}{2^\alpha-1}\mathcal{E}(J_n-I_n)+\frac{1}{2^\alpha-1}\mathcal{E}(-RD_\alpha).$$

Thus,

$$\begin{aligned}&(2^\alpha-1)\mathcal{E}(A_n)-2^\alpha\mathcal{E}(J_n-I_n)\le\mathcal{E}(RD_\alpha)\\ &\le(2^\alpha-1)\mathcal{E}(-A_n)+2^\alpha\mathcal{E}(J_n-I_n).\end{aligned}$$

It is easy to see that $\mathcal{E}(J_n - I_n) = 2(n-1)$. By (2), the following result can be obtained.

$$(2^{\alpha}-1)\left(\frac{8}{3\pi}\sqrt{p(1-p)}+o(1)\right)n^{3/2}-2^{\alpha+1}(n-1)\le \mathcal{E}(RD_{\alpha})$$
$$\le (2^{\alpha}-1)\left(\frac{8}{3\pi}\sqrt{p(1-p)}+o(1)\right)n^{3/2}+2^{\alpha+1}(n-1).$$

Consequently, a.e. graph $G_n(p)$ in $\mathcal{G}_n(p)$ satisfies

$$RD\mathcal{E}_{\alpha}(G_n(p))=\mathcal{E}(RD_{\alpha}) = (2^{\alpha}-1)\left(\frac{8}{3\pi}\sqrt{p(1-p)}+o(1)\right)n^{3/2} \ a.s. \tag{9}$$

iii If $\alpha<0$, (7) is equivalent to

$$RD_{\alpha}(G_n(p)) = 2^{\alpha}(J_n - I_n) + (1-2^{\alpha})A_n \tag{10}$$

and

$$A_n = \frac{RD_{\alpha}}{1-2^{\alpha}} - \frac{2^{\alpha}}{1-2^{\alpha}}(J_n - I_n). \tag{11}$$

The following two results can be obtained from (9), (10) and Lemma 2.

$$\mathcal{E}(RD_{\alpha}) \le 2^{\alpha}\mathcal{E}(J_n - I_n) + (1-2^{\alpha})\mathcal{E}(A_n)$$

and

$$\mathcal{E}(A_n) \le \frac{2^{\alpha}}{1-2^{\alpha}}\mathcal{E}(J_n - I_n) + \frac{1}{1-2^{\alpha}}\mathcal{E}(RD_{\alpha}).$$

Thus,

$$(1-2^{\alpha})\mathcal{E}(A_n) - 2^{\alpha}\mathcal{E}(J_n - I_n) \le \mathcal{E}(RD_{\alpha})$$
$$\le (1-2^{\alpha})\mathcal{E}(A_n) + 2^{\alpha}\mathcal{E}(J_n - I_n),$$

i.e.

$$(1-2^{\alpha})\left(\frac{8}{3\pi}\sqrt{p(1-p)}+o(1)\right)n^{3/2}-2^{\alpha+1}(n-1)\le \mathcal{E}(RD_{\alpha})$$
$$\le (1-2^{\alpha})\left(\frac{8}{3\pi}\sqrt{p(1-p)}+o(1)\right)n^{3/2}+2^{\alpha+1}(n-1).$$

Consequently, a.e. graph $G_n(p)$ in $\mathcal{G}_n(p)$ satisfies

$$RD\mathcal{E}_{\alpha}(G_n(p))=\mathcal{E}(RD_{\alpha}) = (1-2^{\alpha})\left(\frac{8}{3\pi}\sqrt{p(1-p)}+o(1)\right)n^{3/2} \ a.s.$$

This completes the proof.

It is easy to see that

$$RD\mathcal{E}_1(G_n(p)) = \left(\frac{8}{3\pi}\sqrt{p(1-p)}+o(1)\right)n^{3/2} \ a.s.$$
$$= D\mathcal{E}(G_n(p)).$$

The results of Theorem 1 generalize the results of Du et al. [25] on the distance energy of random graph.

The concept of Harary matrix $H(G_n(p))$ of random graph can be introduced such as the Harary matrix $H(G)$ of $G$. The entries of $H(G_n(p))$ satisfy the following

$$H(ij) = RD_{-1}(ij) = \begin{cases} 0, & \text{if } v_i = v_j; \\ 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 1/2, & \text{if } v_i \text{ and } v_j \text{ are nonadjacent,} \end{cases} \tag{12}$$

Clearly, $RD\mathcal{E}_{-1}(G_n(p))$ is Harary energy $H\mathcal{E}(G_n(p))$ for random graphs in $\mathcal{G}_n(p)$.

By Theorem 1, the estimator of Harary energy of a random graph is obtained for the first time.

Corollary 1. Let $G_n(p) \in \mathcal{G}_n(p)$. Then

$$H\mathcal{E}(G_n(p)) = \frac{1}{2}\left(\frac{8}{3\pi}\sqrt{p(1-p)}+o(1)\right)n^{3/2} \ a.s.$$

# 3. General Distance Estrada Index of Random Graphs

In this section, general distance Estrada indices $RDEE_{\alpha}(G_n(p))$ for random graphs shall be estimated. Lemma 3 present the explicit information about the eigenvalues of $A_n(G)$.

Lemma 3 (Chen [25] ). Let $A_n$ is the adjacency matrix $A(G_n(p))$ of the random graph $G_n(p) \in \mathcal{G}_n(p)$. Then, the eigenvalues of $A_n(G)$ satisfies

$$\lambda_1(A_n) = np + O(\sqrt{n}) \quad a.s.$$

and for $i = 2, \dots, n$,

$$\lambda_i(A_n) = O(\sqrt{n}) \ a.s.$$

The result of Lemma 4 will play an important role in the proofs of our main results.

Lemma 4 (Weyl's inequality [30]). Let $X, Y$ and $Z$ be all $n \times n$ real symmetric matrices such that $X = Y + Z$ Suppose that $X, Y, Z$ have eigenvalues, respectively,

$$\lambda_1(X) \ge \lambda_2(X) \ge \cdots \ge \lambda_n(X),\ \lambda_1(Y) \ge \cdots \ge \lambda_n(Y),$$

$\lambda_1(Z) \ge \cdots \ge \lambda_n(Z)$. Then for $i = 1, 2, \dots, n$ the following inequalities hold:

$$\lambda_i(Y) + \lambda_n(Z) \le \lambda_i(X) \le \lambda_i(Y) + \lambda_1(Z)$$

Theorem 2. Let $G_n(p) \in \mathcal{G}_n(p)$ and $\alpha \ne 0$ be a fix real number. Then general distance Estrada indices of

$G_n(p) \in \mathcal{G}_n(p)$ can be given as follows.

i If $\alpha > 0$, then

$$\left[e^{np(2^\alpha-1)}(e^{n2^\alpha}+1)+n-2\right]e^{-2^\alpha-(2^\alpha-1)O(\sqrt{n})} \le RDEE_\alpha(G_n(p)) \le \left[e^{n2^\alpha}(n-1)+1\right]e^{-2^\alpha-(2^\alpha-1)O(\sqrt{n})} \quad a.s.$$

ii If $\alpha < 0$, then

$$\left[e^{n2^\alpha}+n-1\right]e^{-2^\alpha+(1-2^\alpha)O(\sqrt{n})} \le RDEE_\alpha(G_n(p)) \le \left[e^{np(1-2^\alpha)}(e^{n2^\alpha}+1)+(n-2)e^{n2^\alpha}\right]e^{-2^\alpha+(1-2^\alpha)O(\sqrt{n})} \quad a.s.$$

Proof: The general distance Estrada index $RDEE_\alpha(G_n(p))$ can be evaluated once the eigenvalues of $RD_n$ are known.

i If $\alpha > 0$, applying lemma 4 to (7)

$$2^\alpha\lambda_i(J_n-I_n)+(2^\alpha-1)\lambda_n(-A_n) \le \lambda_i(RD_\alpha) \le 2^\alpha\lambda_i(J_n-I_n)+(2^\alpha-1)\lambda_1(-A_n)$$

can be obtained.

That is

$$2^\alpha\lambda_i(J_n-I_n)-(2^\alpha-1)\lambda_1(A_n) \le \lambda_i(RD_\alpha) \le 2^\alpha\lambda_i(J_n-I_n)-(2^\alpha-1)\lambda_n(-A_n).$$

$$2^\alpha(n-1)-(2^\alpha-1)(np+O(\sqrt{n})) \le \lambda_1(RD_\alpha) \le 2^\alpha(n-1)-(2^\alpha-1)O(\sqrt{n}) \tag{13}$$

and

$$-2^\alpha-(2^\alpha-1)(np+O(\sqrt{n})) \le \lambda_n(RD_\alpha) \le -2^\alpha-(2^\alpha-1)O(\sqrt{n}) \tag{14}$$

can be deduced form Lemma 3 and $\lambda_1(J_n-I_n)=n-1$, $\lambda_i(J_n-I_n)=-1$ for $i=2,\ldots,n.$

Owing to Lemma 4 again,

$$2^\alpha\lambda_n(J_n-I_n)-(2^\alpha-1)\lambda_{n+1-i}(A_n) \le \lambda_i(RD_\alpha) \le 2^\alpha\lambda_1(J_n-I_n)-(2^\alpha-1)\lambda_{n+1-i}(A_n)$$

is very easy to get.

Consequently,

$$-2^\alpha-(2^\alpha-1)O(\sqrt{n}) \le \lambda_i(RD_\alpha) \le 2^\alpha(n-1)-(2^\alpha-1)O(\sqrt{n}) \tag{15}$$

for $i=2,\ldots,n-1.$

Combining (13), (14) and (15) the conclusion can be proved. That is

$$\begin{aligned}RDEE_\alpha(G_n(p)) &= \sum_{i=1}^{n} e^{\lambda_i(RD_\alpha)} \\ &\ge e^{2^\alpha(n-1)-(2^\alpha-1)(np+O(\sqrt{n}))} + (n-2)e^{-2^\alpha-(2^\alpha-1)O(\sqrt{n})} + e^{-2^\alpha-(2^\alpha-1)(np+O(\sqrt{n}))} \\ &= \left[e^{np(2^\alpha-1)}(e^{n2^\alpha}+1)+n-2\right]e^{-2^\alpha-(2^\alpha-1)O(\sqrt{n})} \quad a.s.\end{aligned}$$

and

$$\begin{aligned}RDEE_\alpha(G_n(p)) &= \sum_{i=1}^{n} e^{\lambda_i(RD_\alpha)} \\ &\le e^{2^\alpha(n-1)-(2^\alpha-1)O(\sqrt{n})} + (n-2)e^{2^\alpha(n-1)-(2^\alpha-1)O(\sqrt{n})} + e^{-2^\alpha-(2^\alpha-1)O(\sqrt{n})} \\ &= \left[e^{n2^\alpha}(n-1)+1\right]e^{-2^\alpha-(2^\alpha-1)O(\sqrt{n})} \quad a.s.\end{aligned}$$

ii If $\alpha < 0$, applying lemma 4 to (7)

$$2^\alpha\lambda_i(J_n-I_n)+(1-2^\alpha)\lambda_n(A_n) \le \lambda_i(RD_\alpha) \le 2^\alpha\lambda_i(J_n-I_n)+(1-2^\alpha)\lambda_1(A_n)$$

can be obtained.

Then,

$$2^\alpha(n-1)+(1-2^\alpha)O(\sqrt{n}) \le \lambda_1(RD_\alpha) \le 2^\alpha(n-1)+(1-2^\alpha)(np+O(\sqrt{n})) \tag{16}$$

and

$$-2^\alpha+(1-2^\alpha)O(\sqrt{n}) \le \lambda_n(RD_\alpha) \le -2^\alpha+(1-2^\alpha)(np+O(\sqrt{n})) \tag{17}$$

Owing to Lemma 4 again,

$$2^\alpha\lambda_n(J_n-I_n)+(1-2^\alpha)\lambda_{n+1-i}(A_n) \le \lambda_i(RD_\alpha) \le 2^\alpha\lambda_1(J_n-I_n)+(1-2^\alpha)\lambda_{n+1-i}(A_n)$$

is very easy to get.

Consequently,

$$\begin{aligned}&-2^{\alpha}+(1-2^{\alpha})O(\sqrt{n})\\&\le \lambda_i(RD_{\alpha})\\&\le 2^{\alpha}(n-1)+(1-2^{\alpha})O(\sqrt{n})\end{aligned} \tag{18}$$

for $i=2,\ldots,n-1$.

Combining (16), (17) and (18) the conclusion can be proved. That is

$$\begin{aligned}&RDEE_{\alpha}(G_n(p))=\sum_{i=1}^{n}e^{\lambda_i(RD_{\alpha})}\\&\ge e^{2^{\alpha}(n-1)+(1-2^{\alpha})O(\sqrt{n})}+\\&(n-2)e^{-2^{\alpha}+(1-2^{\alpha})O(\sqrt{n})}+e^{-2^{\alpha}+(1-2^{\alpha})O(\sqrt{n})}\\&=\left[e^{n2^{\alpha}}+n-1\right]e^{-2^{\alpha}+(1-2^{\alpha})O(\sqrt{n})}\quad a.s.\end{aligned}$$

and

$$\begin{aligned}&RDEE_{\alpha}(G_n(p))=\sum_{i=1}^{n}e^{\lambda_i(RD_{\alpha})}\\&\le e^{2^{\alpha}(n-1)+(1-2^{\alpha})(np+O(\sqrt{n}))}+(n-2)e^{2^{\alpha}(n-1)+(1-2^{\alpha})O(\sqrt{n})}+\\&e^{-2^{\alpha}+(1-2^{\alpha})(np+O(\sqrt{n}))}\\&=\left[e^{np(1-2^{\alpha})}(e^{n2^{\alpha}}+1)+(n-2)e^{n2^{\alpha}}\right]e^{-2^{\alpha}+(1-2^{\alpha})O(\sqrt{n})}\quad a.s.\end{aligned}$$

This completes the proof.

It is easy to see that

$$\begin{aligned}&\left[e^{np}(e^{n2}+1)+n-2\right]e^{-3O(\sqrt{n})}\\&\le RDEE_1(G_n(p))=DEE(G_n(p))\\&\le\left[e^{n2}(n-1)+1\right]e^{-3O(\sqrt{n})}\quad a.s.\end{aligned}$$

The results of Theorem 2 generalize the results of Shang [27] on the distance Estrada index of random graph.

Clearly, $RDEE_{-1}(G_n(p))$ is Harary Estrada index $HEE(G_n(p))$ for random graphs in $\mathcal{G}_n(p)$. By Theorem 2, the asymptotically inequation of Harary Estrada index of a random graph is obtained for the first time.

Corollary 2. Let $G_n(p)\in\mathcal{G}_n(p)$. Then

$$\begin{aligned}&\left[e^{1/2}+n-1\right]e^{-1/2+1/2 O(\sqrt{n})}\\&\le HEE(G_n(p))\\&\le\left[e^{np/2}(e^{n/2}+1)+(n-2)e^{n/2}\right]e^{-1/2+1/2 O(\sqrt{n})}\quad a.s.\end{aligned}$$

# 4. Conclusion

In 1970s, Gutman introduced the concept of energy $\mathcal{E}(G)$ for a simple graph $G$, which is defined as the sum of the absolute values of the eigenvalues of $G$ and can be used to estimate the total $\pi-$ electron energy in conjugated hydrocarbons. The concept attracted lots of attention and furthermore, some other similar notions were also considered such as Laplacian energy $L\mathcal{E}(G)$, signless Laplacian energy, $L\mathcal{E}^{+}(G)$, incidence energy $I\mathcal{E}(G)$, and distance energy $D\mathcal{E}(G)$. Moreover, many researchers established a large number of upper and lower bounds for those invariants. But there are only a few graphs attaining the equalities of those bounds. In the present paper, however, we present exact estimates of general distance energy for almost all graphs by probabilistic and algebraic approaches.

In spite of the fact that the Estrada index of $G$, $EE(G)$ has numerous practical applications, investigations of its basic properties started only short time ago. The concept attracted lots of attention and furthermore, some other similar notions were also considered such as Laplacian

Estrada index $LEE(G)$, signless Laplacian Estrada index $LEE^{+}(G)$ incidence Estrada index $IEE(G)$, and distance Estrada index $DEE(G)$. It is rather hard, as well-known, to compute the eigenvalues for a large matrix even for $A(G)$. So, in order to estimate these invariants, researchers established some lower and upper bounds by algebraic approaches in the last few years. However, there are, as examples given below, only a few classes of graphs attaining the equalities of those bounds. Consequently, one can hardly see the major behavior of these invariants for most graphs. In this paper, however, we establish bounds to general distance Estrada index for almost all graphs by probabilistic and algebraic approaches.


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