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Modelling Infectiology of Dengue Epidemic

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Abstract: In this paper a mathematical model for the transmission dynamics of dengue fever disease is presented. We present a SITR (susceptible, infected, treated, recovery) and ASI (aquatic, susceptible, infected) epidemic model to describe the interaction between human and dengue fever mosquito populations. In order to assess the transmission of Dengue fever disease, the susceptible population is divided into two, namely, careful and careless human susceptible population. The model presents four possible equilibria: two disease-free and two endemic equilibrium.The results show that the disease-free equilibrium point is locally and globally asymptotically stable if the reproduction number is less than unity. Endemic equilibrium point is locally and globally asymptotically stable under certain conditions using additive compound matrix and Lyapunov method respectively. Sensitivity analysis of the model is implemented in order to investigate the sensitivity of certain key parameters of dengue fever disease with treatment, Careful and Careless Susceptibles on the transmission of Dengue fever Disease.

Keywords: Dengue Fever Disease, Careful, Careless, Susceptibles, Equilibrium, Stability, Reproduction Number

1. Introduction

Dengue is a major health problem found in tropical and sub-tropical climates worldwide, mostly in urban and semi-urban areas [1]. Dengue fever disease can cause a severe flu-like illness, and sometimes Dengue fever can vary from mild to severe. The more severe forms of dengue fever include dengue hemorrhagic fever and dengue shock syndrome. Dengue fever (DF) is a vector-borne disease transmitted by female Aedes aegypti and Aedes albopictus mosquitoes because they require blood meal for the development of their eggs. Four different serotypes can cause dengue fever. A human infected by one serotype, on recovery, gains total immunity to that serotype and only partial and transient immunity with respect to the other three. Preventing or reducing dengue virus transmission depends entirely on the control of mosquito. The spread of dengue is attributed to expanding geographic distribution of the four dengue viruses and their mosquito vectors, the most important of which is the predominantly urban species Aedes aegypti [2].

Mathematical models have played a major role in increasing our understanding of the dynamics of infectious diseases. Several models have been proposed to study the effects of some factors on the transmission dynamics of these infectious diseases including Dengue fever and to provide guidelines as to how the spread can be controlled [3].

Mathematical modelling also became considerable important tool in the study of epidemiology because it helped us to understand the observed epidemiological patterns, disease control and provide understanding of the underlying mechanisms which influence the spread of disease and may suggest control strategies [4,5,6,7,8,2,9,10,11]. Moreover in [12] the authors presented a dynamical model that studied the temporal model for dengue disease with treatment. So far no one considered a dynamical system that incorporates the effects of treated individual, Careful and Careless Susceptibles on the transmission of Dengue fever in the society. In this paper, an extension of the model of [12] is presented to include temporary immunity and Susceptibles with different behaviour i.e. the dynamical system that incorporates the effects of Careful and Careless Susceptibles on the transmission of Dengue fever in the society.

Thus, we study and analyse a non-linear mathematical model showing the effect of Treatment, Careful and Careless Susceptibles on the transmission of dengue fever disease in the population.

2. Formulation of the Model

In this section, a deterministic model is developed that describes the dynamics of Dengue fever of population size N [13].Two types of population are considered: humans and mosquito. The humans are divided into five mutuall-exclusive compartments indexed by h are given by: careful human susceptible, $S_{h_1}(t)$, careless human susceptible, $S_{h_2}(t)$, in which the possibility of careless human Susceptibles contracting the disease is higher than that for careful human Susceptibles, $I_h(t)$, individuals capable of transmitting dengue fever disease to others; $T_h(t)$, individual who are treated and $R_h(t)$, individuals who have acquired immunity at time t. The total number of human is constant, which means that $N(t) = S_{h_1}(t) + S_{h_2}(t) + I_h(t) + T_h(t) + R_h(t)$. Similarly, the model has also three compartments for the mosquito (mosquitoes) indexed by m are given by: $A_m(t)$, which represents the aquatic phase of the mosquito (including egg, pupae and larvae) and the adult phase of the mosquito, with $S_m(t)$ and $I_m(t)$, susceptible and infected, respectively. It is also assumed that $N_m = S_m(t) + I_m(t)$.

Considering the above considerations and assumptions, we then have the following schematic model flow diagram for dengue fever disease with treatment, Careful and Careless Susceptibles:

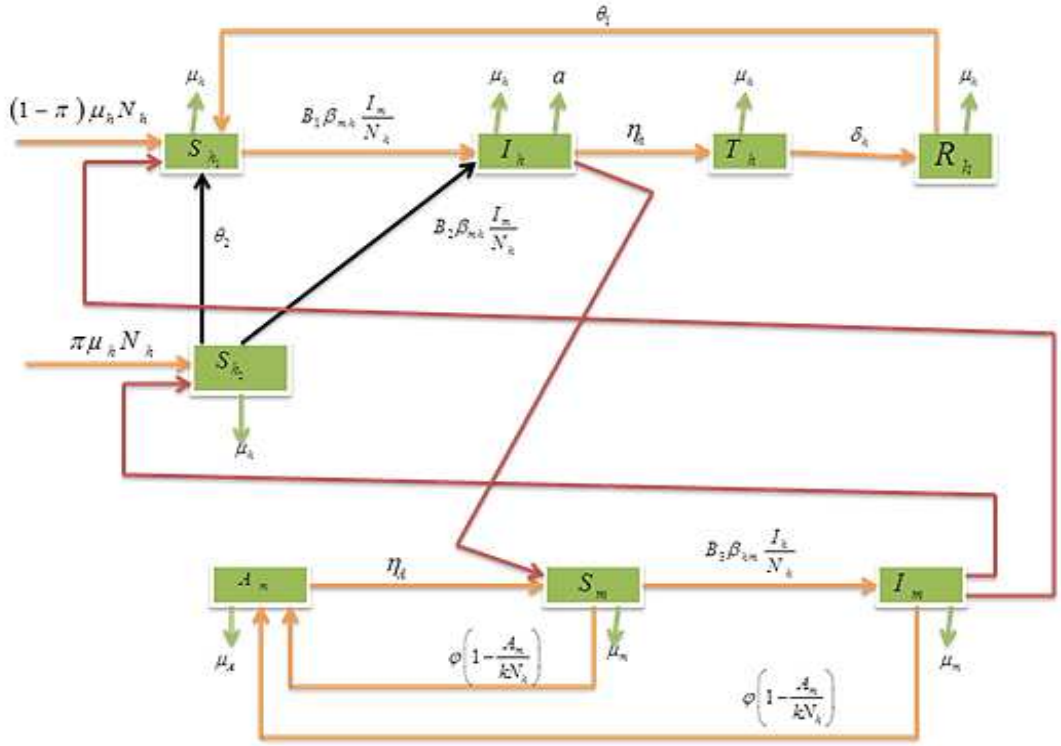


Figure 1. *Model Flow diagram for dengue fever disease with treatment, Careful and Careless human susceptible.*

From the above flow diagram, the model is described by an initial value problem with a system of eight differential equations given as follows:

$$\frac{dS_{h_1}}{dt} = (1-\pi)\mu_h N_h - B_1\beta_{mh}\frac{I_m}{N_h}S_{h_1} - \mu_h S_{h_1} + \theta_1 R_h + \theta_2 S_{h_2}$$

$$\frac{dS_{h_2}}{dt} = \pi\mu_h N_h - B_2\beta_{mh}\frac{I_m}{N_h}S_{h_2} - \mu_h S_{h_2} - \theta_2 S_{h_2} \quad (1)$$

$$\frac{dI_h}{dt} = \left(B_1 S_{h_1} + B_2 S_{h_2}\right)\beta_{mh}\frac{I_m}{N_h} - (\mu_h + \eta_h + a)I_h$$

$$\frac{dT_h}{dt} = \eta_h I_h - (\mu_h + \delta_h)T_h$$

$$\frac{dR_h}{dt} = \delta_h T_h - (\mu_h + \theta_1)R_h$$

$$\frac{dA_m}{dt} = \varphi\left(1 - \frac{A_m}{kN_h}\right)(S_m + I_m) - (\mu_A + \eta_A)A_m$$

$$\frac{dS_m}{dt} = \eta_A A_m - \left(B_3\beta_{hm}\frac{I_h}{N_h} + \mu_m\right)S_m$$

$$\frac{dI_m}{dt} = B_3\beta_{hm}\frac{I_h}{N_h}S_m - \mu_m I_m$$

where $S_{h_1}(0) > 0$, $S_{h_2}(0) > 0$, $I_h(0) \geq 0$, $T_h(0) \geq 0$, $R_h(0) \geq 0$, $A_m(0) \geq 0$, $S_m(0) > 0$, $I_m(0) \geq 0$, for all $t \geq 0$.

3. Mathematical Analysis of the Model

The dynamics of dengue fever disease is determined by the

basic reproduction number R_0 which is a key concept and is defined as the average number of secondary infection arising from a single infected individual introduced into the susceptible class during its entire infectious period in a totally susceptible population [14,15],for if R_0<1 the result is disease free-equilibrium and if R_0>1 means that there exists endemic equilibrium point. The model system of equations (1) will be analysed qualitatively to get a better understanding of the effects of treated individual, careful and Careless human Susceptibles of Dengue fever disease.

3.1. Disease Free Equilibrium (DFE)

For the disease free equilibrium, it is assumed that there is no infection for both populations of human and mosquitoes i.e. $I_h(t)=0$ and $I_m(t)=0$, denoted by 'E_0'. Thus E_0 of the model system (1) is obtained as $E_0=\left(S_{h_1}(t),S_{h_2}(t),0,0,0,A_m(t),S_m(t),0\right)=$

$$\left(\frac{(1-\pi)N_h(\mu_h+\theta_2)+\theta_2\pi N_h}{\mu_h+\theta_2},\frac{\pi\mu_h N_h}{\mu_h+\theta_2},0,0,0,\frac{kN_hq}{\eta_A\varphi},\frac{kN_hq}{\varphi\mu_m},0\right)$$

where $q=-\left((\mu_A+\eta_A)-\eta_A\varphi\right)$

3.2. The Basic Reproduction Number,' R_0 '

The basic reproduction number of the model (1) R_0 is calculated by using the next generation matrix of an ODE [14]. Using the approach of [14]. R_0 is obtaining by taking the largest (dominant) Eigen value (spectral radius) of

$$\left[\frac{\partial F_i(E_0)}{\partial X_j}\right]\left[\frac{\partial V_i(E_0)}{\partial X_j}\right]^{-1},$$

where, F_i is the rate of appearance of new infection in compartment i, V_i^+ is the transfer of individuals out of the compartment i by all other means and E_0 is the disease free equilibrium.

$$\mathrm{F}_i=\begin{bmatrix}F_1\\F_2\end{bmatrix}=\begin{bmatrix}\left(B_1S_{h_1}+B_2S_{h_2}\right)\beta_{mh}\dfrac{I_m}{N_h}\\[2ex] B_3\beta_{hm}\dfrac{I_h}{N_h}S_m\end{bmatrix}$$

Using the linearization method, the associated matrix at DFE is given by

$$\mathrm{F}=\begin{pmatrix}\dfrac{\partial F_1}{\partial I_h}(E_0) & \dfrac{\partial F_1}{\partial I_m}(E_0)\\[2ex]\dfrac{\partial F_2}{\partial I_h}(E_0) & \dfrac{\partial F_2}{\partial I_m}(E_0)\end{pmatrix}$$

This implies that

$$\mathrm{F}=\begin{pmatrix}0 & \dfrac{\left(B_1S_{h_1}+B_2S_{h_2}\right)\beta_{mh}}{N_h}\\[2ex] B_3\beta_{hm}\dfrac{S_m}{N_h} & 0\end{pmatrix}$$

With

$$S_{h_1}=\frac{(1-\pi)N_h(\mu_h+\theta_2)+\theta_2\pi N_h}{\mu_h+\theta_2},$$

$$S_{h_2}=\frac{\pi\mu_h N_h}{\mu_h+\theta_2},\quad S_m=\frac{kN_hq}{\varphi\mu_m}$$

we have

$$\mathrm{F}=\begin{pmatrix}0 & \left(\dfrac{B_1(1-\pi)(\mu_h+\theta_2)+\theta_2\pi}{\mu_h+\theta_2}+B_2\dfrac{\pi\mu_h}{\mu_h+\theta_2}\right)\beta_{mh}\\[2ex]\dfrac{B_3\beta_{hm}kq}{\varphi\mu_m} & 0\end{pmatrix}$$

The transfer of individuals out of the compartment i is given by

$$\mathrm{V}_i=\begin{bmatrix}V_1\\V_2\end{bmatrix}=\begin{bmatrix}(\mu_h+\eta_h+a)I_h\\\mu_mI_m\end{bmatrix}$$

Using the linearization method, the associated matrix at DFE is given by,

$$\mathrm{V}=\begin{pmatrix}\dfrac{\partial V_1}{\partial I_h}(E_0) & \dfrac{\partial V_1}{\partial I_m}(E_0)\\[2ex]\dfrac{\partial V_2}{\partial I_h}(E_0) & \dfrac{\partial V_2}{\partial I_m}(E_0)\end{pmatrix}$$

This gives

$$\mathrm{V}=\begin{pmatrix}\mu_h+\eta_h+a & 0\\0 & \mu_m\end{pmatrix}$$

With

$$\mathrm{V}^{-1}=\begin{pmatrix}\dfrac{1}{\mu_h+\eta_h+a} & 0\\ 0 & \dfrac{1}{\mu_m}\end{pmatrix}$$

Therefore

$$\mathrm{FV}^{-1}=\begin{pmatrix}0 & \left(\dfrac{B_1(1-\pi)(\mu_h+\theta_2)+\theta_2\pi}{\mu_h+\theta_2}+\dfrac{B_2\pi\mu_h}{\mu_h+\theta_2}\right)\beta_{mh}\\ \dfrac{B_3\beta_{hm}kq}{\varphi\mu_m} & 0\end{pmatrix}\begin{pmatrix}\dfrac{1}{\mu_h+\eta_h+a} & 0\\ 0 & \dfrac{1}{\mu_m}\end{pmatrix} \tag{2}$$

Then eigenvalues of the equation (2) is given by

$$\det\left(\mathrm{FV}^{-1}-\mathrm{I}\lambda\right)=\det\begin{pmatrix}0-\lambda & \left(\dfrac{B_1\beta_{mh}(1-\pi)(\mu_h+\theta_2)+\theta_2\pi\beta_{mh}+B_2\beta_{mh}\pi\mu_h}{(\mu_h+\theta_2)\mu_m}\right)\\ \dfrac{kqB_3\beta_{hm}}{\varphi\mu_m(\mu_h+\eta_h+a)} & 0-\lambda\end{pmatrix}$$

This gives

$$\lambda^2=\left(\frac{kqB_3\beta_{hm}}{\varphi\mu_m(\mu_h+\eta_h+a)}\right)\left(\frac{B_1\beta_{mh}(1-\pi)(\mu_h+\theta_2)+\theta_2\pi\beta_{mh}+B_2\beta_{mh}\pi\mu_h}{(\mu_h+\theta_2)\mu_m}\right)$$

consequently

$$\lambda=\sqrt{\left(\frac{kqB_3\beta_1}{\varphi\mu_m(\mu_h+\eta_h+a)}\right)\left(\frac{B_1\beta_2(1-\pi)(\mu_h+\theta_2)+\theta_2\pi\beta_2+B_2\beta_2\pi\mu_h}{(\mu_h+\theta_2)\mu_m}\right)}$$

or

$$\lambda=-\sqrt{\left(\frac{kqB_3\beta_{hm}}{\varphi\mu_m(\mu_h+\eta_h+a)}\right)\left(\frac{B_1\beta_{mh}(1-\pi)(\mu_h+\theta_2)+\theta_2\pi\beta_{mh}+B_2\beta_{mh}\pi\mu_h}{(\mu_h+\theta_2)\mu_m}\right)}$$

It follows that the Basic Reproductive number which is given by the largest Eigen value for model system (1) denoted by R_0 is given as

$$R_0=\sqrt{\left(\frac{kqB_3\beta_{hm}}{\varphi\mu_m(\mu_h+\eta_h+a)}\right)\left(\frac{B_1\beta_{mh}(1-\pi)(\mu_h+\theta_2)+\theta_2\pi\beta_{mh}+B_2\beta_{mh}\pi\mu_h}{(\mu_h+\theta_2)\mu_m}\right)}$$

But

$$q=-\left((\mu_A+\eta_A)\mu_m-\eta_A\varphi\right)$$

It is follows that

$$R_0=\sqrt{\frac{-kB_3\beta_{hm}\beta_{mh}\left((\mu_A+\eta_A)\mu_m-\eta_A\varphi\right)\left(B_1(1-\pi)(\mu_h+\theta_2)+\theta_2\pi+B_2\pi\mu_h\right)}{\varphi\mu_m^2(\mu_h+\eta_h+a)(\mu_h+\theta_2)}}$$

or

$$R_0=\sqrt{\frac{-kB_3\beta_{hm}\beta_{mh}t}{\varphi\mu_m^2(\mu_h+\eta_h+a)(\mu_h+\theta_2)}}$$

where

$$t=\left((\mu_A+\eta_A)\mu_m-\eta_A\varphi\right)\left(B_1(1-\pi)(\mu_h+\theta_2)+\theta_2\pi+B_2\pi\mu_h\right)$$

Model System (1) has infection-free equilibrium E_0 if $R_0 < 1$, otherwise endemic equilibrium.

3.3. Sensitivity Analysis of Model Parameters

In this subsection we would like to know difference factors for disease transmission where this helps to reduce mortality and morbidity due to dengue fever disease.

In order to determine how best human mortality and morbidity due to dengue fever disease is reduced, we calculate the sensitivity indices of the reproduction number R_0 to each parameter in the model using the approach of [14]. These indices tell us which parameters have high impact on R_0 and should be targeted by intervention strategies [13].

Definition 1: The normalised forward sensitivity index of a variable 'p' that depends differentiable on a parameter $'q'$ is defined as:

$$X_q^p = \frac{\partial p}{\partial q} \times \frac{q}{p} \text{ in [16].} \quad (3)$$

Having an explicit formula for R_0 in equation (3), we derive an analytical expression for the sensitivity of R_0 as $X_q^{R_0} = \frac{\partial R_0}{\partial q} \times \frac{q}{R_0}$

where

$$R_0 = \sqrt{\frac{-kB_3\beta_{hm}\beta_{mh}\left(\left(\mu_A+\eta_A\right)\mu_m-\eta_A\varphi\right)\left(B_1\left(1-\pi\right)\left(\mu_h+\theta_2\right)+\theta_2\pi+B_2\pi\mu_h\right)}{\varphi\mu_m^2\left(\mu_h+\eta_h+a\right)\left(\mu_h+\theta_2\right)}}$$

Then analytical expression for the sensitivity of R_0 with respect to each parameter can be calculated using a set of reasonable parameter values. Parameter values are obtained from the different literatures like [12], [13] and [17] (http://www.wavuti.com/2014/05/wizara-ya-afya-kitengo-cha.html). Other parameter values are estimated to vary within realistic means and given as $\beta_{hm} = 0.375$, $\beta_{mh} = 0.45$, $\pi = 0.96$, $B_1 = 0.5$, $B_2 = 0.9$, $B_3 = 0.7$, $\mu_m = \frac{1}{11}$, $k = 3$, $\eta_A = 0.35$, $\mu_A = 0.25$, $\mu_h = \frac{1}{78\times 365}$, $\eta_h = 1/3$, $\varphi = 5$, $\theta_1 = 0.01$, $\theta_2 = 0.6$, $a = 0.001$, $\delta_h = 0.98$.

The sensitivity indices of R_0 with respect to η_A and μ_m are given by $X_{\eta_A}^{R_0} = \frac{\partial R_0}{\partial \eta_A} \times \frac{\eta_A}{R_0} = +0.506702261$ and $X_{\mu_m}^{R_0} = \frac{\partial R_0}{\partial \mu_m} \times \frac{\mu_m}{R_0} = -1.016085624$ respectively.

Other indices $X_{B_3}^{R_0}$, $X_{\beta_{hm}}^{R_0}$, $X_k^{R_0}$, $X_{\beta_{mh}}^{R_0}$, $X_\pi^{R_0}$, $X_{\mu_h}^{R_0}$, $X_{\eta_h}^{R_0}$, $X_\varphi^{R_0}$, $X_{B_1}^{R_0}$, $X_{B_2}^{R_0}$, $X_{\theta_2}^{R_0}$, $X_a^{R_0}$, $X_a^{R_0}$ and $X_{\mu_A}^{R_0}$ are obtained following the same method and tabulated as follows:

Table 1. *Sensitivity Indices of Model Parameters to* R_0.

S/N	Parameter Symbol	Sensitivity index
1	η_A	+0.506702261
2	β_{hm}	+0.500000191
3	B_3	+0.499999846
4	β_{mh}	+0.499999617
5	k	+0.499998468
6	π	+0.244896308
7	φ	+0.016085811
8	B_1	+0.010204166
9	B_2	+0.00002580447057
10	θ_2	+0.000002867004084
11	μ_h	-0.00005539116931
12	a	-0.001495356938
13	μ_A	-0.006702415127
14	η_h	-0.498452312
15	μ_m	-1.016085624

The parameters are ordered from most sensitive to the least.

3.3.1. Interpretation

By analysing sensitivity indices of model parameters to R_0, it is observed that the following parameters η_A, β_{hm}, B_3, β_{mh}, k, π, φ, B_1, B_2 and θ_2 when each one increases keeping the other parameters constant they increase the value of R_0 implying that they increase the endemicity of the disease as they have positive indices. While parameters such as, a, μ_A, η_h and μ_m when each one increases while keeping the other parameters constant they decrease the value of R_0 implying that they decrease the endemicity of the disease as they have negative indices.

But individually, the most sensitive parameter is maturation rate from larvae to adult (per day) η_A, followed by the transmission probability from I_h (per bite) β_{hm}, average daily biting (per day) for mosquito susceptible B_3, transmission probability from I_m (per bite) β_{mh}, number of larvae per human k, Fraction of subpopulation recruited into the population π, number of eggs at each deposit per capita (per day) φ, average daily biting (per day) for careful human susceptible B_1, average daily biting (per day) for careless human susceptible B_2, Positive change in behaviour of Careless individuals θ_2, average lifespan of humans (per day) μ_h, Per capita disease induced death rate for humans a, natural mortality of larvae (per day) μ_A, mean viremic period (per day) η_h and finally the least sensitive parameter is

average lifespan of adult mosquitoes (Per day) μ_m.

3.4. Local Stability of Disease Free Equilibrium Point

Local stability of the disease free equilibrium is determined by the variation matrix J_{E_0} of the model system (1) corresponding to the disease free E_0 as

$$\mathrm{J}_{E_0}=\begin{pmatrix} -\mu_h & \theta_2 & 0 & 0 & \theta_1 & 0 & 0 & A \\ 0 & -\theta_2-\mu_h & 0 & 0 & 0 & 0 & 0 & -\dfrac{\pi B_2\beta_{mh}\mu_h}{\theta_2+\mu_h} \\ 0 & 0 & -a-\eta_h-\mu_h & 0 & 0 & 0 & 0 & B \\ 0 & 0 & \eta_h & -\delta_h-\mu_h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_h & -\theta_1-\mu_h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\eta_A-\mu_A-\dfrac{q}{\mu_m} & \varphi-\dfrac{q}{\eta_A} & \varphi-\dfrac{q}{\eta_A} \\ 0 & 0 & -\dfrac{kqB_3\beta_{hm}}{\varphi\mu_m} & 0 & 0 & \eta_A & -\mu_m & 0 \\ 0 & 0 & \dfrac{kqB_3\beta_{hm}}{\varphi\mu_m} & 0 & 0 & 0 & 0 & -\mu_m \end{pmatrix} \tag{4}$$

where $A=-\dfrac{B_1\beta_{hm}(\theta_2-(-1+\pi)\mu_h)}{\theta_2+\mu_h}$,

$B=\dfrac{\beta_{mh}(\pi B_2\mu_h+B_1(\theta_2-(-1+\pi)\mu_h))}{\theta_2+\mu_h}$

Thus the stability of the disease free equilibrium point is clarified by studying the behaviour of J_{E_0} in which for local stability of DFE we seek for its all eigenvalues to have negative real parts. It follows that, the characteristic function of the matrix (4) with λ being the eigenvalues of the Jacobian matrix, J_{E_0}. The Jacobian matrix has the following eigenvalues:

$$\lambda_1=-\mu_h,\ \lambda_2=-\theta_2-\mu_h$$

The other eigenvalues are given as

$$\lambda_3=-\frac{1}{2}\left(a+\eta_h+\mu_h+\mu_m+\frac{1}{\sqrt{\varphi}\sqrt{\theta_2+\mu_h}\sqrt{\mu_m}}\left(\sqrt{\sigma}\right)\right)$$

when $\sqrt{\sigma}$ is not a real number

$$\lambda_4=-\delta_h-\mu_h,\ \lambda_5=-\theta_1-\mu_h$$

$$\lambda_6=-\frac{1}{2}\left(a+\eta_h+\mu_h+\mu_m-\frac{1}{\sqrt{\varphi}\sqrt{\theta_2+\mu_h}\sqrt{\mu_m}}\left(\sqrt{\sigma}\right)\right)$$

when $\sqrt{\sigma}$ is not a real number

$$\lambda_7=-\frac{1}{2\mu_m}(q+\mu_m(\eta_A+\mu_A+\mu_m)+\sqrt{\alpha})$$

when $\sqrt{\alpha}$ is not a real number and finally

$$\lambda_8=-\frac{1}{2\mu_m}\left(q+\mu_m(\eta_A+\mu_A+\mu_m)-\sqrt{\alpha}\right)$$

when $\sqrt{\alpha}$ is not a real number

where $\sigma=4k\pi qB_2B_3\beta_{hm}\beta_{mh}\mu_h+4kqB_1B_3\beta_{hm}\beta_{mh}\left(\theta_2-(-1+\pi)\mu\right)_h+\varphi(\theta_2+\mu_h)(a+\eta_h+\mu_h-\mu_m)^2\mu_m$

$$\alpha=q^2+2q(\eta_A+\mu_A)\mu_m+\left(-6q+\eta_A^2+\mu_A^2+2\eta_A(2\varphi+\mu_A)\right)\mu_m^2-2(\eta_A+\mu_A)\mu_m^3+\mu_m^4$$

Therefore the system is stable since all the eight eigenvalues are negative. This implies that at $R_0<1$ the Disease-free Equilibrium point is locally asymptotically stable.

3.5. Global Stability of Disease Free Equilibrium Point

In this subsection, we adopt the idea of [8], to analyse the global behaviour of the equilibria for system (1). The following theorem provides the global property of the disease free equilibrium E_0 of the system. The results are obtained by means of Lyapunov function.

Theorem 1: If $R_0\le 1$, then the infection-free equilibrium is globally asymptotically stable in the interior of Ω

Proof:

To determine the global stability of the disease-free equilibrium point, we construct the following Lyapunov function:

$$L(t) = -kB_3\beta_{hm}tI_h(t) + \varphi(\mu_h+\eta_h+a)(\mu_h+\theta_2)\mu_m I_m(t) \quad (5)$$

Calculating the time derivative of L along (5), we obtain

$$L'(t) = -kB_3\beta_{hm}tI_h'(t) + \varphi(\mu_h+\eta_h+a)(\mu_h+\theta_2)\mu_m I_m'(t)$$

Then we substitute $I_h'(t)\ \&\ I_m'(t)$ from system (1) to obtain

$$L'(t) = -kB_3\beta_{hm}t\left(\left(B_1S_{h_1}+B_2S_{h_2}\right)\beta_{mh}\frac{I_m}{N_h} - (\mu_h+\eta_h+a)I_h\right) + \varphi(\mu_h+\eta_h+a)(\mu_h+\theta_2)\mu_m\left(B_3\beta_{hm}\frac{I_h}{N_h}S_m - \mu_m I_m\right)$$

Consequently

$$L'(t) = \varphi(\mu_h+\eta_h+a)(\mu_h+\theta_2)\mu_m\mu_m I_m\left(\frac{-kB_3\beta_{hm}t\left(B_1S_{h_1}+B_2S_{h_2}\right)\beta_{mh}}{N_h\varphi(\mu_h+\eta_h+a)(\mu_h+\theta_2)\mu_m\mu_m} - 1\right)$$

$$(\mu_h+\eta_h+a)B_3\beta_{hm}\left(ktI_h + \varphi(\mu_h+\theta_2)\mu_m\frac{I_h}{N_h}S_m\right)$$

But $R_0^2 = \dfrac{-kB_3\beta_{hm}\beta_{mh}t}{\varphi\mu_m(\mu_h+\theta_2)\mu_m(\mu_h+\eta_h+a)}$ it follows that

$$L'(t) = \varphi(\mu_h+\eta_h+a)(\mu_h+\theta_2)\mu_m\mu_m I_m\left(\frac{\left(B_1S_{h_1}+B_2S_{h_2}\right)}{N_h}R_0^2 - 1\right) - \frac{kB_3^2\beta_{hm}^2\beta_{mh}t}{\varphi\mu_m(\mu_h+\theta_2)\mu_m R_0^2}\left(ktI_h + \varphi(\mu_h+\theta_2)\mu_m\frac{I_h}{N_h}S_m\right)$$

implying that

$$L'(t) = \varphi(\mu_h+\eta_h+a)(\mu_h+\theta_2)\mu_m\mu_m I_m\left(\sqrt{f}R_0+1\right)\left(\sqrt{f}R_0-1\right) - \frac{kB_3^2\beta_{hm}^2\beta_{mh}t}{\varphi\mu_m(\mu_h+\theta_2)\mu_m R_0^2}\left(ktI_h + \varphi(\mu_h+\theta_2)\mu_m\frac{I_h}{N_h}S_m\right)$$

Therefore

$$L'(t) = -\varphi(\mu_h+\eta_h+a)(\mu_h+\theta_2)\mu_m\mu_m I_m\left(\sqrt{f}R_0+1\right)\left(1-\sqrt{f}R_0\right) - \frac{kB_3^2\beta_{hm}^2\beta_{mh}t}{\varphi\mu_m(\mu_h+\theta_2)\mu_m R_0^2}\left(ktI_h + \varphi(\mu_h+\theta_2)\mu_m\frac{I_h}{N_h}S_m\right)$$

where $f = \dfrac{\left(B_1S_{h_1}+B_2S_{h_2}\right)}{N_h}$

Thus, $L'(t)$ is negative if $R_0 \le 1$ and $L' = 0$ if and only if $I_h = I_m = 0$ is reduced to the DFE. Consequently, the largest compact invariant set in $\{(S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m) \in \Omega,\ L' = 0\}$ when $R_0 \le 1$ is the singleton $\{E_0\}$. Hence, by LaSalle's invariance principle it implies that "E_0" is globally asymptotically stable in Ω [18]. This completes the proof.

3.6. Existence and Stability of Endemic Equilibrium

Since we are dealing with presence of dengue fever disease in human population, we can reduce system (1) to a 4-dimensional system by eliminating $T_h, R_h, A_m\ \&\ S_m$ respectively, in the feasible region Ω. The values of S_m can be determined by setting $S_m = mN_h\ ?\ I_m$ to obtain

$$\frac{dS_{h_1}}{dt} = (1-\pi)\mu_h N_h - B_1\beta_{mh}\frac{I_m}{N_h}S_{h_1} - \mu_h S_{h_1} + \theta_1 R_h + \theta_2 S_{h_2}$$

$$\frac{dS_{h_2}}{dt} = \pi\mu_h N_h - B_2\beta_{mh}\frac{I_m}{N_h}S_{h_2} - \mu_h S_{h_2} - \theta_2 S_{h_2} \quad (6)$$

$$\frac{dI_h}{dt} = \left(B_1S_{h_1}+B_2S_{h_2}\right)\beta_{mh}\frac{I_m}{N_h} - (\mu_h+\eta_h+a)I_h$$

$$\frac{dI_m}{dt} = B_3\beta_{hm}\frac{I_h}{N_h}S_m - \mu_m I_m$$

3.6.1. The Endemic Equilibrium and Its Stability

Here, we study the existence and stability of the endemic equilibrium points. If $R_0 > 1$, then the host-vector model system (6) has a unique endemic equilibrium given by $E^* = \left(S_{h_1}^*, S_{h_2}^*, I_h^*, I_m^*\right)$ in Ω with

$$S_{h_1}^* = \left(m + B_3N_h^2\beta_{hm}\beta_{mh}\mu_h\left(B_2\left(2\theta_2+(2-\pi)\mu_h\right) - B_1\left(\theta_2+(1-\pi)\mu_h\right)\right) + N_h\beta_{mh}\left((a+\eta_h+\mu_h)\left(B_2\left(2\theta_2+\mu_h\right) - B_1\left(\theta_2+\mu_h\right)\right)\mu_m + m + (2B_2 - B_1)\right.$$

$$B_3\beta_{hm}\theta_1(\theta_2+\mu_h)R_h^*)+\sqrt{(N_h^2\beta_{mh}^2(4B_1B_2\mu_h(a+\eta_h+\mu_h)(\theta_2+\mu_h)\mu_m$$
$$\left(\frac{kB_3\beta_{hm}\beta_{mh}t}{\varphi(\mu_h+\theta_2)\mu_m R_0^2}+m+B_3\beta_{hm}\left(N_h\mu_h+\theta_1R_h^*\right)\right)+(B_2\mu_h(-m\pi-B_3N_h\beta_{hm}\mu_h$$
$$+(a+\eta_h+\mu_h)\mu_m)+B_1\left((a+\eta_h+\mu_h)(\theta_2+\mu_h)\mu_m-m-B_3\beta_{hm}\left(N_h\mu_h(\theta_2-\right.\right.$$
$$\left.\left.(-1+\pi)\mu_h)+\theta_1(\theta_2+\mu_h)R_h^*)\right)^2\right)\Big)\Big)/\left(2m+(B_2-B_1)B_3N_h\beta_{hm}\beta_{mh}\mu_h(\theta_2+\mu_h)\right),$$

$$S_{h_2}^*=\Bigg(m+B_3N_h^2\beta_{hm}\beta_{mh}\mu_h\pi B_2\mu_h+\frac{\varphi\mu_m^2(\mu_h+\eta_h+a)(\mu_h+\theta_2)R_0^2}{k\beta_{mh}t}N_h^2\beta_{mh}\mu_hB_1$$
$$(\theta_2+(1+\pi)\mu_h)+N_h\beta_{mh}\Bigg((a+\eta_h+\mu_h)\Bigg(B_2\mu_h+B_1\frac{kB_3\beta_{hm}\beta_{mh}t}{\mu_m(\mu_h+\eta_h+a)\varphi\mu_mR_0^2}\Bigg)\mu_m+m+$$
$$\frac{\varphi\mu_m^2(\mu_h+\eta_h+a)(\mu_h+\theta_2)R_0^2}{k\beta_{mh}t}B_1\theta_1(\theta_2+\mu_h)R_h^*\Bigg)+\sqrt{(N_h^2\beta_2^2(4B_1B_2\mu_h(a+\eta_h+\mu_h)(\theta_2+\mu_h)\mu_m\left(\frac{kB_3\beta_{hm}\beta_{mh}t}{\varphi(\mu_h+\theta_2)\mu_mR_0^2}+m+B_3\beta_{hm}\left(N_h\mu_h+\theta_1R_h^*\right)\right)+(B_2\mu_h(-m\pi-$$
$$B_3N_h\beta_{hm}\mu_h+(a+\eta_h+\mu_h)\mu_m)+B_1\left((a+\eta_h+\mu_h)(\theta_2+\mu_h)\mu_m-m-B_3\beta_{hm}\left(N_h\mu_h(\theta_2-\right.\right.$$
$$\left.\left.(-1+\pi)\mu_h)+\theta_1(\theta_2+\mu_h)R_h^*)\right)^2\right)\Big)\Big)/\left(2m+(B_2-B_1)B_3N_h\beta_{hm}\beta_{mh}\mu_h(\theta_2+\mu_h)\right)$$

$$I_h^*=\frac{\mu_mk\beta_{mh}t}{k\beta_{mh}tm+\varphi\mu_m^2(\mu_h+\eta_h+a)(\mu_h+\theta_2)R_0^2}$$

$$I_m^*=\Bigg(m+\frac{\varphi\mu_m^2(\mu_h+\eta_h+a)(\mu_h+\theta_2)R_0^2}{k\beta_{mh}t}N_h^2\beta_{mh}\mu_h\left(\pi B_2\mu_h+B_1\left(\theta_2+(1-\pi)\mu_h\right)\right)+N_h\beta_2\Bigg(\frac{kB_3\beta_{hm}\beta_{mh}t}{\varphi(\mu_h+\theta_2)\mu_mR_0^2}\left(B_2\mu_h+B_1(\theta_2+\mu_h)\right)+m+\frac{\varphi\mu_m^2(\mu_h+\eta_h+a)(\mu_h+\theta_2)R_0^2}{k\beta_{mh}t}$$
$$B_1\theta_1(\theta_2+\mu_h)R_h^*)+\sqrt{\Bigg(N_h^2\beta_{mh}^2\Bigg(4B_1B_2\mu_h(a+\eta_h+\mu_h)(\theta_2+\mu_h)\mu_m\Bigg(\frac{kB_3\beta_{hm}\beta_{mh}t}{\varphi(\mu_h+\theta_2)\mu_mR_0^2}$$
$$+m+B_3\beta_{hm}\left(N_h\mu_h+\theta_1R_h^*\right)+\left(B_2\mu_h\left(-m\pi-B_3N_h\beta_{hm}\mu_h+(a+\eta_h+\mu_h)\mu_m\right)+\right.$$
$$B_1\Big((a+\eta_h+\mu_h)(\theta_2+\mu_h)\mu_m-m-B_3\beta_{hm}\Big(N_h\mu_h\left(\theta_2-(-1+\pi)\mu_h\right)+$$
$$\left.\theta_1(\theta_2+\mu_h)R_h^*)\right)^2\Bigg)\Bigg)\Bigg)/\left(2B_1B_2\beta_{mh}^2\left((a+\eta_h+\mu_h)\mu_m+m\left(\sqrt{T}R_0+1\right)\left(\sqrt{T}R_0-1\right)\right)\right)$$

where $T=\dfrac{\varphi\mu_m^2(\mu_h+\eta_h+a)(\mu_h+\theta_2)\left(N_h\mu_h+\theta_1R_h^*\right)}{mk\beta_{mh}t}$

$$R_0^2=\frac{-kB_3\beta_{hm}\beta_{mh}t}{\varphi\mu_m(\mu_h+\theta_2)\mu_m(\mu_h+\eta_h+a)}$$

the additive compound matrices approach is used, using the idea of [19]. Local stability of the endemic equilibrium point is determined by the variational matrix $\mathrm{J}\left(E^*\right)$ of the nonlinear system (6) corresponding to E^* and get the matrix

3.6.2. Local Stability of the Endemic Equilibrium

In order to analyse the stability of the endemic equilibrium,

$$\mathrm{J}\left(E^*\right)=\begin{pmatrix}-\mu_h-\dfrac{B_1\beta_{mh}I_m^*}{N_h} & \theta_2 & 0 & -\dfrac{B_1\beta_{mh}S_{h_1}^*}{N_h}\\ 0 & -\theta_2-\mu_h-\dfrac{B_2\beta_{mh}I_m^*}{N_h} & 0 & -\dfrac{B_2\beta_{mh}S_{h_2}^*}{N_h}\\ \dfrac{B_1\beta_{mh}I_m^*}{N_h} & \dfrac{B_2\beta_{mh}I_m^*}{N_h} & -a-\eta_h-\mu_h & \dfrac{\beta_{mh}(B_1S_{h_1}^*+B_2S_{h_2}^*)}{N_h}\\ 0 & 0 & mB_3\beta_{hm}-\dfrac{B_3\beta_{hm}I_m^*}{N_h} & -\mu_m-\dfrac{B_3\beta_{hm}I_h^*}{N_h}\end{pmatrix} \tag{7}$$

From (7) the second additive compound matrix is given by

$$J^{[2]}_{(E^*)} = \begin{bmatrix} a_{11} & 0 & -\frac{B_2\beta_{mh}S^*_{h_2}}{N_h} & 0 & \frac{B_1\beta_{mh}S^*_{h_1}}{N_h} & 0 \\ \frac{B_2\beta_{mh}I^*_m}{N_h} & a_{22} & a_{23} & \theta_2 & 0 & \frac{B_1\beta_{mh}S^*_{h_1}}{N_h} \\ 0 & mB_3\beta_1 - \frac{B_3\beta_{hm}I^*_m}{N_h} & a_{33} & 0 & \theta_2 & 0 \\ -\frac{B_1\beta_{mh}I^*_m}{N_h} & 0 & 0 & a_{44} & a_{45} & \frac{B_2\beta_{mh}S^*_{h_2}}{N_h} \\ 0 & 0 & 0 & mB_3\beta_{hm} - \frac{B_3\beta_{hm}I^*_m}{N_h} & a_{55} & 0 \\ 0 & 0 & \frac{B_1\beta_{mh}I^*_m}{N_h} & 0 & \frac{B_2\beta_{mh}I^*_m}{N_h} & a_{66} \end{bmatrix} \text{ in [11]}$$

where

$$a_{11} = -\mu_h - \frac{B_1\beta_{mh}I^*_m}{N_h} - \theta_2 - \mu_h - \frac{B_2\beta_{mh}I^*_m}{N_h},\ a_{22} = -\frac{B_1\beta_{mh}I^*_m}{N_h} - a - \eta_h - 2\mu_h, a_{23} = \frac{\beta_{mh}\left(B_1S^*_{h_1} + B_2S^*_{h_2}\right)}{N_h}\ a_{33} = -\mu_h - \frac{B_1\beta_{mh}I^*_m}{N_h} - \mu_m - \frac{B_3\beta_{hm}I^*_h}{N_h}$$

$$a_{44} = -\theta_2 - \mu_h - \frac{B_2\beta_{mh}I^*_m}{N_h} - a - \eta_h - \mu_h, a_{45} = \frac{\beta_{mh}\left(B_1S^*_{h_1} + B_2S^*_{h_2}\right)}{N_h}, a_{55} = -\theta_2 - \mu_h - \frac{B_2\beta_{mh}I^*_m}{N_h} - \mu_m - \frac{B_3\beta_{hm}I^*_h}{N_h},$$

$$a_{66} = -a - \eta_h - \mu_h - \mu_m - \frac{B_3\beta_{hm}I^*_h}{N_h}$$

The following lemma was stated and proved by [20] to demonstrate the local stability of endemic equilibrium point E^*.

Lemma 3.2:

Let $J\left(E^*\right)$ be a 4×4 real matrix. If $tr\left(J\left(E^*\right)\right)$, $\det\left(J\left(E^*\right)\right)$ and $\det\left(J^{[2]}\left(E^*\right)\right)$ are all negative, then all eigenvalues of $J\left(E^*\right)$ have negative real parts.

Using the above Lemma, we will study the stability of the endemic equilibrium.

Theorem 3.3: If $R_0 > 1$, the endemic equilibrium E^* of the model (6) is locally asymptotically stable in Ω

Proof:

From the Jacobian matrix $J\left(E^*\right)$ in (7), we have

$$tr\left(J\left(E^*\right)\right) = -\mu_h - \frac{B_1\beta_{mh}I^*_m}{N_h} - \theta_2 - \mu_h - \frac{B_2\beta_{mh}I^*_m}{N_h} - a - \eta_h - \mu_h - \mu_m - \frac{B_3\beta_{hm}I^*_h}{N_h} < 0$$

$$\det\left(J\left(E^*\right)\right) = -\frac{1}{N_h^3}\left(\frac{kB_3\beta_{hm}\beta_{mh}t}{\varphi\mu_m^2(\mu_h+\theta_2)R_0^2}\left(N_h\mu_m + B_3\beta_{hm}I^*_h\right)\left(N_h\mu_h + B_1\beta_{mh}I^*_m\right)\right.$$

$$\left.\left(N_h\left(\theta_2+\mu_h\right) + B_2\beta_{mh}I^*_m\right) + B_3\beta_{hm}\beta_{mh}\mu_h\left(mN_h - I^*_m\right)\left(B_2N_h\left(\theta_2+\mu_h\right)S^*_{h_2} + \ B_1\left(N_h\left(\theta_2+\mu_h\right)S^*_{h_1} + B_2\beta_2I^*_m\left(S^*_{h_1}+S^*_{h_2}\right)\right)\right)\right) < 0$$

$$\det(J^{[2]}(E^*)) = -\frac{1}{N_h^6}\left(B_3\beta_{hm}\beta_{mh}\left(mN_h - I^*_m\right)\left(fR_0^2\left(mN_h - I^*_m\right)F + \left(N_h(a+\eta_h+2\mu_h) + B_1\beta_{mh}I^*_m\right)\left(B_2\beta_{mh}I^*_mV + \left(N_h(a+\eta_h+\mu_h+\mu_m) + B_3\beta_{hm}I^*_h\right)\left(\left(N_h(\mu_h+\mu_m)\right.\right.\right.\right.$$

$$\left.\left.\left.\left. + B_3\beta_{hm}I^*_h + B_1\beta_{mh}I^*_m\right)\left(N_h(\theta_2+2\mu_h) + (B_1+B_2)\beta_{mh}I^*_m\right)\left(B_1S^*_{h_1} + B_2S^*_{h_2}\right) + \ \left(\beta_{mh}B_1I^*_mB_1\left(N_h(\mu_h+\mu_m) + B_3\beta_{hm}I^*_h\right)S^*_{h_1}\left(\sqrt{\delta}R_0+1\right)\left(\sqrt{\delta}R_0-1\right) + \beta_{mh}B_1I^*_m\right.\right.\right.$$

$$\left.\left.\left. B_2N_h\theta_2S^*_{h_2}\right)\right)\right) + \left(N_h\left(\theta_2+\mu_h+\mu_m\right) + B_3\beta_{hm}I^*_h + B_2\beta_{mh}I^*_m\right)\left(G + N_h\left(a+\eta_h+\right.\right.$$

$$\left.\left.\mu_h+\mu_m\right)\left(\sqrt{y}R_0+1\right)\left(\sqrt{y}R_0-1\right)\left(fR_0^2B_1B_2N_h\beta_{mh}\theta_2I^*_m\left(mN_h - I^*_m\right)S^*_{h_2} + \right.\right.$$

$$\left(N_h\left(a+\eta_h+\theta_2+2\mu_h\right)+B_2\beta_{mh}I_m^*\right)\left(\left(N_h\left(a+\eta_h+2\mu_h\right)+B_1\beta_{mh}I_m^*\right)Q+\right.$$
$$\left.\left(B_2^2B_3\beta_{hm}\beta_{mh}\left(mN_h-I_m^*\right)\beta_{mh}I_m^*S_{h_2}^*+fR_0^2\left(mN_h-I_m^*\right)J\right)\right)\Big)\Big)<0$$

where $f=\dfrac{(\mu_h+\eta_h+a)\varphi\mu_m(\mu_h+\theta_2)\mu_m}{kt}$, $\delta=\dfrac{B_1I_m^*S_{h_1}^*f}{\left(N_h\left(\mu_h+\mu_m\right)+B_3\beta_{hm}I_h^*\right)S_{h_1}^*B_3kt\beta_{hm}}$

$$F=(B_1\left(N_h\left(a+\eta_h+\mu_h+\mu_m\right)+B_3\beta_{hm}I_h^*\right)S_{h_1}^*+B_1^2\beta_{mh}I_m^*S_{h_1}^*+B_2\left(N_h\left(a+\eta_h+\mu_h+\mu_m\right)+B_3\beta_{hm}I_h^*+B_2\beta_{mh}I_m^*\right)S_{h_2}^*\right)$$
$$\left(B_2N_h\left(\theta_2+2\mu_h\right)S_{h_2}^*+B_1(N_h\left(\theta_2+2\mu_h\right)S_{h_1}^*+B_2\beta_{mh}I_m^*\left(S_{h_1}^*+S_{h_2}^*\right)\right)\Big)$$

$$G=B_1B_3\beta_{hm}\beta_{mh}^2\left(mN_h-I_m^*\right)I_m^*\left(N_h\left(\theta_2+2\mu_h\right)+\left(B_1+B_2\right)\beta_{mh}I_m^*\right)\ \left(B_1\left(N_h\left(a+\eta_h+\theta_2+2\mu_h\right)+B_2\beta_{mh}I_m^*\right)S_{h_1}^*+B_2N_h\theta_2S_{h_2}^*\right)$$

$$J=\left(N_h\left(\theta_2+2\mu_h\right)+\left(B_1+B_2\right)\beta_{mh}I_m^*\right)\left(B_1S_{h_1}^*+B_2S_{h_2}^*\right)$$

$$V=\left(N_h\left(\theta_2+2\mu_h\right)+\left(B_1+B_2\right)\beta_{mh}I_m^*\right)(B_2(N_h\left(\mu_h+\mu_m\right)+B_3\beta_{hm}I_h^*)+B_1\left(N_h\theta_2+B_2\beta_{mh}I_m^*\right))S_{h_2}^*$$

$$Q=\left(N_h\left(\mu_h+\mu_m\right)+B_3\beta_{hm}I_h^*+B_1\beta_{mh}I_m^*\right)(N_h\left(\theta_2+2\mu_h\right)+\left(B_1+B_2\right)\beta_{mh}I_m^*)$$

$$y=\frac{fI_h^*}{N_h\left(a+\eta_h+\mu_h+\mu_m\right)kt\beta_{mh}},$$
$$R_0^2=\frac{-kB_3\beta_{hm}\beta_{mh}t}{\varphi\mu_m\left(\mu_h+\theta_2\right)\mu_m\left(\mu_h+\eta_h+a\right)}$$

Thus, from the lemma 1, the endemic equilibrium E^* of the model system (6) is locally asymptotically stable in Ω.

3.6.3. Global Stability of Endemic Equilibrium Point (EEP)

Theorem 3: If $R_0>1$ the endemic equilibrium E^* of the model system (1) is globally asymptotically stable

Proof: To establish the global stability of endemic equilibrium E^* we construct the following positive Lyapunov function V as follows;

$$V\left(S_{h_1}^*,S_{h_2}^*,I_h^*,T_h^*,R_h^*,A_m^*,S_m^*\ \&\ \mathrm{I}_m^*\right)=\left(S_{h_1}-S_{h_1}^*\ln S_{h_1}\right)+\left(S_{h_2}-S_{h_2}^*\ln S_{h_2}\right)+\left(I_h-I_h^*\ln I_h\right)\left(T_h-T_h^*\ln T_h\right)+\left(R_h-R_h^*\ln R_h\right)$$
$$+\left(A_m-A_m^*\ln A_m\right)+\left(S_m-S_m^*\ln S_m\right)+\left(I_m-I_m^*\ln I_m\right) \quad (8)$$

Direct calculation of the derivative of V along the solutions of (8) gives,

$$\frac{dV}{dt}\left(S_{h_1}^*,S_{h_2}^*,I_h^*,T_h^*,R_h^*,A_m^*,S_m^*\ \&\ \mathrm{I}_m^*\right)=\left(1-\frac{S_{h_1}^*}{S_{h_1}}\right)\frac{dS_{h_1}}{dt}+\left(1-\frac{S_{h_2}^*}{S_{h_2}}\right)\frac{dS_{h_2}}{dt}+\left(1-\frac{I_h^*}{I_h}\right)\frac{dI_h}{I_h}+\left(1-\frac{T_h^*}{T_h}\right)\frac{dT_h}{dt}+\left(1-\frac{R_h^*}{R_h}\right)\frac{dR_h}{dt}$$
$$+\left(1-\frac{A_m^*}{A_m}\right)\frac{dA_m}{dt}+\left(1-\frac{S_m^*}{S_m}\right)\frac{dS_m}{S_m}+\left(1-\frac{I_m^*}{I_m}\right)\frac{dI_m}{dt}.$$

Consequently

$$\frac{dV}{dt}=\left(\frac{S_{h_1}-S_{h_1}^*}{S_{h_1}}\right)\frac{dS_{h_1}}{dt}+\left(\frac{S_{h_2}-S_{h_2}^*}{S_{h_2}}\right)\frac{dS_{h_2}}{dt}+\left(\frac{I_h-I_h^*}{I_h}\right)\frac{dI_h}{dt}+\left(\frac{T_h-T_h^*}{T_h}\right)\frac{dT_h}{dt}+\left(\frac{R_h-R_h^*}{R_h}\right)\frac{dR_h}{dt}+\left(\frac{A_m-A_m^*}{A_m}\right)\frac{dA_m}{dt}$$
$$+\left(\frac{S_m-S_m^*}{S_m}\right)\frac{dS_m}{dt}+\left(\frac{I_m-I_m^*}{I_m}\right)\frac{dI_m}{dt}$$

which gives

$$\frac{dV}{dt}=A-B \quad (9)$$

where

$$A=\mu_h N_h+\pi\mu_h N_h\frac{S_{h_1}^*}{S_{h_1}}+\frac{B_1\beta_{mh}I_m^*}{N_h}\frac{\left(S_{h_1}-S_{h_1}^*\right)^2}{S_{h_1}}+\theta_1 R_h+\theta_1 R_h^*\frac{S_{h_1}^*}{S_{h_1}}+\theta_2 S_{h_2}+\theta_2 S_{h_2}^*\frac{S_{h_1}^*}{S_{h_1}}$$

$$+\pi\mu_h N_h+\frac{B_2\beta_{mh}I_m^*}{N_h}\frac{\left(S_{h_2}-S_{h_2}^*\right)^2}{S_{h_2}}+\frac{\beta_{mh}I_m}{N_h}B_1S_{h_1}+\frac{\beta_{mh}I_m}{N_h}B_1S_{h_1}^*\frac{I_h^*}{I_h}+\frac{\beta_{mh}I_m^*}{N_h}B_1S_{h_1}\frac{I_h^*}{I_h}$$

$$+\frac{\beta_{mh}I_m^*}{N_h}B_1S_{h_1}^*+\frac{\beta_{mh}I_m}{N_h}B_2S_{h_2}+\frac{\beta_{mh}I_m}{N_h}B_2S_{h_2}^*\frac{I_h^*}{I_h}+\frac{\beta_{mh}I_m^*}{N_h}B_2S_{h_2}\frac{I_h^*}{I_h}+\frac{\beta_{mh}I_m^*}{N_h}B_2S_{h_2}^*$$

$$+\eta_h I_h+\eta_h I_h^*\frac{T_h^*}{T_h}+\delta_h T_h+\delta_h T_h^*\frac{R_h^*}{R_h}+\varphi S_m+\varphi S_m^*\frac{A_m^*}{A_m}+\varphi I_m+\frac{\varphi\left(A_m-A_m^*\right)^2}{kN_hA_m}S_m^*+$$

$$\varphi I_m^*\frac{A_m^*}{A_m}+\frac{\varphi\left(A_m-A_m^*\right)^2}{kN_hA_m}I_m^*+\eta_A A_m+\eta_A A_m^*\frac{S_m^*}{S_m}+\frac{B_3\beta_{hm}I_h^*}{N_h}\frac{\left(S_m-S_m^*\right)^2}{S_m}+$$

$$\frac{B_3\beta_{hm}I_h}{N_h}S_m+\frac{B_3\beta_{hm}I_h^*}{N_h}S_m\frac{I_m^*}{I_m}+\frac{B_3\beta_{hm}I_h^*}{N_h}S_m^*+\frac{B_3\beta_{hm}I_h}{N_h}S_m^*\frac{I_m^*}{I_m}$$

$$B=-\mu_h N_h\frac{S_{h_1}^*}{S_{h_1}}-\pi\mu_h N_h-\frac{B_1\beta_{mh}I_m}{N_h}\frac{\left(S_{h_1}-S_{h_1}^*\right)^2}{S_{h_1}}-\mu_h\frac{\left(S_{h_1}-S_{h_1}^*\right)^2}{S_{h_1}}-$$

$$\theta_1 R_h^*-\theta_1 R_h\frac{S_{h_1}^*}{S_{h_1}}-\theta_2 S_{h_2}^*\frac{S_{h_1}^*}{S_{h_1}}-\theta_2 S_{h_2}-\pi\mu_h N_h\frac{S_{h_2}^*}{S_{h_2}}-\frac{B_2\beta_{mh}I_m}{N_h}\frac{\left(S_{h_2}-S_{h_2}^*\right)^2}{S_{h_2}}-$$

$$\mu_h\frac{\left(S_{h_2}-S_{h_2}^*\right)^2}{S_{h_2}}-\theta_2\frac{\left(S_{h_2}-S_{h_2}^*\right)^2}{S_{h_2}}-\frac{\beta_{mh}I_m}{N_h}B_1S_{h_1}\frac{I_h^*}{I_h}-\frac{\beta_{mh}I_m}{N_h}B_1S_{h_1}^*-\frac{\beta_{mh}I_m^*}{N_h}B_1S_{h_1}-\frac{\beta_{mh}I_m^*}{N_h}B_1S_{h_1}^*\frac{I_h^*}{I_h}-\frac{\beta_{mh}I_m}{N_h}B_2S_{h_2}\frac{I_h^*}{I_h}-\frac{\beta_{mh}I_m}{N_h}B_2S_{h_2}^*-$$

$$\frac{\beta_{mh}I_m^*}{N_h}B_2S_{h_2}-\frac{\beta_{mh}I_m^*}{N_h}B_2S_{h_2}^*\frac{I_h^*}{I_h}-\left(\mu_h+\eta_h+a\right)\frac{\left(I_h-I_h^*\right)^2}{I_h}-\eta_h I_h^*-$$

$$\eta_h I_h\frac{T_h^*}{T_h}-\left(\mu_h+\delta_h\right)\frac{\left(T_h-T_h^*\right)^2}{T_h}-\delta_h T_h\frac{R_h^*}{R_h}-\delta_h T_h^*-\left(\mu_h+\theta_1\right)\frac{\left(R_h-R_h^*\right)^2}{R_h}-$$

$$\varphi S_m\frac{A_m^*}{A_m}-\varphi S_m^*-\frac{\varphi\left(A_m-A_m^*\right)^2}{kN_hA_m}S_m-\varphi I_m\frac{A_m^*}{A_m}-\varphi I_m^*-\frac{\varphi\left(A_m-A_m^*\right)^2}{kN_hA_m}I_m-$$

$$\left(\mu_A+\eta_A\right)\frac{\left(A_m-A_m^*\right)^2}{A_m}-\eta_A A_m\frac{S_m^*}{S_m}-\eta_A A_m^*-\frac{B_3\beta_{hm}I_h}{N_h}\frac{\left(S_m-S_m^*\right)^2}{S_m}-\mu_m\frac{\left(S_m-S_m^*\right)^2}{S_m}-\frac{B_3\beta_{hm}I_h}{N_h}S_m\frac{I_m^*}{I_m}-\frac{B_3\beta_{hm}I_h}{N_h}S_m^*-\frac{B_3\beta_{hm}I_h^*}{N_h}S_m-$$

$$\frac{B_3\beta_{hm}I_h^*}{N_h}S_m^*\frac{I_m^*}{I_m}-\mu_m\frac{\left(I_m-I_m^*\right)^2}{I_m}$$

Thus from equation (9), if $A<\mathrm{B}$ then $\frac{dV}{dt}$ will be negative definite, meaning that $\frac{dV}{dt}<0$. It follows that $\frac{dV}{dt}=0$ if and only if $S_{h_1}=S_{h_1}^*$ $S_{h_2}=S_{h_2}^*$, $I_h=I_h^*$, $T_h=T_h^*$, $R_h=R_h^*$, $A_m=A_m^*$, $S_m=S_m^*$ and $I_m=I_m^*$. Therefore the largest compact invariant set in $\left\{S_{h_1}^*,S_{h_2}^*,I_h^*,T_h^*,R_h^*,A_m^*,S_m^*,I_m^*\in\Omega:\frac{dV}{dt}=0\right\}$ is the singleton $\left\{E^*\right\}$ where E^* is the endemic equilibrium of the model system (1). By LaSalle's invariant principle, then it implies that E^* is globally asymptotically stable in Ω if $A<\mathrm{B}$. This completes the proof.

4. Numerical Simulations

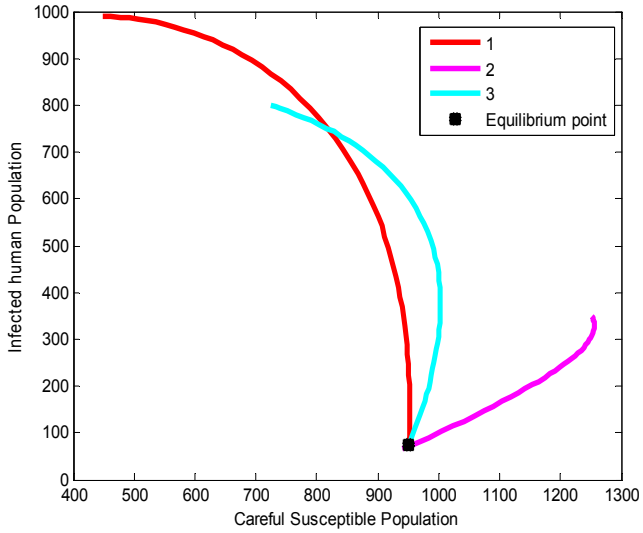
In this section, we illustrate the analytical results of the study by carrying out numerical simulations of the model system (1) using a set of reasonable parameter values. Parameter values are obtained from the different literatures like [12], [13] and [17] (http://www.wavuti.com/2014/05/wizara- ya- afya- kitengo-cha.html). Other parameter values are estimated to vary within realistic means and given

as shown below.

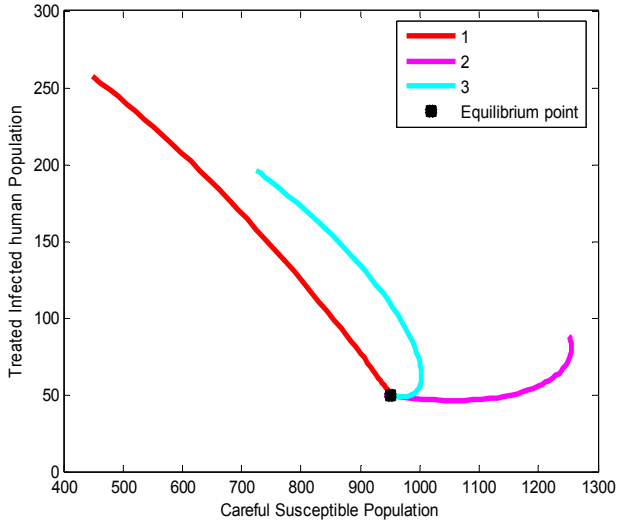
$$\beta_{hm}=0.375,\ \beta_{mh}=0.45,\ \pi=0.96,\ B_1=0.5,\ B_2=0.9,$$
$$B_3=0.7,\ \mu_m=\frac{1}{11},\ k=3,\ \eta_A=0.35,\ \mu_A=0.25,$$
$$\mu_h=\frac{1}{78\times365},\ \eta_h=1/3,\ \varphi=5,\ \theta_1=0.01,\theta_2=0.6,$$
$$a=0.001,\delta_h=0.98 \quad (10)$$

Figures 2 (i)-(vi) show the proportion of Dengue fever disease infectives, treated and recovery proportion all plotted against the proportion of susceptible population. This shows the dynamic behaviour of the endemic equilibrium of the model system (1) using the parameter values in (10) for different initial starting values in three cases as shown below [16].

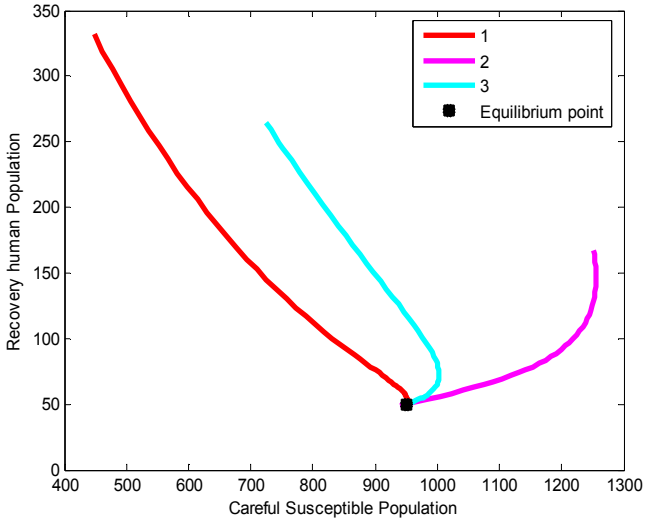
1. $S_{h_1}(0)=951$, $S_{h_2}(0)=950$, $I_h(0)=63$, $T_h(0)=50$, $R_h(0)=49$, $A_m(0)=15000$, $S_m(0)=10000$ and $I_m(0)=5000$.
2. $S_h(0)=940$, $S_{h_2}(0)=945$, $I_h(0)=65$, $T_h(0)=50$, $R_h(0)=49$, $A_m(0)=10000$, $S_m(0)=2000$ and $I_m(0)=1000$
3. $S_h(0)=949$, $S_{h_2}(0)=950$, $I_h(0)=65$, $T_h(0)=50$, $R_h(0)=49$, $A_m(0)=20000$, $S_m(0)=10000$ and $I_m(0)=3000$



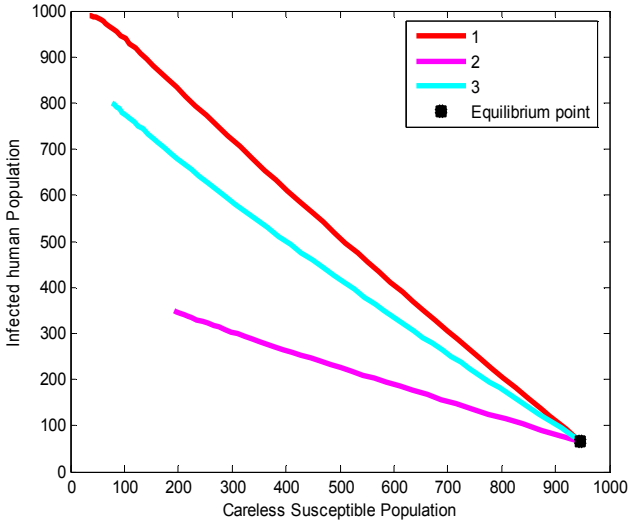
(i)



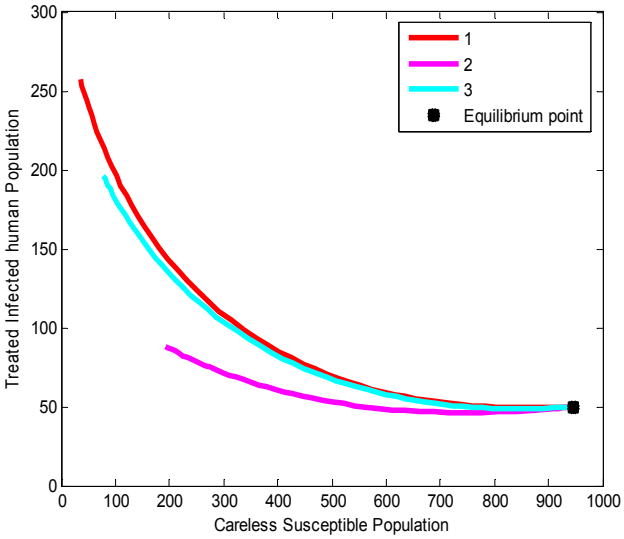
(ii)



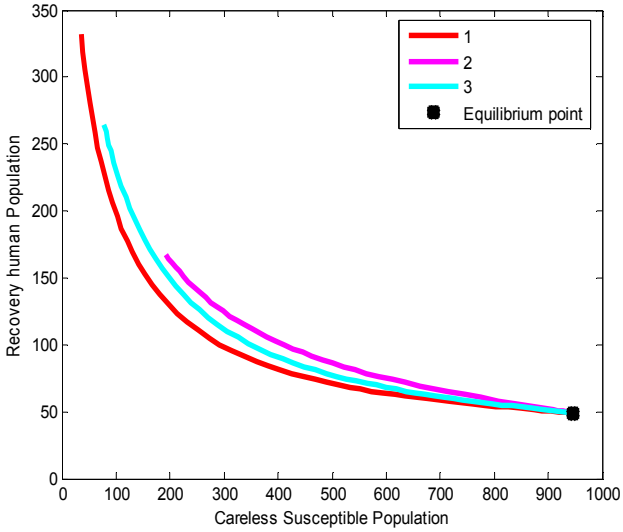
(iii)



(iv)



(v)



(vi)

Figures 2. *(i)-(vi): Variation of proportion of Dengue fever disease infectives, treated and recovery proportion all plotted against the proportion of susceptible population.*

The equilibrium point of the endemic equilibrium E^* was obtained as $S_{h_1}^* = 950$, $I_h^* = 75$, $T_h^* = 50$ and $R_h^* = 50$ and then $S_{h_2}^* = 946$ $I_h^* = 66$, $T_h^* = 50$ and $R_h^* = 49$

It is observed from figures 2(i)-(vi) that for any starting initial value, the solution curves tend to the equilibrium E^*. Therefore we conclude that the model system (1) is globally stable about this endemic equilibrium point E^* for the parameters displayed in (10).

Figure 3 shows the variation of population in different classes.

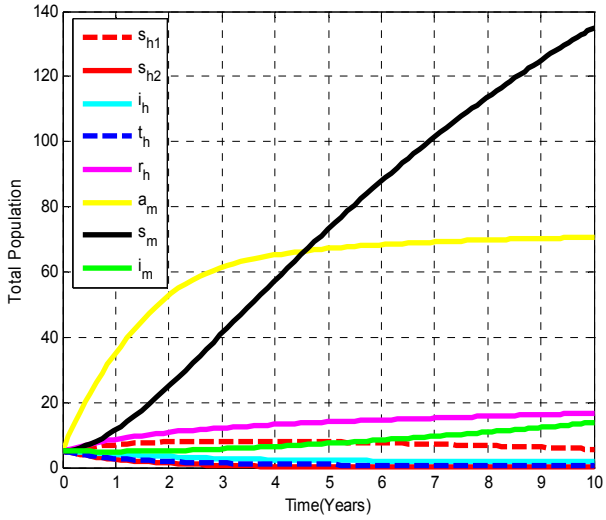
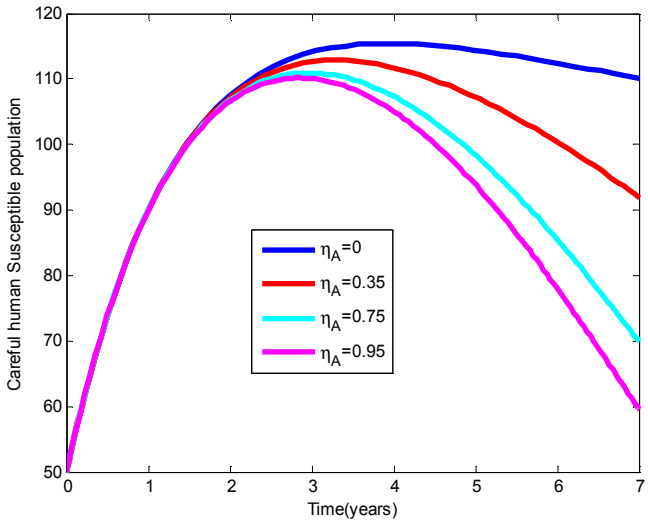


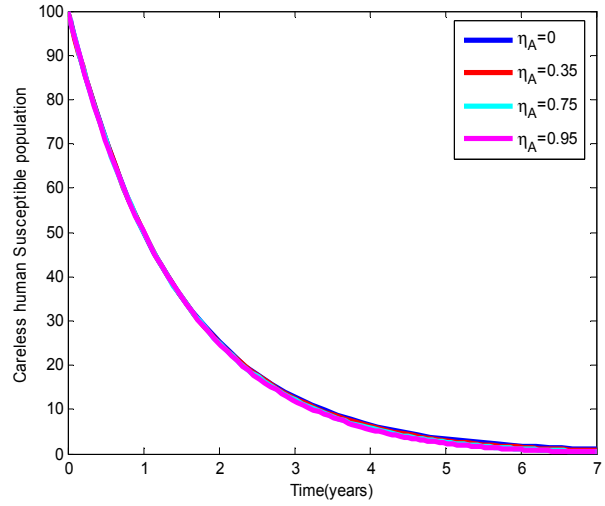
Figure 3. *Distribution of population with time in all classes of human and mosquito.*

From figure 3, it is observed that careful human susceptible population increases in time reaching its equilibrium position due to treatment and change of behaviour of careless susceptible. Moreover, careless human susceptible population decreases with time, due to careless individual moving to other classes. Dengue fever disease infected population decreases in time then reaches equilibrium due to the increase in the number of population changing behaviour and become careful and increase of recovered population. Treated infected population decrease due to the increase of the recovered population. Furthermore aquatic phase increases and then reach the equilibrium point due to its short life span and other move to susceptible class. Mosquito susceptible increases with time and reaches its equilibrium point due to its short life span and others move to infected class.

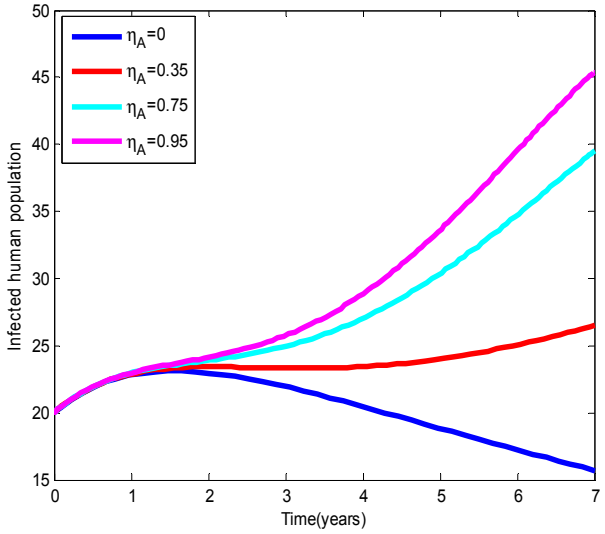
Figures 4(i)-(iv) show the variation of careful and careless human susceptible, infected human and infected mosquito population for different values of maturation rate from larvae to adult (per day) η_A



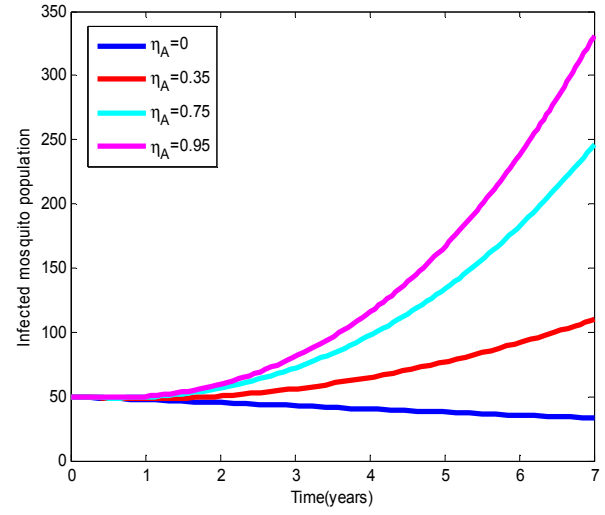
(i)



(ii)



(iii)



(iv)

Figure 4. *(i)-(iv) variation of careful and careless human susceptible, infected human and infected mosquito population for different values of maturation rate from larvae to adult (per day)* η_A .

From figure 4(i)-(iv) the maturation rate from larvae to adult (per day) η_A is varied, and it is observed that when maturation rate from larvae to adult (per day) increases, careful human susceptible increases and then decreases with time due to the increase of production of infected mosquito. Moreover careless human susceptible decrease with time while infected human and mosquito population increase.

5. Conclusion

A compartmental model for Dengue fever disease was presented, based on two populations, humans (with temporary immunity, careful and careless susceptible) and mosquitoes with treatment. Sensitivity analysis revealed that the most sensitive parameter is maturation rate from larvae to adult (per day). Simulation shows that when maturation rate from larvae to adult (per day) increase, the number of infected individual increase while careful and careless susceptible decrease. This indicates that on the reduction of maturation rate from larvae to adult (per day), it is possible to maintain the basic reproduction number below unity and the disease can be eradicated from the community.

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