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On a Subclass of Close-to-Convex Functions Associated with Fixed Second Coefficient

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Abstract: We consider a subclass of univalent functions $f(z)$ for which there corresponds a convex function $g(z)$ of order α such that $\mathrm{Re}(zf'(z) / g(z)) \geq \beta$. We investigate the influence of the second coefficient of $g(z)$ on this class. We also prove distortion, covering, and radius of convexity theorems.

Keywords: Analytic Function, Univalent Function, Convex Function of Order α, Close-to-Convexity, Fixed Second Coefficient, Radius of Convexity

1. Introduction

Let A be the class of regular functions on the unit disk $E = \{z : |z| < 1\}$. Let S be the class of univalent functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

normalized by the conditions $f(0) = 0 = f'(0) - 1$. Let $C_b(\alpha)$ denote the subclass of S consisting of functions of the form

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.2}$$

such that $\mathrm{Re}\left\{1 + \frac{zg''(z)}{g'(z)}\right\} \geq \alpha, \quad z \in E, \quad |b_2| = b,$ and $0 \leq \alpha < 1$. Cb(α) is the class of convex functions of order α. It is known that 0 ≤ b ≤ 1 [2]. Moreover, g(z) ∈ C1(α) if and only if $g(z) = \frac{z}{1 - \varepsilon z}$, $|\varepsilon| = 1$.

Definition 1.1: A normalized regular function of the form (1.1) is said to belong to the class $C'_b(\alpha, \beta)$ if there exists a function $g(z) \in C_b(\alpha)$ of the form (1.2) such that

$$\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\right\} \geq \beta, \quad \beta \geq 0.$$

It is clear that for $\alpha \leq \alpha'$ and $\beta \leq \beta'$, we have $C'_b(\alpha', \beta) \subset C'_b(\alpha, \beta)$ and $C'_b(\alpha, \beta') \subset C'_b(\alpha, \beta)$.

Note that $C'_b(0, \beta) \subset C'_b(\beta)$. In [14] the author has defined a subclass C' consisting of functions $f(z)$ of the form (1.1) for which $\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > 0$ where $g(z) \in C$, C being the class of convex functions and obtained the inclusion relation K* ⊂ C′ ⊂ K ⊂ S*, K* is the subclass known as the class of quasi-convex functions introduced and studied by K.I. Noor [12]. Thus $C'_b(\alpha, \beta) \subset C'$ and hence every member of $C'_b(\alpha, \beta)$ is univalent. By specializing α and β we obtain some important subclasses. If $f(z) \in C'_b(0,0)$ then $f(z)$ is close-to-convex; if $f(z) \in C'_b(1, \beta)$, then $\mathrm{Re}\{f'(z)\} \geq \beta$ that is in $\mathbb{R}(\beta)$ as in [13]; if $f(z) \in C'_b(\alpha, 1)$ then $f(z) = g(z)$ so that $f(z)$ is convex of order α. If $f(z) \in C'_b(1,1)$ then $\mathrm{Re}\{f'(z)\} \geq 0$ so $g(z) = z$.

In this paper we prove distortion, covering and radius of convexity theorems for the class $C'_b(\alpha, \beta)$.

In what follows, we assume $f(z)$ is in $C'_b(\alpha, \beta)$ with

$g(z)$ its associated function in $C_b(\alpha)$.

First we give an example for $f(z)\in C_b'(\alpha,\beta)$ by using the following.

Lemma: [16] Let $Q(z)$ be analytic for z ∈ C with $Q(0)=1$. Then $\operatorname{Re} Q(z)\geq\beta$ if and only if $Q(z)=\frac{1+(1-2\beta)G(z)}{1-G(z)}$, where G(z) is analytic,

G(0) = 0, and $|G(z)|<1$ for $z\in E$.

Example: Let

$$f(z)=\frac{1}{2}\int_0^z\frac{1}{t}\left[t\psi(t)+\int_0^t\psi(t)dt\right]\left(\frac{1+(1-2\beta t)}{1-t}\right)dt \text{ and}$$

$$g(z)=\frac{1}{2}\left[z\psi(z)+\int_0^z\psi(t)dt\right]=\frac{1}{2}\left(z\int_0^z\psi(t)dt\right)' \text{ where}$$

$$\psi(t)=\left(\frac{1}{1-t^2}\right)^{1-\alpha}\left(\frac{1+t}{1-t}\right)^{\frac{2b}{3}}.$$

Since

$$\frac{zf'(z)}{g(z)}=\frac{1+(1-2\beta)z}{1-z} \tag{1.3}$$

has real part ≥ β, it suffices to show that $g(z)\in C_b(\alpha)$.

If $\phi(z)=\int_0^z\psi(t)dt$, then we have

$$1+\frac{z\phi''(z)}{\phi'(z)}=\frac{1+\frac{4b}{3}z+(1-2\alpha)z^2}{1-z^2}=\frac{1+(1-2\alpha)G(z)}{1-G(z)}.$$

Solving for G(z), we obtain

$$G(z)=\frac{z\left[z+\frac{2b}{3(1-\alpha)}\right]}{\left[1+\frac{2b}{3(1-\alpha)}z\right]}=zh(z).$$

Since α + b ≤ 1, h(z) maps E → E, and $|G(z)|\leq|z|<1$ for z ∈ E. Since $G(z)$ satisfies the conditions of Lemma, $\phi(z)\in C_b(\alpha)$. So,

$$g(z)=\frac{1}{2}\left[z\psi(z)+\int_0^z\psi(t)dt\right]=\frac{1}{2}\left(z\int_0^z\psi(t)dt\right)'$$

is convex for $|z|<\frac{1}{2}$ by Livingston [12]. Thus existence of $f(z)\in C_b'(\alpha,\beta)$ is asserted.

2. Distortion Theorem for $C_b'(\alpha,\beta)$

Theorem 2.1: Let $f(z)\in C_b'(\alpha,\beta)$. Then

$$|f'(z)|\leq\frac{1+(1-2\beta)r}{r(1-r)}\int_0^r\left(\frac{1}{1-t^2}\right)^{1-\alpha}\left(\frac{1+t}{1-t}\right)^b dt \tag{2.1}$$

and

$$|f'(z)|\geq\frac{1-(1-2\beta)r}{r(1+r)}\int_0^r\left(1+\frac{2bt}{1-\alpha}+t^2\right)^{1-\alpha} dt \tag{2.2}$$

where the integrand on the right hand side of (2.2) is taken to be 1 for α= 1.

Equality holds in (2.1) for the function

$$f_1(z)=\int_0^z\frac{1+(1-2\beta)s}{s(1-s)}\int_0^s\left(\frac{1}{1-t^2}\right)^{1-\alpha}\left(\frac{1+t}{1-t}\right)^b dtds$$

and equality holds in (2.2) for the function

$$f_2(z)=\int_0^z\frac{1-(1-2\beta)s}{s(1+s)}\int_0^s\left(1+\frac{2bt}{1-\alpha}+t^2\right)^{1-\alpha} dtds.$$

Proof: From (1.3), by Lemma we obtain

$$\frac{zf'(z)}{g(z)}=\frac{1+(1-2\beta)G(z)}{1-G(z)} \tag{2.3}$$

where G(0) = 0 and $|G(z)|<1$ for z ∈ E. Since G(z) satisfies the conditions of Schwarz's lemma, (1.6) yields

$$\frac{1-(1-2\beta)r}{1+r}\leq\left|\frac{zf'(z)}{g(z)}\right|\leq\frac{1+(1-2\beta)r}{1-r} \tag{2.4}$$

That is, $\frac{1-(1-2\beta)r}{r(1+r)}\leq\left|\frac{f'(z)}{g(z)}\right|\leq\frac{1+(1-2\beta)r}{r(1-r)}$.

We have [2, p. 105],

$$\int_0^r\left(1+\frac{2bt}{1-\alpha}+t^2\right)^{\alpha-1}dt\leq|g(z)|\leq\int_0^r\left(\frac{1}{1-t^2}\right)^{1-\alpha}\left(\frac{1+t}{1-t}\right)^b dt. \tag{2.5}$$

Combining (1.7) and (1.8), the result follows. Clearly $f_1(z)$ and $f_2(z)\in C_b(\alpha,\beta)$ with

$$g_1(z)=\int_0^z\left(\frac{1}{1-t^2}\right)^{1-\alpha}\left(\frac{1+t}{1-t}\right)^b dt \text{ and}$$

$$g_2(z)=\int_0^z\left(1+\frac{2bt}{1-\alpha}+t^2\right)^{\alpha-1}dt.$$

Theorem 2.2. Let $f(z)\in C_b'(\alpha,\beta)$. Then

$$\int_0^r\frac{1-(1-2\beta)s}{s(1-s)}\left[\int_0^s\left(1+\frac{2bt}{1-\alpha}+t^2\right)^{\alpha-1}dt\right]ds\leq|f(z)|$$

$$\leq\int_0^r\frac{1+(1-2\beta)s}{s(1+s)}\left[\int_0^s\left(\frac{1}{1-t^2}\right)^{1-\alpha}\left(\frac{1+t}{1-t}\right)^b dt\right]ds.$$

Equality holds on the right-hand side for $f_1(z)$ in Theorem 2.1 and on the left-hand side for $f_2(z)$ in Theorem 2.1.

Proof: Integrating along the straight line segment from the origin to $z = re^{i\theta}$ and applying Theorem 2.1 we obtain

$$|f(z)| \le \int_0^r |f'(se^{i\theta})| ds$$

$$\le \int_0^r \frac{1+(1-2\beta)s}{s(1+s)} \left[\int_0^s \left(\frac{1}{1-t^2} \right)^{1-\alpha} \left(\frac{1+t}{1-t} \right)^b dt \right] ds,$$

which proves the right-hand inequality.

To prove the left-hand inequality, for every r we choose z_0, $|z_0| = r$, such that

$$|f(z_0)| = \min_{|z|=r} |f(z)|.$$

If $L(z_0)$ is the pre-image of segment $[0, f(z_0)]$, then

$$|f(z)| \ge |f(z_0)| \ge \int_{L(z_0)} |f'(z)| |dz|$$

$$\ge \int_0^r \frac{1-(1-2\beta)s}{s(1-s)} \left[\int_0^s \left(1 + \frac{2bt}{1-\alpha} + t^2 \right)^{\alpha-1} dt \right] ds.$$

This completes the proof.

3. Covering Theorem for $C'_b(\alpha, \beta)$

Theorem 3.1: Let $f(z) \in C'_b(\alpha, \beta)$ with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. If $f(z) \ne w$ for z ∈ E, then

$$|w| \ge 2/(6 - 2\beta + b).$$

Proof: First we establish that $|a_2| \le 1 - \beta + \frac{b}{2}$.

Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Then, for

$$\phi(z) = \frac{1}{1-\beta}\left[\frac{zf'(z)}{g(z)} - \beta \right] = 1 + \frac{(2a_2 - b_2)z}{1-\beta} + \sum_{n=2}^{\infty} c_n z^n,$$

we have [9, p. 15],

$$\left| \frac{2a_2 - b_2}{1-\beta} \right| \le 2, \quad so \quad |a_2| \le 1 - \beta + \frac{b}{2}. \tag{3.1}$$

Since f(z) does not assume the value w,

$$\frac{wf(z)}{w - f(z)} = z + \left(a_2 + \frac{1}{w} \right) z^2 + \cdots$$

is in the class *S*. Therefore

$$\left| a_2 + \frac{1}{w} \right| \le 2. \tag{3.2}$$

Now, using (3.1) and (3.2),

$$\left| \frac{1}{w} \right| = \left| a_2 + \frac{1}{w} - a_2 \right| \le 2 + 1 - \beta + \frac{b}{2},$$

and this completes the proof.

4. A Radius of Convexity Theorem for $C'_b(\alpha, \beta)$

Lemma 4.1: If $f \in C'_b(\beta)$, then $f(z)$ maps the disk $|z| < r_1$ onto a convex domain, where r1 is the least positive root of the equation $\gamma(r, b, \beta) = 0$, where

$$\gamma(r, b, \beta) = [1 + (1-2\beta)r](1+r)^b + 2(1-\beta)r(1-r)^b.$$

Proof: Let $f(z) \in C'_b(\beta)$. Then for $g(z) \in C_b(0)$, $\frac{zf'(z)}{g(z)} = Q(z)$ with $\mathrm{Re}\{\phi(z)\} \ge \beta$.

From $zf'(z) = g(z)Q(z)$, it follows that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zg'(z)}{g(z)} + \frac{zQ'(z)}{Q(z)}. \tag{4.1}$$

So, the radius of convexity of $f(z)$ is at least equal to the smallest positive root of

$$\min \mathrm{Re}\left(\frac{zg'(z)}{g(z)} \right) + \min \mathrm{Re}\left(\frac{zQ'(z)}{Q(z)} \right) = 0.$$

For α = 0, from the inequalities in [2, p. 104] and [14, p. 384] we obtain

$$\mathrm{Re}\left(\frac{zg'(z)}{g(z)} \right) \ge -\left| \frac{zg'(z)}{g(z)} \right| \ge -\frac{(1+r)^b}{(1-r)^{b+1}}. \tag{4.2}$$

Now, let $Q(z) = (1-\beta)P(z) + \beta$, where P(z) is analytic, P(0) = 1, and Re{P(z)} > 0 in E.

Then $\frac{zQ'(z)}{Q(z)} = \frac{P'(z)}{P(z) + \frac{\beta}{(1-\beta)}}$. Using the lemma of Libera [11, p. 150] we obtain

$$\left| \frac{zQ'(z)}{Q(z)} \right| \le \frac{2(1-\beta)r}{(1-r)[1+(1-2\beta)r]}. \tag{4.3}$$

Using (4.2) and (4.3) in (4.1) we get

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \geq -\frac{(1+r)^b}{(1-r)^{b+1}} - \frac{2(1-\beta)r}{(1-r)[1+(1-2\beta)r]}$$
$$\geq -\frac{\gamma(r,b,\beta)}{(1-r)^{b+1}[1+(1-2\beta)r]}.$$

Hence, $f(z)$ is convex in $|z| < r_1$ where r1 is the least positive root of the equation $\gamma(r, b, \beta) = 0$ for given b, β. This lemma improves the result obtained in [16].

Theorem 4.2: If $f(z) \in C'_b(\alpha,\beta)$, then $f(z)$ maps the disk $|z| < R$ onto a convex domain, where R is the least positive root of the equation $\delta(r,b,\alpha,\beta) = 0$, where

$$\delta(r,b,\alpha,\beta) = (1-2\alpha)(1-2\beta)r(1+r)^{b+\alpha-1}(1-r)^{\alpha-b}$$
$$+(1-2\alpha)(1+r)^{b+\alpha-1}(1-r)^{\alpha-b}$$
$$-2(1-\beta)(1+r)^{2\alpha-1} + 2(1-\beta).$$

Proof: Let $f(z) \in C'_b(\alpha,\beta)$. Then for $\frac{zf'(z)}{g(z)} = Q(z)$ with $\operatorname{Re}\{\phi(z)\} \geq \beta.$

From $zf'(z) = g(z)Q(z)$, it follows that

$$1+\frac{zf''(z)}{f'(z)} = \frac{zg'(z)}{g(z)} + \frac{zQ'(z)}{Q(z)}. \tag{4.4}$$

So, the radius of convexity of $f(z)$ is at least equal to the smallest positive root of

$$\min \operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) + \min \operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) = 0.$$

Using the inequalities in [2, p. 104] and [14, p. 384] we obtain

$$\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) \geq -\left|\frac{zg'(z)}{g(z)}\right|$$
$$\geq -\frac{(2\alpha-1)r(1+r)^b}{(1-r^2)^{1-\alpha}(1-r)^b[(1+r)^{2\alpha-1}-1]} \tag{4.5}$$

Now, let $Q(z) = (1-\beta)P(z) + \beta$, where P(z) is analytic, P(0) = 1, and Re{P(z)} > 0 in E.

Then $\frac{zQ'(z)}{Q(z)} = \frac{P'(z)}{P(z)+\frac{\beta}{(1-\beta)}}$. Using the lemma of Libera [11, p. 150] we obtain

$$\left|\frac{zQ'(z)}{Q(z)}\right| \leq \frac{2(1-\beta)r}{(1-r)[1+(1-2\beta)r]}. \tag{4.6}$$

Using (4.5) and (4.6) in (4.4) we get

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \geq -\frac{(2\alpha-1)r(1+r)^b}{(1-r^2)^{1-\alpha}(1-r)^b[(1+r)^{2\alpha-1}-1]}$$
$$-\frac{2(1-\beta)r}{(1-r)[1+(1-2\beta)r]}$$
$$\geq -\frac{r\delta(r,b,\alpha,\beta)}{(1-r)[(1+r)^{2\alpha-1}][1+(1-2\beta)r]}.$$

Hence, $f(z)$ is convex in $|z| < R$ where R is the least positive root of the equation $\delta(r,b,\alpha,\beta) = 0$, for given b, α, β.

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