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Axiomatization of Aristotelian Syllogistic Logic Based on Generalized Quantifier Theory

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Abstract: Syllogistic reasoning is important due to the prominence of syllogistic arguments in human reasoning, and also to the role they have played in theory of reasoning from Aristotle onwards. Aristotelian syllogistic logic is a formal study of the meaning of four Aristotelian quantifiers and of their properties. This paper focuses on logical system based on syllogistic reasoning. It firstly formalized the 24 valid Aristotle's syllogisms, and then has proven that the other 22 valid Aristotle's syllogisms can be derived from the syllogisms 'Barbara' AAA-1 and 'Celarent' EAE-1 by means of generalized quantifier theory and set theory, so the paper has completed the axiomatization of Aristotelian syllogistic Logic. This axiomatization needs to make full use of symmetry and transformable relations between/among the monotonicity of the four Aristotelian quantifiers from the perspective of generalized quantifier theory. In fact, these innovative achievements and the method in this paper provide a simple and reasonable mathematical model for studying other generalized syllogisms. It is hoped that the present study will make contributions to the development of generalized quantifier theory, and to bringing about consequences to natural language information processing as well as knowledge representation and reasoning in computer science.

Keywords: Generalized Quantifier Theory, Aristotelian Syllogisms, Aristotelian Quantifiers, Axiomatization

1. Introduction

Most reasoning theorists agree that the appropriate theory of inference should be provided by formal logic. The logic can provide a computational level or competence theory of reasoning, in other words, the theory of what inferences people should draw ([1], p. 192). Syllogistic reasoning is the most intensively researched and theoretically important task in the study of logical reasoning [2-5]. It is important due to the prominence of syllogistic arguments in human reasoning, and also to the role they have played in theory of reasoning from Aristotle onwards. The completeness of various formulation of syllogistic logic has already been shown, for example by in Łukasiewicz [6], by in Martin [7] and Moss [8]. Syllogistic logic has already been studied from the perspective of generalized quantifier theory [9-12]. Although there are many other articles about Aristotelian syllogisms [13]-[18], we are not aware of axiomatization of Aristotelian syllogisms by means of generalized quantifier theory, and so this is a goal of the paper.

This paper focuses on logical system based on syllogistic reasoning. A syllogism is a particular instantiation of a syllogistic scheme. One can interpret a syllogism such as the following example:

All students in class 3 are running on the playground.
Some students in red clothes are students in class 3.
Some students in red clothes are running on the playground.

The syllogism means that the sentences above the line semantically entail the one below the line. In other words, in every context or model in which *All students in class 3 are running on the playground* and *Some students in red clothes are students in class 3* are true, it must be the case that *Some students in red clothes are running on the playground* is also true.

A syllogism has two premises, one conclusion. It has the form $Q_1\ (P, M) \wedge Q_2\ (M, S) \Rightarrow Q_3\ (S, P)$, where S is the set of things that the subject term denotes, P is the set of things that the predicate term expresses, and M is the set of things that the middle term signifies, each of Q_1, Q_2, Q_3 in a Aristotelian syllogism is one of the four Aristotelian quantifiers *all*, *some*, *no*, *not all*. The above example can be denoted by $all\ (M, P) \wedge some\ (S, M) \Rightarrow some\ (S, P)$. The other cases are similar.

Aristotelian syllogistic logic is a formal study of the meaning of the four Aristotelian quantifiers and of their properties. For example, the validity of the syllogism $not\ all\ (S, M) \wedge all\ (P, M) \Rightarrow not\ all\ (S, P)$ show that the quantifier *not all* is monotone decreasing in the second argument. Aristotle derived all valid syllogisms from the two syllogisms 'Barbara' (i.e. $all\ (M, P) \wedge all\ (S, M) \Rightarrow all\ (S, P)$) and 'Celarent' (i.e. $no\ (M, P) \wedge no\ (S, M) \Rightarrow all\ (S, P)$) ([19], p. 228]). In other words, Aristotelian syllogistic logic can be axiomatized on the basis of 'Barbara' and 'Celarent'. How can one do that? The writer of this paper applies generalized quantifier theory to formalize Aristotelian syllogisms, and then to axiomatize the logic. To full appreciate the paper below, one will need basic familiarity with the language of first-order logic, with generalized quantifier theory, and with elementary set theoretic terminology.

2. Preliminaries

If Q is a generalized quantifiers, there are three important forms of its negation that appear in natural and logical languages, that is, outer negation $\neg Q$, inner negation $Q\neg$, and dual negation Q^d. There are two important generalized quantifiers in English, i.e., type $\langle 1\rangle$ and type $\langle 1, 1\rangle$. These two kinds of quantifiers are ubiquitous in the natural languages. The four Aristotelian quantifiers *all*, *some*, *no*, *not all* are just four instances of type $\langle 1, 1\rangle$ generalized quantifiers. It is important to recognize that the latter are more basic than the former in the natural languages ([20], p. 12). The type $\langle 1\rangle$ quantifiers are properties of sets of things. The type $\langle 1, 1\rangle$ quantifiers are binary relations between sets of things or stuff.

For instance, a quantified sentence '*All cars of our brothers are running quickly*' states that $all\ (S, P)$ holds, where S is the set of cars of our brothers, P is the set of things that are running quickly, and *all* is a relation between sets. The type $\langle 1, 1\rangle$ quantifier *all* is a particularly simple relation to describe: it is just the subset relation $\subseteq$, i.e., $S\subseteq P$. In other words, *all* signifies the inclusion relation.

By the same token, each of the other Aristotelian quantifiers stands for a particular binary relation between properties, i.e., a binary relation between of individuals. When S, P are arbitrary sets, these relations can be given in standard set-theoretic notations as the following:

Definition 1:

(1) $all\ (S, P) \Leftrightarrow S\subseteq P$; (2) $no\ (S, P) \Leftrightarrow S\cap P = \varnothing$;

(3) $some\ (S, P) \Leftrightarrow S\cap P \neq \varnothing$; (4) $not\ all\ (S, P) \Leftrightarrow S-P\neq\varnothing$.

Definition 2: three forms of negation for type $\langle 1, 1\rangle$ quantifiers

Let E be a given universe, and S, $P\subseteq E$, for a type $\langle 1, 1\rangle$ quantifier Q,

(1) $(\neg Q)_E\ (S, P) \Leftrightarrow not\ Q_E\ (S, P)$;

(2) $(Q\neg)_E\ (S, P) \Leftrightarrow Q_E\ (S, E-P)$;

(3) $(Q^d)_E\ (S, P) \Leftrightarrow \neg\ (Q\neg)_E\ (S, P) \Leftrightarrow (\neg Q)_E\neg\ (S, P)$.

For example, $\neg all = not\ all$, $\neg some = no$, $all\neg = no$, $some\neg = not\ all$, $all^d = some$, $no^d = not\ all$. The modern square of opposition (in which *all* is used without existential import) is composed of the four quantifiers Q, $\neg Q$, $Q\neg$, and Q^d. For example, the Aristotelian square of opposition is composed of the four Aristotelian quantifiers, *all*, *not all*, *no* and *some* as in figure 1 below. The modern square of opposition is closed under these forms of negation, namely, applying any number of these operations to a quantifiers in the square will not lead outside it ([20], p. 12), for example, $\neg\neg\ (some^d)\neg = (some^d)\neg = all\neg = no$.

Definition 3: monotonicity for type $\langle 1, 1\rangle$ quantifiers

Let E be a given universe, and S, S', P, $P'\subseteq E$, for a type $\langle 1, 1\rangle$ quantifier Q,

(1) Q_E is right monotone increasing (denoted by $Mon\uparrow$ or $Q\uparrow$) iff the following holds:

if $P\subseteq P'\subseteq E$, then $Q_E\ (S, P) \Rightarrow Q_E\ (S, P')$.

(2) Q_E is right monotone decreasing (denoted by $Mon\downarrow$ or $Q\downarrow$) iff the following holds:

if $P\subseteq P'\subseteq E$, then $Q_E\ (S, P') \Rightarrow Q_E\ (S, P)$.

(3) Q_E is left monotone increasing (denoted by $\uparrow Mon$ or $\uparrow Q$) iff the following holds:

if $S\subseteq S'\subseteq E$, then $Q_E\ (S, P) \Rightarrow Q_E\ (S', P)$.

(4) Q_E is left monotone decreasing (denoted by $\downarrow Mon$ or $\downarrow Q$) iff the following holds:

if $S\subseteq S'\subseteq E$, then $Q_E\ (S', P) \Rightarrow Q_E\ (S, P)$.

These local notions can be immediately extended to the global case: Q is right (or left) increasing (or decreasing) if each Q_E is.

For example,

(1) *All* cars of our brothers are running quickly. $\Rightarrow$ *All* cars of our brothers are running.

Then one can say that *all* is right monotone increasing, denoted by $all\uparrow$.

(2) *No* cars of our brothers are running. $\Rightarrow$ *No* cars of our brothers are running quickly.

Then one can say that *no* is right monotone decreasing, denoted by $no\downarrow$.

(3) *Some* black cars of our brothers are running. $\Rightarrow$ *Some* cars of our brothers are running.

Then one can say that *some* is left monotone increasing, denoted by $\uparrow some$.

(4) *All* cars of our brothers are running. $\Rightarrow$ *All* black cars of our brothers are running.

Then one can say that *all* is left monotone decreasing, denoted by $\downarrow all$.

The (right or left) monotonicity behavior of a type $\langle 1, 1\rangle$ quantifier completely determines the monotonicity behavior of the other negation quantifiers in its square of opposition, that is, as the following Fact 1 ([20], pp. 170-171).

Fact 1: Let Q be any type $\langle 1, 1\rangle$ quantifier:

(1) Q is $Mon\uparrow$ iff $\neg Q$ is $Mon\downarrow$;

(2) Q is $Mon\uparrow$ iff $Q\neg$ is $Mon\downarrow$;

(3) Q is $Mon\uparrow$ iff Q^d is $Mon\uparrow$;

(4) Q is $Mon\downarrow$ iff $\neg Q$ is $Mon\uparrow$;

(5) Q is $Mon\downarrow$ iff $Q\neg$ is $Mon\uparrow$;

(6) Q is $Mon\downarrow$ iff Q^d is $Mon\downarrow$;

(7) Q is $\uparrow Mon$ iff $\neg Q$ is $Mon\downarrow$;

(8) Q is $\uparrow Mon$ iff $Q\neg$ is $\uparrow Mon$;
(9) Q is $\uparrow Mon$ iff Q^d is $\downarrow Mon$;
(10) Q is $\downarrow Mon$ iff $\neg Q$ is $\uparrow Mon$;
(11) Q is $\downarrow Mon$ iff $Q\neg$ is $\downarrow Mon$;
(12) Q is $\downarrow Mon$ iff Q^d is $\uparrow Mon$.

Proof For (1), let E be a given universe, let S, P, and P' be any subsets of E. If a type $\langle 1, 1\rangle$ quantifier Q is $Mon\uparrow$, this means that Q is right monotone increasing, then for all $P\subseteq P'\subseteq E$, $Q_E\ (S, P) \Rightarrow Q_E\ (S, P')$ according to the clause (1) of Definition 3, thus for all $P\subseteq P'\subseteq E$, $\neg Q_E\ (S, P') \Rightarrow \neg Q_E\ (S, P)$, therefore $\neg Q$ is right monotone decreasing by the clause (2) of Definition 3. That is, $\neg Q$ is $Mon\downarrow$, as desired.

The proof of the other direction is similar. If $\neg Q$ is $Mon\downarrow$, this means that $\neg Q$ is right monotone decreasing, then for all $P\subseteq P'\subseteq E$, $\neg Q_E\ (S, P') \Rightarrow \neg Q_E\ (S, P)$ according to the clause (2) of Definition 3, then for all $P\subseteq P'\subseteq E$, $Q_E\ (S, P) \Rightarrow Q_E\ (S, P')$, hence Q is right monotone increasing by the clause (1) of Definition 3. That is, Q is $Mon\uparrow$, just as desired.

The proofs of the other cases are similarly to this.

As a result of the four Aristotelian quantifiers have been found to be just four instances of generalized quantifiers, therefore, the conclusion of Fact 1 is also suitable for the four quantifiers. The monotonicity of the four Aristotelian quantifies and their interrelations are as in Figure 1.

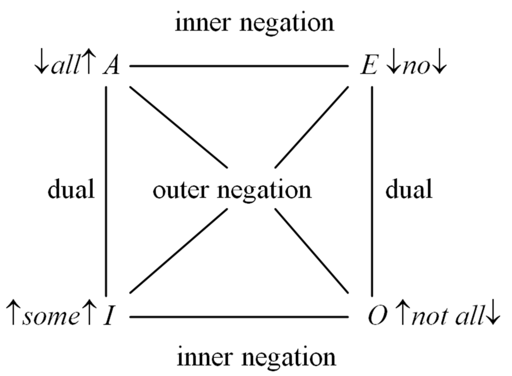

Figure 1. *Monotonicity of the four Aristotelian quantifies and their interrelations.*

Definition 4: symmetry for type $\langle 1, 1\rangle$ quantifiers

Let Q be a type $\langle 1, 1\rangle$ quantifier, Q is symmetric if and only if for all universes E, and S, $P\subseteq E$, $Q_E\ (S, P) \Leftrightarrow Q_E\ (P, S)$.

For example, *Some* doctors are women. $\Rightarrow$ *Some* women are doctors.

Some women are doctors. $\Rightarrow$ *Some* doctors are women.

Therefore *some* is symmetric according to Definition 4.

3. Formalization of Aristotelian Syllogisms

In order to keep the symbolic language concise, the proposition '*All S are P*' is denoted by *all (S, P)* and called by *A* proposition, '*No S are P*' is denoted by *no (S, P)* and called by *E* proposition, '*Some S are P*' is denoted by *some (S, P)* and called by *I* proposition, and '*Not all S are P*' is denoted by *no (S, P)* and called by *O* proposition.

According to the possible configurations of variables, Aristotelian syllogisms can be grouped into four different 'figures'(as shown in Table 1):

Table 1. *Four different 'figures' of Aristotelian syllogisms.*

(1) first figure	(2) second figure	(3) third figuree	(4) fourth figure
Q_1 (M, P)	Q_1 (P, M)	Q_1 (M, P)	Q_1 (P, M)
Q_2 (S, M)	Q_2 (S, M)	Q_2 (M, S)	Q_2 (M, S)
Q_3 (S, P)	Q_3 (S, P)	Q_3 (S, P)	Q_3 (S, P)

Here Q can be chosen among the quantifiers *all*, *some*, *no*, *not all*, so there are 4×4×4×4 = 256 syllogisms. A syllogism is valid if each instantiation of S, M and P verifying the premises also verifies the conclusion. For what choices of quantifiers are the above figures valid? For example, in the second figure, if we let Q_1=*all* and Q_2=Q_2=*no*, then the syllogism *all (P, M)* $\wedge$ *no (S, M)* $\Rightarrow$ *no (S, P)* is valid. The syllogism can be denoted as *AEE-2*. Similarly, the syllogism 'Barbara' *all (M, P)* $\wedge$ *all (S, M)* $\Rightarrow$ *all (S, P)* can be denoted as *AAA-1*.

Let E be any given universe. Now the 24 valid Aristotelian syllogisms can be formalized as follows ([21]):

(01) *AAA-1*: $all_E\ (M, P) \wedge all_E\ (S, M) \Rightarrow all_E\ (S, P)$
(02) *AAI-1*: $all_E\ (M, P) \wedge all_E\ (S, M) \Rightarrow some_E\ (S, P)$
(03) *AII-1*: $all_E\ (M, P) \wedge some_E\ (S, M) \Rightarrow some_E\ (S, P)$
(04) *EIO-1*: $no_E\ (M, P) \wedge some_E\ (S, M) \Rightarrow not\ all_E\ (S, P)$
(05) *EAE-1*: $no_E\ (M, P) \wedge all_E\ (S, M) \Rightarrow no_E\ (S, P)$
(06) *EAO-1*: $no_E\ (M, P) \wedge all_E\ (S, M) \Rightarrow not\ all_E\ (S, P)$
(07)*AEE-2*: $all_E\ (P, M) \wedge no_E\ (S, M) \Rightarrow no_E\ (S, P)$
(08) *AEO-2*: $all_E\ (P, M) \wedge no_E\ (S, M) \Rightarrow not\ all_E\ (S, P)$
(09) *EAE-2*: $no_E\ (P, M) \wedge all_E\ (S, M) \Rightarrow no_E\ (S, P)$
(10) *EAO-2*: $no_E\ (P, M) \wedge all_E\ (S, M) \Rightarrow not\ all_E\ (S, P)$
(11) *EIO-2*: $no_E\ (P, M) \wedge some_E\ (S, M) \Rightarrow not\ all_E\ (S, P)$
(12) *AOO-2*: $all_E\ (P, M) \wedge not\ all_E\ (S, M) \Rightarrow not\ all_E\ (S, P)$
(13) *EIO-3*: $no_E\ (M, P) \wedge some_E\ (M, S) \Rightarrow not\ all_E\ (S, P)$
(14) *OAO-3*: $not\ all_E\ (M, P) \wedge all_E\ (M, S) \Rightarrow not\ all_E\ (S, P)$
(15) *IAI-3*: $some_E\ (M, P) \wedge all_E\ (M, S) \Rightarrow some_E\ (S, P)$
(16) *AII-3*: $all_E\ (M, P) \wedge some_E\ (M, S) \Rightarrow some_E\ (S, P)$
(17) *AAI-3*: $all_E\ (M, P) \wedge all_E\ (M, S) \Rightarrow some_E\ (S, P)$
(18) *EAO-3*: $no_E\ (M, P) \wedge all_E\ (M, S) \Rightarrow not\ all_E\ (S, P)$
(19) *IAI-4*: $some_E\ (P, M) \wedge all_E\ (M, S) \Rightarrow some_E\ (S, P)$
(20) *EIO-4*: $no_E\ (P, M) \wedge some_E\ (M, S) \Rightarrow not\ all_E\ (S, P)$
(21) *AAI-4*: $all_E\ (P, M) \wedge all_E\ (M, S) \Rightarrow some_E\ (S, P)$
(22) *AEE-4*: $all_E\ (P, M) \wedge no_E\ (M, S) \Rightarrow no_E\ (S, P)$
(23) *AEO-4*: $all_E\ (P, M) \wedge no_E\ (M, S) \Rightarrow not\ all_E\ (S, P)$
(24) *EAO-4*: $no_E\ (P, M) \wedge all_E\ (M, S) \Rightarrow not\ all_E\ (S, P)$

In Aristotelian syllogisms, it must be emphasized that the quantifier *all* in *A* proposition is used with existential import. So *all (S, P)* in effect means that all *S*s are *P* and there are some *S*s. Therefore the quantifier *all* in (02), (06), (08), (10) and (23) above is used with existential import, otherwise the corresponding syllogisms are invalid.

4. Axiomalization of Aristotelian Syllogistic Logic

The 24 above valid Aristotelian syllogisms can be derived from the two syllogisms 'Barbara' and 'Celarent' ([9], p. 228).

Is that true? How to prove? If it is true and the conclusion is proved, then one can say that she has completed the axiomatization of Aristotelian syllogistic logic. We are not aware of proof this conclusion by means of generalized quantifier theory, and so this is a goal of the paper.

The validities of the two syllogisms 'Barbara' and 'Celarent' are proved by means of generalized quantifier theory.

Proof: (1) The syllogism 'Barbara''form is $all_E\ (M, P) \wedge all_E\ (S, M) \Rightarrow all_E\ (S, P)$. Suppose that $all_E\ (M, P)$ and $all_E\ (S, M)$ hold, then $all_E\ (M, P) \Leftrightarrow M{\subseteq}P{\subseteq}E$ and $all_E\ (S, M) \Leftrightarrow S{\subseteq}M{\subseteq}E$ according to the clause (1) of Definition 1. Since $M{\subseteq}P{\subseteq}E$ and $S{\subseteq}M{\subseteq}E$, and hence $S{\subseteq}P{\subseteq}E$. So it follows that $S{\subseteq}P{\subseteq}E \Leftrightarrow all_E\ (S, P)$ according to the clause (1) of Definition 1 again. This proves the claim that $all_E\ (M, P) \wedge all_E\ (S, M) \Rightarrow all_E\ (S, P)$, as desired.

(2) The validity of the syllogism 'Celarent' can be similarly proved. The syllogism 'Celarent''s form is $no\ (M, P) \wedge no\ (S, M) \Rightarrow all\ (S, P)$. If $no_E\ (M, P)$ and $all_E\ (S, M)$ hold, then $no_E\ (M, P) \Leftrightarrow M{\cap}P{=}\varnothing$ and $all_E\ (S, M) \Leftrightarrow S{\subseteq}M{\subseteq}E$ by the clause (2) and (1) of Definition 1 respectively. So it follows that $M{\cap}P{=}\varnothing$ and $S{\subseteq}M{\subseteq}E$, then $S{\cap}P{=}\varnothing$. Hence $S{\cap}P{=}\varnothing \Leftrightarrow no_E\ (S, P)$ according to the clause (2) of Definition 1. So $no_E\ (M, P) \wedge all_E\ (S, M) \Rightarrow no_E\ (S, P)$, just as desired.

Now the paper tries to axiomatize Aristotlian syllogistic logic by means of the above Definition 1-4 and Fact 1.

4.1. The Valid Syllogisms Can Be Derived from the Syllogism 'Barbara' AAA-1

That the syllogism 'Barbara' *AAA-1* is valid means that $all_E\ (M, P) \wedge all_E\ (S, M) \Rightarrow all_E\ (S, P)$. It follows that $all_E\ (M, P) \Leftrightarrow M{\subseteq}P{\subseteq}E$ according to the clause (1) of Definition 1. Hence that $all_E\ (M, P) \wedge all_E\ (S, M) \Rightarrow all_E\ (S, P)$ is equivalent to that if $M{\subseteq}P{\subseteq}E$, then $all_E\ (S, M) \Rightarrow all_E\ (S, P)$, and therefore *all* is right monotone increasing by the clause (1) of Definition 3. That is to say that the syllogism *AAA-1* is valid if and only if that *all* is right monotone increasing. Then the following 15 Aristotlian syllogisms can be derived from the validity of the syllogism 'Barbara' *AAA-1*.

(1) It is easy to observe that $all = \neg not$ all by the clause (1) of Definition 2. So one has the following: *all* is right monotone increasing, iff, *not all* is right monotone decreasing according to the clause (1) of Fact 1, iff, if $P{\subseteq}M{\subseteq}E$, then not $all_E\ (S, M) \Rightarrow not\ all_E\ (S, P)$ by the clause (2) of Definition 3, iff, $all_E\ (P, M) \wedge not\ all_E\ (S, M) \Rightarrow not\ all_E\ (S, P)$ by the clause (1) of Definition 1. Therefore the syllogism *AOO-2* is valid, just as desired.

(2) One can similarly prove that the syllogism *AOO-2* is valid. One has that $all = no\neg$ by Definition 2. Then one has the following: *all* is right monotone increasing, iff, *no* is right monotone decreasing by virtue of the clause (2) of Fact 1, iff, if $P{\subseteq}M{\subseteq}E$, thus $no_E\ (S, M) \Rightarrow no_E\ (S, P)$ by the clause (2) of Definition 3, iff, $all_E\ (P, M) \wedge no_E\ (S, M) \Rightarrow no_E\ (S, P)$ by the clause (1) of Definition 1. Hence the syllogism *AEE-2* is valid.

(3) It is easy to observe that $no_E\ (S, P) \Rightarrow not\ all_E\ (S, P)$ since the *O* proposition is subalternate to the *E* proposition. And (2) has proven that $all_E\ (P, M) \wedge no_E\ (S, M) \Rightarrow no_E\ (S, P)$. Therefore $all_E\ (P, M) \wedge no_E\ (S, M) \Rightarrow not\ all_E\ (S, P)$. That is to say that the syllogism *AEO-2* is valid.

(4) One can easily check that $no(S, M) \Leftrightarrow no(M, S)$, i.e., *no* is symmetric by Definition 4. If one substitutes $no_E\ (M, S)$ for $no_E\ (S, M)$ in '$all_E\ (P, M) \wedge no_E\ (S, M) \Rightarrow no_E\ (S, P)$' proved in (2), it follows that $all_E\ (P, M) \wedge no_E\ (M, S) \Rightarrow no_E\ (S, P)$. Therefore the syllogism *AEE-4* is valid.

(5) The proof of validity of *AEO-4* is similar to that of *AEO-2* in (3). It is easy to observe that $no_E\ (S, P) \Rightarrow not\ all_E\ (S, P)$. And the paper has proven that $all_E\ (P, M) \wedge no_E\ (M, S) \Rightarrow no_E\ (S, P)$ in (4). Hence $all_E\ (P, M) \wedge no_E\ (M, S) \Rightarrow not\ all_E\ (S, P)$. In other words, the syllogism *AEO-4* is valid.

(6) The proof of validity of *AII-1* is similar to that of *AOO-2* in (1). It is easy to show that all = some^{d} all by the clause (3) of Definition 2. Hence it can be proved the following: *all* is right monotone increasing, iff, *some* is right monotone increasing according to the clause (3) of Fact 1, iff, if $M{\subseteq}P{\subseteq}E$, then $some_E\ (S, M) \Rightarrow some_E\ (S, P)$ by the clause (1) of Definition 3, iff, $all_E\ (M, P) \wedge some_E\ (S, M) \Rightarrow some_E\ (S, P)$ by the clause (1) of Definition 1. That is to say that the syllogism *AII-1* is valid, as desired.

(7) It is known to that $(\neg r \wedge p \rightarrow \neg q)$ can be derived from $(p \wedge q \rightarrow r)$ in which p, q and r are proposition variables. It follows that $\neg some_E\ (S, P) \wedge some_E\ (S, M) \Rightarrow \neg all_E\ (M, P))$can be derived from '$all_E\ (M, P) \wedge some_E\ (S, M) \Rightarrow some_E\ (S, P)$' proved in (6). Then since *M*, *S*, and *P* are any variables, this is semantically equivalent to that $\neg some_E\ (M, P) \wedge some_E(M, S) \Rightarrow \neg all_E(S, P)$ by changing variables. It is clear that $\neg some = no$ and $\neg all = not\ all$, hence $no_E(M, P) \wedge some_E(M, S) \Rightarrow not\ all_E(S, P)$. This means that the syllogism *EIO-3* is valid.

(8) The proof of validity of *EIO-4* is similar to that of *AEE-4* in (4). It is intuitively clear that $no_E(M, P) \Leftrightarrow no_E(P, M)$, that is, no is symmetric by Definition 4. Now if one substitutes $no_E(P, M)$ for $no_E(M, P)$ in '$no_E(M, P) \wedge some_E(M, S) \Rightarrow not\ all_E(S, P)$' proved in (7), it follows that $no_E(P, M) \wedge some_E(M, S) \Rightarrow not\ all_E(S, P)$. So the syllogism *EIO-4* is valid, as desired.

(9) The proof of validity of *EIO-1* is similar to that of *EIO-4* in (8). It is easily to check that $some_E(M, S) \Leftrightarrow some_E(S, M)$, i.e., some is symmetric. Now if one substitutes $some_E(S, M)$ for $some_E(M, S)$ in '$no_E(M, P) \wedge some_E(M, S) \Rightarrow not\ all_E(S, P)$' proved in (7), it follows that $no_E(M, P) \wedge some_E(S, M) \Rightarrow not\ all_E(S, P)$. Hence the syllogism *EIO-1* is valid, as desired.

(10) One can observe that no is symmetric since it satisfies the scheme $no_E(M, P) \Leftrightarrow no_E(P, M)$ *as above*. If one replaces $no_E(M, P)$ by $no_E(P, M)$ in '$no_E(M, P) \wedge some_E(S, M) \Rightarrow not\ all_E(S, P)$' proved in (9), it follows that $no_E(P, M) \wedge some_E(S, M) \Rightarrow not\ all_E(S, P)$. In other words, the syllogism *EIO-2* is valid.

(11) The proof of validity of *AAI-4* is similar to that of *EIO-3* in (7). It is can be showed that $\neg not\ all_E(S, P) \wedge all_E(P, M) \Rightarrow \neg no_E(M, S)$ can be derived from $all_E(P, M) \wedge no_E(M, S) \Rightarrow not\ all_E(S, P)$. Therefore $all_E(S, P)) \wedge all_E(P, M) \Rightarrow some_E(M, S)$ since $\neg not\ all = all$ and $\neg no = some$. It is intuitively clear that $all_E(P, M) \wedge all_E(M, S) \Rightarrow some_E(S, P)$. Hence the syllogism *AAI-4* is valid, as desired.

(12) The proof of validity of *AAI-1* is similar to that of

AEO-2 in (3). It is intuitively clear that $all_E(S, P) \Rightarrow some_E(S, P)$ since the *I* proposition is subalternate to the *A* proposition. Then the validity of $all_E(M, P) \wedge all_E(S, M) \Rightarrow some_E(S, P)$ can be derived from that of *AAA-1* $all_E(M, P) \wedge all_E(S, M) \Rightarrow all_E(S, P)$. In other words, the syllogism *AAI-1* is valid.

(13) The proof of validity of *EAO-3* is similar to that of *EIO-3* in (7). That $\neg some_E(S, P) \wedge all_E(S, M) \Rightarrow \neg all_E(M, P)$ is implied by that $all_E(M, P) \wedge all_E(S, M) \Rightarrow some_E(S, P)$ proved in (12). Then $no_E(M, P)) \wedge all_E(M, S) \Rightarrow not\ all_E(S, P)$ since $\neg some = no$ and $\neg all = not\ all$. It is semantically equivalent to that $no_E(M, P)) \wedge all_E(M, S) \Rightarrow not\ all_E(S, P)$ by changing variables. This shows that the syllogism *EIO-3* is valid.

(14) The proof of validity of *AII-3* is similar to that of *AEE-4* in (4). One can observe that *some* is symmetric since $some_E(S, M) \Leftrightarrow some_E(M, S)$. If one replaces $some_E(S, M)$ by $some_E(M, S)$ in '$all_E(M, P) \wedge some_E(S, M) \Rightarrow some_E(S, P)$' proved in (6), it follows that $all_E(M, P) \wedge some_E(M, S) \Rightarrow some_E(S, P)$. So the syllogism *AII-3* is valid, just as desired.

(15) The proof of validity of *EAO-4* is similar to that of *EIO-3* in (7). That $\neg some_E(S, P) \wedge all_E(P, M) \Rightarrow \neg all_E(M, S)$ can be implied by that $all_E(P, M) \wedge all_E(M, S) \Rightarrow some_E(S, P)$ proved in (11). Therefore $no_E(S, P) \wedge all_E(P, M) \Rightarrow not\ all_E(M, S)$ since $\neg some = no$ and $\neg all = not\ all$. Hence $no_E(P, M)) \wedge all_E(M, S) \Rightarrow not\ all_E(S, P)$ by changing variables. That is, the syllogism *EAO-4* is valid, as desired.

4.2. The Valid Syllogisms Can Be Derived from the Syllogism 'Celarent' EAE-1

That the syllogism 'Celarent' EAE-1 is valid means that $no_E(M, P) \wedge all_E(S, M) \Rightarrow no_E(S, P)$. It follows that $all_E(S, M) \Leftrightarrow S \subseteq M \subseteq E$ according to the clause (1) of Definition 1. Then that $no_E(M, P) \wedge all_E(S, M) \Rightarrow no_E(S, P)$ is equivalent to that if $S \subseteq M \subseteq E$, then $no_E(M, P) \Rightarrow no_E(S, P)$, and therefore *no* is left monotone decreasing by the clause (4) of Definition 3. That is to say that the syllogism *EAE-1* is valid if and only if that *no* is left monotone decreasing. Then the following 7 Aristotlian syllogisms can be derived from the validity of the syllogism 'Celarent' *EAE-1*.

(16) The proof of validity of *IAI-3* is similar to that of *AOO-2* in (1). It follows that $no = \neg some$ according to the clause (1) of Definition 2. So one has the following: *no* is left monotone decreasing, iff, *some* is left monotone increasing according to the clause (1) of Fact 1, iff, if $M \subseteq S \subseteq$E, then $some_E(M, P) \Rightarrow some_E(S, P)$ by the clause (3) of Definition 3, iff, $some_E(M, P) \wedge all_E(M, S) \Rightarrow some_E(S, P)$ by the clause (1) of Definition 1. Therefore the syllogism *IAI-3* is valid, as desired.

(17) The proof of validity of *IAI-4* is similar to that of *AEE-4* in (4). The paper has proven that $some_E(M, P) \Leftrightarrow some_E(P, M)$ as above. If one replaces $some_E(S, M)$ by $some_E(M, S)$ in '$all_E(M, P) \wedge some_E(S, M) \Rightarrow some_E(S, P)$' proved in (6), it follows that $all_E(M, P) \wedge some_E(M, S) \Rightarrow some_E(S, P)$. So the syllogism *IAI-4* is valid.

(18) The proof of validity of *OAO-3* is similar to that of *AOO-2* in (1). It is easy to check that $no\neg = not\ all$ according to the clause (2) of Definition 2. So it follows the following: *no* is left monotone decreasing, iff, *not all* is left monotone increasing according to the clause (8) of Fact 1, iff, if $M \subseteq S \subseteq E$, then $not\ all_E(M, P) \Rightarrow not\ all_E(S, P)$ by the clause (4) of Definition 3, iff, $not\ all_E(M, P) \wedge all_E(M, S) \Rightarrow not\ all_E(S, P)$ by the clause (1) of Definition 1. Hence the syllogism *OAO-3* is valid, as desired.

(19) The proof of validity of *EAE-2* is similar to that of *AEE-4* in (4). It follows that $no_E(M, P) \Leftrightarrow no_E(P, M)$ as above. If we substitute $no_E(P, M)$ for $no_E(M, P)$ in *EAE-1* '$no_E(M, P) \wedge all_E(S, M) \Rightarrow no_E(S, P)$', one can obtain that $no_E(P, M) \wedge all_E(S, M) \Rightarrow no_E(S, P)$. Hence the syllogism *IAI-4* is valid.

(20) The proof of validity of *EAO-2* is similar to that of *AEO-2* in (3). It follows that $no_E(S, P) \Rightarrow not\ all_E(S, P)$ as above. And (19) has proven that $no_E(P, M) \wedge all_E(S, M) \Rightarrow no_E(S, P)$. Then $no_E(P, M) \wedge all_E(S, M) \Rightarrow not\ all_E(S, P)$. Therefore the syllogism *EAO-2* is valid.

(21) The proof of validity of *EAO-1* is similar to that of *EAO-2* in (20). It is clear that $no_E(S, P) \Rightarrow not\ all_E(S, P)$. Then that $no_E(P, M) \wedge all_E(S, M) \Rightarrow not\ all_E(S, P)$ can be derived from *EAE-1* $no_E(M, P) \wedge all_E(S, M) \Rightarrow no_E(S, P)$. That is to say that the syllogism *EAO-1* is valid.

(22) The proof of validity of *AAI-3* is similar to that of *EIO-3* in (7). That $\neg not\ all_E(S, P) \wedge all_E(S, M) \Rightarrow \neg\ no_E(P, M)$ can be implied by that $no_E(P, M) \wedge all_E(S, M) \Rightarrow not\ all_E(S, P)$ proved in (20). Then $all_E(S, P) \wedge all_E(S, M) \Rightarrow some_E(P, M)$ since $\neg not\ all = all$ and $\neg no = some$. It is equivalent to that $all_E(M, P)) \wedge all_E(M, S) \Rightarrow some_E(S, P)$ by changing variables. Hence the syllogism *AAI-3* is valid.

Now the paper has derived the other 22 valid Aristotelian syllogisms just from the two syllogisms *AAA-1* and *EAE-1*. In other words, it has completed the axiomatiztion of Aristotelian syllogistic logic, just as desired.

5. Conclusion

This paper firstly formalized the 24 valid Aristotle's syllogisms, and then has proven that the *other* 22 valid Aristotle's syllogisms can be derived from the syllogisms *AAA-1* and *EAE-1* by means of generalized quantifier theory and set theory, so the paper has completed the axiomatization of Aristotelian syllogistic logic. In fact, these innovative achievements and the method in this paper provide a simple and reasonable mathematical model for studying other generalized syllogisms. It is hoped that the present study will make contributions to the development of generalized quantifier theory, and to bringing about consequences to natural language information processing as well as knowledge representation and reasoning in computer science.

As it turns out, generalized quantifiers are an extremely versatile syntactic and semantic tool. As a future work, it would be interesting to formally study the validity of generalized syllogisms, and then to formally discuss on the validity of discourse reasoning in natural languages nested by two or more Aristotelian syllogisms or generalized ones.

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