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Numerical Solution of an Optimal Control Problem Governed by Two Dimensional Schrodinger Equation

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Abstract: In this study, the finite difference method is applied to an optimal control problem controlled by two functions which are in the coefficients of two-dimensional Schrodinger equation. Convergence of the finite difference approximation according to the functional is proved. We have used the implicit method for solving the two-dimensional Schrodinger equation. Although the implicit scheme obtained from solution of the system of the linear equations is generally numerically stable and convergent without time-step condition, the solution of considered equation is numerically stable with time-step condition, due to the gradient term.

Keywords: Optimal Control, Schrodinger Operator, Finite Difference Methods, Stability, Convergence of Numerical Methods

1. Introduction and Statement of Problem

We consider the following optimal control problem:

$$J(v)=\int_0^{l_1}\int_0^{l_2}\left|\psi(x_1,x_2,T)-y(x_1,x_2)\right|^2 dx_2dx_1 \to \min \tag{1}$$

on the set

$$V=\left\{v=v(t)=(v_1(t),v_2(t)),\ v_p\in L_2(0,T),\ p=1,2,\ \left\|v_p\right\|_{L_2(0,T)}\le b_p,\ p=1,2\right\}$$

subject to the problem;

$$i\frac{\partial\psi}{\partial t}+a_0\left(\frac{\partial^2\psi}{\partial x_1^2}+\frac{\partial^2\psi}{\partial x_2^2}\right)+i\sum_{j=1}^{2}a_j(x_1,x_2)\frac{\partial\psi}{\partial x_j}+\sum_{j=1}^{2}b_j(x_1,x_2)v_j(t)\psi-a(x_1,x_2)\psi=f(x_1,x_2,t),\ (x_1,x_2,t)\in\Omega, \tag{2}$$

$$\psi(x_1,x_2,0)=\varphi(x_1,x_2),\ (x_1,x_2)\in D \tag{3}$$

$$\psi(0,x_2,t)=\psi(l_1,x_2,t)=\psi(x_1,0,t)=\psi(x_1,l_2,t)=0,\ t\in(0,T], \tag{4}$$

where $i^2=-1$; $a_0>0$, $T>0$ and $b_p>0$, $p=1,2$ are given numbers. $x=(x_1,x_2)\in D=(0,l_1)\times(0,l_2)\subset R^2$; $\Omega_t=D\times(0,t)$; $\Omega=\Omega_T$; $y\in W_2^1(D)$ is a given function; $a,\ a_j,\ b_j\in L_\infty(D),\ j=1,2$ are given functions satisfying the following conditions:

$$0 \le a(x) \le \mu_0,\ \left|\frac{\partial a(x)}{\partial x_k}\right| \le \mu_1,\ \left|\frac{\partial^2 a(x)}{\partial x_k \partial x_p}\right| \le \mu_2,\ k,p=1,2,\ \forall x \in \overset{0}{D},\ \mu_0,\mu_1,\mu_2 = \text{const.} > 0, \tag{5}$$

$$0 \le a_j(x) \le \upsilon_0,\ \left|\frac{\partial a_j(x)}{\partial x_k}\right| \le \upsilon_1,\ \left|\frac{\partial^2 a_j(x)}{\partial x_k \partial x_p}\right| \le \upsilon_2,\ j,p,k=1,2, \forall x \in \overset{0}{D},\ \upsilon_0,\upsilon_1,\upsilon_2 = \text{const.} > 0, \tag{6}$$

$$0 \le b_j(x) \le \eta_0,\ \left|\frac{\partial b_j(x)}{\partial x_k}\right| \le \eta_1,\ \left|\frac{\partial^2 b_j(x)}{\partial x_k \partial x_p}\right| \le \eta_2,\ j,p,k=1,2, \forall x \in \overset{0}{D},\ \eta_0,\eta_1,\eta_2 = \text{const.} > 0. \tag{7}$$

Moreover we assume that the function a_j satisfy the following condition:

$$a_j(0,x_2) = a_j(l_1,x_2) = a_j(x_1,0) = a_j(x_1,l_2) = 0,\ j=1,2. \tag{8}$$

$\varphi(x_1,x_2)$ and $f(x_1,x_2,t)$ are given functions satisfying the following conditions:

$$\varphi \in \overset{0}{W}{}_2^2(D),\ f \in \overset{0}{W}{}_2^{2,0}(\Omega). \tag{9}$$

Definitions of the given function spaces above are such as given in [9].

The optimal control problems for the Schrödinger equation have been investigated by different authors. Yetişkin and Subaşı have studied the problem of determining of the potential in the Schrodinger equation from the measured final data [1]. Yıldız at all.have investigated the existence and uniqueness of the solution of the optimal control problem for non-stationary Schrödinger equation and given necessary and sufficient conditions for the solution [2]. Beauchard and Laurent have investigated the exact controllability for the system which contain the linear Schrodinger equation, on a bounded interval, with a bilinear control in any positive time, locally around the ground state [3]. Yıldız and Subaşı have proved two estimates the solution of the optimal control problem for the linear Schrödinger equation and obtained necessary and sufficient conditions for the optimal solution [4]. Baudouin at all.have obtained that the problem $i\partial_t u + \Delta u + |x-a(t)|^{-1} u + V_1(x,t)u = 0$ with $u(x,0)=u_0(x)$ is well-posed and the regularity of the initial data is conserved for the solution if the electric potential V_1 is regular enough and at most quadratic at infinity [5].

Yagubov and Musayeva investigated finite-difference method solution of variation formulation of an inverse problem for nonlinear Schrodinger equation in [6].

This paper is organized as follows: In this section we state a theorem for the generalized solution of Schrödinger equation and then we discretize the given optimal control problem. In Section 2 we state and prove a theorem for the stability of the solution of finite difference approximations. In Section 3 we state a theorem for the error of the finite difference approximations. In the last section we state and prove two theorems for convergence of the finite difference approximations according to the functional.

Theorem 1: Suppose that the functions $a(x_1,x_2)$, $a_j(x_1,x_2)$, $b_j(x_1,x_2)$, $j=1,2$, $\varphi(x_1,x_2)$, $f(x_1,x_2,t)$ hold the conditions (5)-(9). Then the initial-boundary value problem (2)-(4) has a unique solution belonging to the space $\overset{0}{W}{}_2^{2,1}(\Omega)$ for each $v \in V$ and this solution satisfies the following estimate:

$$\|\psi\|^2_{\overset{0}{W}{}_2^{2,1}(\Omega)} \le c_0 \left(\|\varphi\|^2_{\overset{0}{W}{}_2^{2}(D)} + \|f\|^2_{\overset{0}{W}{}_2^{2,0}(\Omega)} \right) \tag{10}$$

where the number $c_0 > 0$ is independent of φ and f.

Proof of this problem can be obtained by similar processes given in [9].

Suppose that conditions (5)-(9) holds, then there exist the solution of the optimal control problem (1)-(4) according to [9].In other words, we can write the following:

$$V_* \equiv \left\{ v^* \in V : J(v^*) = J_* = \inf_{v \in V} J(v) \right\} \ne \varnothing .$$

Now we solve the optimal control problem by using finite difference method. Firstly let us separate the domain $\bar{\Omega} = [0,l_1]\times[0,l_2]\times[0,T]$ using the meshes $0 = x_{10} < x_{11} ... < x_{1M_1} = l_1$ and $0 = x_{20} < x_{21} ... < x_{2M_2} = l_2$ in space and using a mesh $0 = t_0 < t_1 < ... < t_N = T$ in time. Here the mesh points are following:

$$\left\{ \left(x_{1j_1}, x_{2j_2}, t_k\right)_n \right\},\ n=1,2,...,\ x_{1j_1} = j_1 h_{1n} - h_{1n}/2,\ j_1 = \overline{1, M_{1n}-1},\ x_{2j_2} = j_2 h_{2n} - h_{2n}/2,$$
$$j_2 = \overline{1, M_{2n}-1},\ x_{11} - \frac{h_{1n}}{2} = 0,\ x_{1M_{1n}-1} + \frac{h_{1n}}{2} = \ell_1,\ x_{21} - \frac{h_{2n}}{2} = 0,\ x_{2M_{2n}-1} + \frac{h_{2n}}{2} = \ell_2,\ t_k = k\tau_n,$$
$$k = \overline{0, N_n},\ h_1 = h_{1n} = \ell_1/(M_{1n}-1),\ h_2 = h_{2n} = \ell_2/(M_{2n}-1),$$
$$\tau = \tau_n = T/N_n,\ M_1 = M_{1n},\ M_2 = M_{2n},\ N = N_n.$$

Finite differences for the space derivatives and the time derivatives in (2) are define in the following way according to [7,8,10].

$$\delta_{\bar t}\phi_{j_1j_2k} = \left(\phi_{j_1j_2k} - \phi_{j_1j_2k-1}\right)/\tau,$$

$$\delta_{\bar x_1}\phi_{j_1j_2k} = \left(\phi_{j_1j_2k} - \phi_{j_1-1j_2k}\right)/h_1, \qquad \delta_{\bar x_2}\phi_{j_1j_2k} = \left(\phi_{j_1j_2k} - \phi_{j_1j_2-1k}\right)/h_2,$$

$$\delta_{x_1}\phi_{j_1j_2k} = \left(\phi_{j_1+1j_2k} - \phi_{j_1j_2k}\right)/h_1, \qquad \delta_{x_2}\phi_{j_1j_2k} = \left(\phi_{j_1j_2+1k} - \phi_{j_1j_2k}\right)/h_2,$$

$$\delta_{x_1\bar x_1}\phi_{j_1j_2k} = \left(\phi_{j_1+1j_2k} - 2\phi_{j_1j_2k} + \phi_{j_1-1j_2k}\right)/h_1^2, \quad \delta_{x_2\bar x_2}\phi_{j_1j_2k} = \left(\phi_{j_1j_2+1k} - 2\phi_{j_1j_2k} + \phi_{j_1j_2-1k}\right)/h_2^2,$$

$$\delta_{\bar x_1}\phi_{1j_2k} = \delta_{x_1}\phi_{0j_2k} = \frac{\left(\phi_{1j_2k} - \phi_{0j_2k}\right)}{(h_1/2)}, \qquad \delta_{\bar x_1}\phi_{M_1j_2k} = \delta_{x_1}\phi_{M_1-1j_2k} = \frac{\left(\phi_{M_1j_2k} - \phi_{M_1-1j_2k}\right)}{(h_1/2)},$$

$$\delta_{\bar x_2}\phi_{j_11k} = \delta_{x_2}\phi_{j_10k} = \frac{\left(\phi_{j_11k} - \phi_{j_10k}\right)}{(h_2/2)}, \qquad \delta_{\bar x_2}\phi_{j_1M_2k} = \delta_{x_2}\phi_{j_1M_2-1k} = \frac{\left(\phi_{j_1M_2k} - \phi_{j_1M_2-1k}\right)}{(h_2/2)}.$$

Hence finite difference approximations are substituted for the derivatives to convert the optimal control problem (1)-(4) to an algebraic form for any integer $n \ge 1$ as following problem according to [10]:

$$I_n\left([v]_n\right) = h_1h_2\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\phi_{j_1j_2N} - y_{j_1j_2}\right|^2 \to \min \tag{11}$$

on the set

$$V_n \equiv \Big\{[v]_n : [v]_n = \left([v_1]_n, [v_2]_n\right),\ \left[v_p\right]_n = \left(v_{p1}, v_{p2}, \dots, v_{pN}\right),\ p = 1,2,$$
$$\left(\tau\sum_{k=1}^{N}\left|v_{pk}\right|^2\right)^{1/2} \le b_p,\ p = 1,2,\ k = \overline{1,N}\Big\}$$

subject to the problem

$$\begin{gathered} i\delta_{\bar t}\phi_{j_1j_2k} + a_0\left(\delta_{x_1\bar x_1}\phi_{j_1j_2k} + \delta_{x_2\bar x_2}\phi_{j_1j_2k}\right) + i\sum_{j=1}^{2}a_j^{j_1j_2}\delta_{\bar x_j}\phi_{j_1j_2k} + \sum_{j=1}^{2}b_j^{j_1j_2}v_{jk}\phi_{j_1j_2k} - a^{j_1j_2}\phi_{j_1j_2k} = f_{j_1j_2k}, \\ j_1 = \overline{1, M_1-1},\ j_2 = \overline{1, M_2-1},\ k = \overline{1,N}, \end{gathered} \tag{12}$$

$$\phi_{j_1j_20} = \varphi_{j_1j_2},\ j_1 = \overline{0, M_1},\ j_2 = \overline{0, M_2}, \tag{13}$$

$$\phi_{0j_2k} = \phi_{M_1j_2k} = 0,\ j_2 = \overline{0, M_2},\ k = \overline{1,N}, \tag{14}$$

$$\phi_{j_10k} = \phi_{j_1M_2k} = 0,\ j_1 = \overline{0, M_1},\ k = \overline{1,N}. \tag{15}$$

Here $a_j^{j_1j_2}$, $b_j^{j_1j_2}$, $a^{j_1j_2}$, $y_{j_1j_2}$, $\varphi_{j_1j_2}$, $f_{j_1j_2k}$ are mesh functions which are defined as following:

$$a_j^{j_1j_2} = \frac{1}{h_1h_2}\int_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2}\int_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2} a_j\left(x_1,x_2\right)dx_2dx_1,\ j = 1,2,\ j_1 = \overline{1, M_1-1},\ j_2 = \overline{1, M_2-1}, \tag{16}$$

$$b_j^{j_1j_2} = \frac{1}{h_1h_2}\int_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2}\int_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2} b_j\left(x_1,x_2\right)dx_2dx_1,\ j = 1,2,\ j_1 = \overline{1, M_1-1},\ j_2 = \overline{1, M_2-1}, \tag{17}$$

$$a^{j_1j_2} = \frac{1}{h_1h_2}\int_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2}\int_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2} a\left(x_1,x_2\right)dx_2dx_1,\ j_1 = \overline{1, M_1-1},\ j_2 = \overline{1, M_2-1}, \tag{18}$$

$$\begin{gathered} \varphi_{j_1j_2} = \frac{1}{h_1h_2}\int_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2}\int_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2} \varphi\left(x_1,x_2\right)dx_2dx_1,\ j_1 = \overline{1, M_1-1},\ j_2 = \overline{1, M_2-1}, \\ \varphi_{00} = \varphi_{0M_2} = \varphi_{M_10} = \varphi_{M_1M_2} = 0, \end{gathered} \tag{19}$$

$$f_{j_1j_2k} = \frac{1}{h_1h_2\tau}\int_{t_{k-1}}^{t_k}\int_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2}\int_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2} f\left(x_1,x_2,t\right)dx_2dx_1dt,\ j_1 = \overline{1, M_1-1},\ j_2 = \overline{1, M_2-1},\ k = \overline{1,N}, \tag{20}$$

$$y_{j_1 j_2} = \frac{1}{h_1 h_2} \int_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2} \int_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2} y(x_1,x_2)\,dx_2 dx_1,\ j_1=\overline{1,M_1-1},\ j_2=\overline{1,M_2-1}. \tag{21}$$

Using the equalities (16)-(18) and the conditions on $a,\ a_j,\ b_j$ given with (5)-(8), we find

$$0 \le a^{j_1 j_2} \le \mu_0,\ \left|\delta_{\bar{x}_k} a^{j_1 j_2}\right| \le \mu_1,\ \left|\delta_{x_k \bar{x}_p} a^{j_1 j_2}\right| \le \mu_2,\ k,p=1,2, \tag{22}$$

$$\begin{gathered} 0 \le a_j^{j_1 j_2} \le \upsilon_0,\ \left|\delta_{\bar{x}_k} a_j^{j_1 j_2}\right| \le \upsilon_1,\ \left|\delta_{x_k \bar{x}_p} a_j^{j_1 j_2}\right| \le \upsilon_2,\ j,k,p=1,2, \\ a_j^{0 j_2} = a_j^{M_1 j_2} = 0,\ a_j^{j_1 0} = a_j^{j_1 M_2} = 0, \end{gathered} \tag{23}$$

$$0 \le b_j^{j_1 j_2} \le \eta_0,\ \left|\delta_{\bar{x}_k} b_j^{j_1 j_2}\right| \le \eta_1,\ \left|\delta_{x_k \bar{x}_p} b_j^{j_1 j_2}\right| \le \eta_2,\ j,k,p=1,2. \tag{24}$$

2. Stability of the Finite Difference Approximations

In this section, we examine whether the solution of finite difference approximations is stable or not.

Theorem 2: Suppose that the time-step τ satisfy the condition $0<\tau\le\frac{1}{8}(\upsilon_1)^{-1}$. Then the solution of finite difference approximations (12)-(15) satisfies the following estimate for each $[v]_n \in V_n$:

$$h_1 h_2 \sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\phi_{j_1 j_2 m}\right|^2 \le c_1\left(h_1 h_2 \sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\varphi_{j_1 j_2}\right|^2 + h_1 h_2 \tau \sum_{k=1}^{N}\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|f_{j_1 j_2 k}\right|^2\right),\ m\in\{1,2,\dots,N\}. \tag{25}$$

Here the number $c_1>0$ is independent of h_1, h_2 and τ.

Proof: The scheme (12)-(15) is equivalent to the following identity for each $t=t_k$:

$$\begin{aligned} & h_1 h_2 \sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left(i\delta_t \phi_{j_1 j_2 k}\bar{\eta}_{j_1 j_2 k}\right) - a_0 h_1 h_2 \sum_{j_1=1}^{M_1}\sum_{j_2=1}^{M_2-1}\left(\delta_{\bar{x}_1}\phi_{j_1 j_2 k}\delta_{\bar{x}_1}\bar{\eta}_{j_1 j_2 k}\right) - a_0 h_1 h_2 \sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2}\left(\delta_{\bar{x}_2}\phi_{j_1 j_2 k}\delta_{\bar{x}_2}\bar{\eta}_{j_1 j_2 k}\right) + h_1 h_2 \sum_{j=1}^{2}\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left(i a_j^{j_1 j_2}\delta_{\bar{x}_j}\phi_{j_1 j_2 k}\bar{\eta}_{j_1 j_2 k}\right) \\ & + h_1 h_2 \sum_{j=1}^{2}\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left(b_j^{j_1 j_2} v_{jk}\phi_{j_1 j_2 k}\bar{\eta}_{j_1 j_2 k}\right) - h_1 h_2 \sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left(a^{j_1 j_2}\phi_{j_1 j_2 k}\bar{\eta}_{j_1 j_2 k}\right) = h_1 h_2 \sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left(f_{j_1 j_2 k}\bar{\eta}_{j_1 j_2 k}\right),\ k=\overline{1,N}. \end{aligned} \tag{26}$$

Here the mesh function $\bar{\eta}_{j_1 j_2 k}$ is complex conjugate of the arbitrary mesh function $\eta_{j_1 j_2 k}$ which is defined on the mesh sequence $\left\{\left(x_{1j_1}, x_{2j_2}, t_k\right)_n\right\}$ and satisfies the conditions $\eta_{0 j_2 k}=\eta_{M_1 j_2 k}=0,\ \eta_{j_1 0 k}=\eta_{j_1 M_2 k}=0,\ j_1=\overline{0,M_1},\ j_2=\overline{0,M_2},\ \mathrm{k}=\overline{1,N}$. Here substituting $\bar{\eta}_{j_1 j_2 k}$ with $\tau\bar{\phi}_{j_1 j_2 k}$ and then subtracting complex conjugate of obtained equality from itself, we have the following equation:

$$\begin{aligned} & h_1 h_2 \tau \sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left(\delta_t \phi_{j_1 j_2 k}\bar{\phi}_{j_1 j_2 k} + \delta_t \bar{\phi}_{j_1 j_2 k}\phi_{j_1 j_2 k}\right) + h_1 h_2 \tau \sum_{j=1}^{2}\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left[a_j^{j_1 j_2}\left(\delta_{\bar{x}_j}\phi_{j_1 j_2 k}\bar{\phi}_{j_1 j_2 k} + \delta_{\bar{x}_j}\bar{\phi}_{j_1 j_2 k}\phi_{j_1 j_2 k}\right)\right] \\ & = 2 h_1 h_2 \tau \sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\operatorname{Im}\left(f_{j_1 j_2 k}\bar{\phi}_{j_1 j_2 k}\right),\ \forall k\in\{1,2,\dots,N\}. \end{aligned} \tag{27}$$

Using

$$\tau\left(\delta_t \phi_{j_1 j_2 k}\bar{\phi}_{j_1 j_2 k} + \delta_t \bar{\phi}_{j_1 j_2 k}\phi_{j_1 j_2 k}\right) = \left|\phi_{j_1 j_2 k}\right|^2 - \left|\phi_{j_1 j_2 k-1}\right|^2 + \left|\phi_{j_1 j_2 k} - \phi_{j_1 j_2 k-1}\right|^2 \tag{28}$$

$$h_1\left(\delta_{\bar{x}_1}\phi_{j_1 j_2 k}\bar{\phi}_{j_1 j_2 k} + \delta_{\bar{x}_1}\bar{\phi}_{j_1 j_2 k}\phi_{j_1 j_2 k}\right) = \left|\phi_{j_1 j_2 k}\right|^2 - \left|\phi_{j_1-1 j_2 k}\right|^2 + \left|\phi_{j_1 j_2 k} - \phi_{j_1-1 j_2 k}\right|^2 \tag{29}$$

$$h_2\left(\delta_{\bar{x}_2}\phi_{j_1 j_2 k}\bar{\phi}_{j_1 j_2 k} + \delta_{\bar{x}_2}\bar{\phi}_{j_1 j_2 k}\phi_{j_1 j_2 k}\right) = \left|\phi_{j_1 j_2 k}\right|^2 - \left|\phi_{j_1 j_2-1 k}\right|^2 + \left|\phi_{j_1 j_2 k} - \phi_{j_1 j_2-1 k}\right|^2 \tag{30}$$

in (27), adding index k from 1 through $m\le N$ and then calculating absolute value of both sides, and we use the condition (23), we have the following:

$$h_1h_2\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\phi_{j_1j_2m}\right|^2 \le h_1h_2\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\varphi_{j_1j_2}\right|^2 + h_1h_2\tau\sum_{k=1}^{m}\sum_{j_1=2}^{M_1}\sum_{j_2=1}^{M_2-1}\left|\delta_{\bar{x}_1}a_1^{j_1j_2}\right|\left|\phi_{j_1-1j_2k}\right|^2$$
$$+h_1h_2\tau\sum_{k=1}^{m}\sum_{j_1=1}^{M_1-1}\sum_{j_2=2}^{M_2}\left|\delta_{\bar{x}_2}a_2^{j_1j_2}\right|\left|\phi_{j_1j_2-1k}\right|^2 + 2h_1h_2\tau\sum_{k=1}^{m}\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|f_{j_1j_2k}\right|\left|\phi_{j_1j_2k}\right|$$
$$\le h_1h_2\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\varphi_{j_1j_2}\right|^2 + 2\upsilon_1h_1h_2\tau\sum_{k=1}^{m}\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\phi_{j_1j_2k}\right|^2$$
$$+2h_1h_2\tau\sum_{k=1}^{m}\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|f_{j_1j_2k}\right|\left|\phi_{j_1j_2k}\right|,\ \forall m\in\{1,2,...,N\}$$

The mth term of the last term on the right side of this inequality writing a separate and then applying $\varepsilon-$Cauchy and Cauchy-Bunyakovsky inequalities, we get the following inequality:

$$h_1h_2\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\phi_{j_1j_2m}\right|^2 \le h_1h_2\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\varphi_{j_1j_2}\right|^2 + 2\upsilon_1h_1h_2\tau\sum_{k=1}^{m}\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\phi_{j_1j_2k}\right|^2 + \varepsilon h_1h_2\tau\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|f_{j_1j_2m}\right|^2 + \frac{1}{\varepsilon}h_1h_2\tau\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\phi_{j_1j_2m}\right|^2$$
$$+h_1h_2\tau\sum_{k=1}^{m-1}\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|f_{j_1j_2k}\right|^2 + h_1h_2\tau\sum_{k=1}^{m-1}\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\phi_{j_1j_2k}\right|^2,\ \forall m\in\{1,2,...,N\} \quad (31)$$

Writing $\varepsilon = 2\tau$, we obtain

$$h_1h_2\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\phi_{j_1j_2m}\right|^2 \le 2h_1h_2\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\varphi_{j_1j_2}\right|^2 + (4T+2)h_1h_2\tau\sum_{k=1}^{m}\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|f_{j_1j_2k}\right|^2$$
$$+4\upsilon_1h_1h_2\tau\sum_{k=1}^{m}\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\phi_{j_1j_2k}\right|^2 + 2h_1h_2\tau\sum_{k=1}^{m-1}\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\phi_{j_1j_2k}\right|^2,\ m=\overline{1,N} \quad (32)$$

and then the mth term of the third term on the right side writing a separate and using the condition $0<\tau\le\frac{1}{8}(\upsilon_1)^{-1}$ for time-step τ, we have the following:

$$h_1h_2\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\phi_{j_1j_2m}\right|^2 \le 4h_1h_2\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\varphi_{j_1j_2}\right|^2 + (8T+4)h_1h_2\tau\sum_{k=1}^{N}\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|f_{j_1j_2k}\right|^2 + (8\upsilon_1+4)h_1h_2\tau\sum_{k=0}^{m-1}\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\phi_{j_1j_2k}\right|^2,\quad m\in\{1,2,...,N\}$$

Using Gronwall Lemma in this inequality, we get the following inequality:

$$h_1h_2\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\phi_{j_1j_2m}\right|^2 \le c_2\left(h_1h_2\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\varphi_{j_1j_2}\right|^2 + h_1h_2\tau\sum_{k=1}^{N}\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|f_{j_1j_2k}\right|^2\right),\ m\in\{1,2,...,N\}. \quad (33)$$

Here the number $c_2>0$ is independent of h_1, h_2 and τ.

3. An Estimate for the Error of the Finite Difference Approximations

Firstly, let us define the solution $\psi=\psi(x,t;v)$ of the optimal control problem (1)-(4) for each $v\in V$ such as following:

$$\left[\psi(x_1,x_2,t;v)\right]_n = \left\{\psi_{j_1j_2k}\right\},$$
$$\psi_{j_1j_2k} = \frac{1}{h_1h_2\tau}\int_{t_{k-1}}^{t_k}\int_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2}\int_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2}\psi(x_1,x_2,t)\,dx_2dx_1dt,\ j_1=\overline{1,M_1-1},\ j_2=\overline{1,M_2-1},\ k=\overline{1,N},$$
$$\psi_{j_1j_20}=\varphi_{j_1j_2},\ j_1=\overline{0,M_1},\ j_2=\overline{0,M_2},$$
$$\psi_{0j_2k}=\psi_{M_1j_2k}=0,\ j_2=\overline{0,M_2},\ k=\overline{1,N},$$
$$\psi_{j_10k}=\psi_{j_1M_2k}=0,\ j_1=\overline{0,M_1},\ k=\overline{1,N}. \quad (34)$$

Let us define the operator Q_n on the set V for each $v\in V$ such as following:

$$Q_n(v):V\to V_n,\ Q_n(v)=[w]_n=\left([w_1]_n,[w_2]_n\right),\ Q_n(v)=\left(w_{p1},w_{p2},...,w_{pN}\right),\ p=1,2,$$
$$w_{pk}=\frac{1}{\tau}\int_{t_{k-1}}^{t_k}v_p(t)\,dt,\ p=1,2,\ k=\overline{1,N} \quad (35)$$

$[Z]_n=\{Z_{j_1j_2k}\}=\{\phi_{j_1j_2k}-\psi_{j_1j_2k}\}$ is error of the finite difference approximations. The mesh function $\{Z_{j_1j_2k}\}$ satisfies the following problem:

$$i\delta_t Z_{j_1j_2k}+a_0\left(\delta_{x_1\bar{x}_1}Z_{j_1j_2k}+\delta_{x_2\bar{x}_2}Z_{j_1j_2k}\right)+i\sum_{j=1}^{2}a_j^{j_1j_2}\delta_{\bar{x}_j}Z_{j_1j_2k}+\sum_{j=1}^{2}b_j^{j_1j_2}v_{jk}Z_{j_1j_2k}-a^{j_1j_2}Z_{j_1j_2k}=F_{j_1j_2k},$$
$$j_1=\overline{1,M_1-1},\ j_2=\overline{1,M_2-1},\ k=\overline{1,N}, \tag{36}$$

$$Z_{j_1j_20}=0,\ j_1=\overline{0,M_1},\ j_2=\overline{0,M_2}, \tag{37}$$

$$Z_{0j_2k}=Z_{M_1j_2k}=0,\ j_2=\overline{0,M_2},\ k=\overline{1,N},\ Z_{j_10k}=Z_{j_1M_2k}=0,\ j_1=\overline{0,M_1},\ k=\overline{1,N}. \tag{38}$$

Here the function $F_{j_1j_2k}$ is defined such as the following:

$$F_{j_1j_2k}=\frac{1}{h_1h_2\tau}\int_{t_{k-1}}^{t_k}\int_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2}\int_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2}\left[i\frac{\partial\psi}{\partial t}+a_0\left(\frac{\partial^2\psi}{\partial x_1^2}+\frac{\partial^2\psi}{\partial x_2^2}\right)+i\sum_{j=1}^{2}a_j(x_1,x_2)\frac{\partial\psi}{\partial x_j}\right.$$
$$\left.+\sum_{j=1}^{2}b_j(x_1,x_2)v_j(t)\psi-a(x_1,x_2)\psi\right]dx_2dx_1dt-i\delta_t\psi_{j_1j_2k}-a_0\left(\delta_{x_1\bar{x}_1}\psi_{j_1j_2k}+\delta_{x_2\bar{x}_2}\psi_{j_1j_2k}\right) \tag{39}$$
$$-i\sum_{j=1}^{2}a_j^{j_1j_2}\delta_{\bar{x}_j}\psi_{j_1j_2k}-\sum_{j=1}^{2}b_j^{j_1j_2}v_{jk}\psi_{j_1j_2k}+a^{j_1j_2}\psi_{j_1j_2k},\quad j_1=\overline{1,M_1-1},\ j_2=\overline{1,M_2-1},\ k=\overline{1,N}.$$

Theorem 3: Suppose that the time-step τ satisfy the condition $0<\tau\le\frac{1}{8}(\upsilon_1)^{-1}$ and space-steps h_1,h_2 satisfy the adaptation conditions $c_3\le\frac{h_1}{\tau}\le c_4$ and $c_5\le\frac{h_2}{\tau}\le c_6$. Here the numbers $c_3>0,\ c_4>0,\ c_5>0,\ c_6>0$ are independent of $\tau,\ h_1$ and h_2. Then the following estimate is valid:

$$h_1h_2\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|Z_{j_1j_2m}\right|^2\le c_7\left(\beta_{\tau h_1h_2}+\left\|Q_n(v)-[v]_n\right\|^2\right),\ \forall m\in\{1,2,...,N\}. \tag{40}$$

Here the number $c_7>0$ is independent of $h_1,\ h_2$ and τ. $\beta_{\tau h_1h_2}>0,\ \ \beta_{\tau h_1h_2}\to 0$ for $\tau\to 0$, $h_1\to 0$ and $h_2\to 0$. Moreover the following equality is satisfied;

$$\left\|Q_n(v)-[v]_n\right\|^2=\left(\tau\sum_{k=1}^{N}\sum_{p=1}^{2}\left|w_{pk}-v_{pk}\right|^2\right).$$

4. Convergence of the Finite Difference Approximations According to the Functional

In this section, we prove convergence of the finite difference approximations according to the functional. First, let us estimate the finite difference between the functionals $J(v)$ and $I_n([v]_n)$.

Theorem 4: Suppose that the conditions of Theorem 3 are hold. Then the following inequality is valid for $\forall v\in V$ and $\forall[v]_n\in V_n$:

$$\left|J(v)-I_n([v]_n)\right|\le c_8\left(\sqrt{\beta_{h_1h_2\tau}}+\left\|Q_n(v)-[v]_n\right\|\right). \tag{41}$$

Here the number $c_8>0$ is independent $h_1,\ h_2$ and τ.

Proof: We consider the difference $J(v)-I_n([v]_n)$. Using (1) and (11) we can write the following equality:

$$J(v)-I_n([v]_n)=\int_0^{l_1}\int_0^{l_2}\left|\psi(x_1,x_2,T)-y(x_1,x_2)\right|^2dx_2dx_1-h_1h_2\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\phi_{j_1j_2N}-y_{j_1j_2}\right|^2$$
$$=\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\int_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2}\int_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2}\Big[\left(\left|\psi(x_1,x_2,T)-y(x_1,x_2)\right|+\left|\phi_{j_1j_2N}-y_{j_1j_2}\right|\right)$$
$$\times\left(\left|\psi(x_1,x_2,T)-y(x_1,x_2)\right|-\left|\phi_{j_1j_2N}-y_{j_1j_2}\right|\right)\Big]dx_2dx_1.$$

Applying Cauchy-Bunyakovsky inequality and using (10) and (25) in this equality, we have:

$$\left|J(v)-I_n\left([v]_n\right)\right| \le c_9\left[\left(\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\int\limits_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2}\int\limits_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2}\left|\psi(x_1,x_2,T)-\phi_{j_1j_2N}\right|^2 dx_2dx_1\right)^{1/2} + \left(\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\int\limits_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2}\int\limits_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2}\left|y(x_1,x_2)-y_{j_1j_2}\right|^2 dx_2dx_1\right)^{1/2}\right] \le c_{10}\left[J_1+J_2\right] \tag{42}$$

Here the number $c_{10}>0$ is independent $h_1,\ h_2$ and τ.

Firstly let us evaluate the term J_1. To this, the following inequality can be written:

$$\begin{aligned} J_1^2 &= \sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\int\limits_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2}\int\limits_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2}\left|\psi(x_1,x_2,T)-\phi_{j_1j_2N}\right|^2 dx_2dx_1 \\ &\le 2\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\int\limits_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2}\int\limits_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2}\left|\psi(x_1,x_2,T)-\psi_{j_1j_2N}\right|^2 dx_2dx_1 + 2h_1h_2\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\left|\phi_{j_1j_2N}-\psi_{j_1j_2N}\right|^2 \\ &= J_{11}+J_{12}. \end{aligned} \tag{43}$$

Using (34), we can write the following inequality:

$$\begin{aligned} \left|\psi(x_1,x_2,T)-\psi_{j_1j_2N}\right|^2 &\le \frac{3h_1}{h_2\tau}\int\limits_{t_{N-1}}^{t_N}\int\limits_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2}\int\limits_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2}\left|\frac{\partial\psi(\eta_1,\xi_2,\theta)}{\partial\eta_1}\right|^2 d\xi_2d\eta_1d\theta \\ &+\frac{3h_2}{\tau}\int\limits_{t_{N-1}}^{t_N}\int\limits_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2}\left|\frac{\partial\psi(x_1,\eta_2,\theta)}{\partial\eta_2}\right|^2 d\eta_2d\theta+3\tau\int\limits_{t_{N-1}}^{t_N}\left|\frac{\partial\psi(x_1,x_2,\sigma)}{\partial\sigma}\right|^2 d\sigma. \end{aligned}$$

Using the last inequality in the term J_{11}, we have:

$$\begin{aligned} J_{11} &\le \frac{6h_1^2}{\tau}\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\int\limits_{t_{N-1}}^{t_N}\int\limits_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2}\int\limits_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2}\left|\frac{\partial\psi(x_1,x_2,t)}{\partial x_1}\right|^2 dx_2dx_1dt+\frac{6h_2^2}{\tau}\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\int\limits_{t_{N-1}}^{t_N}\int\limits_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2}\int\limits_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2}\left|\frac{\partial\psi(x_1,x_2,t)}{\partial x_2}\right|^2 dx_2dx_1dt \\ &+6\tau\sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\int\limits_{t_{N-1}}^{t_N}\int\limits_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2}\int\limits_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2}\left|\frac{\partial\psi(x_1,x_2,t)}{\partial t}\right|^2 dx_2dx_1dt. \end{aligned}$$

Considering adaptation conditions and (10) in last inequality, we get the following:

$$J_{11}\le c_{11}\left(h_1+h_2+\tau\right). \tag{44}$$

Here the number $c_{11}>0$ is independent $h_1,\ h_2$ and τ.

Using $Z_{j_1j_2N}=\phi_{j_1j_2N}-\psi_{j_1j_2N}$ and (40), we have:

$$J_{12}\le c_9\left(\beta_{\tau h_1h_2}+\left\|Q_n(v)-[v]_n\right\|^2\right) \tag{45}$$

Here the number $c_9>0$ is independent $h_1,\ h_2$ and τ.

Now let us evaluate the term J_2. Considering the formula of $y_{j_1j_2}$, we can write the following:

$$\begin{aligned} J_2^2 &= \sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\int\limits_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2}\int\limits_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2}\left|y(x_1,x_2)-y_{j_1j_2}\right|^2 dx_2dx_1 \\ &= \sum_{j_1=1}^{M_1-1}\sum_{j_2=1}^{M_2-1}\int\limits_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2}\int\limits_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2}\left|\frac{1}{h_1h_2}\int\limits_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2}\int\limits_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2}\left(\int\limits_{\xi_1}^{x_1}\frac{\partial y(\eta_1,\xi_2)}{\partial\eta_1}d\eta_1+\int\limits_{\xi_2}^{x_2}\frac{\partial y(x_1,\eta_2)}{\partial\eta_2}d\eta_2\right)d\xi_2d\xi_1\right|^2 dx_2dx_1. \end{aligned}$$

Applying Cauchy-Bunyakovsky inequality, we get following:

$$J_2^2 \le 2h_1^2 \sum_{j_1=1}^{M_1-1} \sum_{j_2=1}^{M_2-1} \int_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2} \int_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2} \left| \frac{\partial y(x_1,x_2)}{\partial x_1} \right|^2 dx_2 dx_1 + 2h_2^2 \sum_{j_1=1}^{M_1-1} \sum_{j_2=1}^{M_2-1} \int_{x_{1j_1}-h_1/2}^{x_{1j_1}+h_1/2} \int_{x_{2j_2}-h_2/2}^{x_{2j_2}+h_2/2} \left| \frac{\partial y(x_1,x_2)}{\partial x_2} \right|^2 dx_2 dx_1 .$$

We easily have the following from the last inequality:

$$J_2^2 \le c_{12}\left(h_1^2 + h_2^2\right). \tag{46}$$

Here the number $c_{12} > 0$ is independent h_1 and h_2.

Using (44)-(46) in inequality (42), we get the following:

$$\left| J(v) - I_n\left([v]_n\right) \right| \le c_{10}\left[J_1 + J_2\right] \le c_{13}\left(\sqrt{\beta_{\tau h_1 h_2}} + \left\| Q_n(v) - [v]_n \right\|\right). \tag{47}$$

Here the number $c_{13} > 0$ is independent h_1, h_2 and τ.

Lemma 1: Accept that the conditions of Theorem 4 are hold. Furthermore let the operator $Q_n(v)$ is defined by the formula (35). Then $Q_n(v) \in V_n$ for $\forall v \in V$ and the following estimate is valid:

$$\left| J(v) - I_n\left(Q_n(v)\right) \right| \le c_8 \sqrt{\beta_{\tau h_1 h_2}}. \tag{48}$$

Lemma 2: Accept that the conditions of Theorem 4 are hold and let the operator P_n is defined by the following formula:

$$\begin{aligned} & P_n\left([v]_n\right) = \tilde{v}(t) = \left(\tilde{v}_1(t), \tilde{v}_2(t)\right), \\ & \tilde{v}_p(t) = v_{pk},\ t_{k-1} \le t \le t_k,\ k = \overline{1,N},\ p = 1,2. \end{aligned} \tag{49}$$

Then $P_n\left([v]_n\right) \in V$ for $\forall [v]_n \in V_n$ and the following estimate is valid:

$$\left| J\left(P_n\left([v]_n\right)\right) - I_n\left([v]_n\right) \right| \le c_8 \sqrt{\beta_{h_1 h_2 \tau}}. \tag{50}$$

Theorem 5: Accept that the conditions of Lemma 1 and Lemma 2 are hold, $v^* \in V$ and $[v]_n^* \in V_n$ are solutions of problems (1)-(4) and (11)-(15), respectively. In other words let the equality satisfy the following:

$$J\left(v^*\right) = J_* = \inf_{v \in V} J(v) \text{ ve } I_{n*} = I_n\left([v]_n^*\right) = \inf_{[v]_n \in V_n} I_n\left([v]_n\right).$$

Then the problem (11)-(15) is approximation of the optimal control problem (1)-(4). Namely the condition

$$\lim_{n \to \infty} I_{n*} = J_* \tag{51}$$

is satisfied and the estimate

$$\left| I_{n*} - J_* \right| \le c_8 \sqrt{\beta_{h_1 h_2 \tau}} \tag{52}$$

is valid for the convergence according to the functional.

Proof: Suppose that $v^* \in V$ is the arbitrary solution of the optimal control problem (1)-(4). $Q_n\left(v^*\right) \in V_n$ and $\left| I_n\left(Q_n\left(v^*\right)\right) - J\left(v^*\right) \right| \le c_8 \sqrt{\beta_{\tau h_1 h_2}},\ n = 1,2,...$ from Lemma 1. Using this inequality, we have the following:

$$I_{n*} \le I_n\left(Q_n\left(v^*\right)\right) \le J\left(v^*\right) + c_8\sqrt{\beta_{\tau h_1 h_2}} = J_* + c_8 \sqrt{\beta_{\tau h_1 h_2}},\ n = 1,2,...$$

Hence we write the following inequality:

$$I_{n*} - J_* \le c_8 \sqrt{\beta_{\tau h_1 h_2}},\ n = 1,2,... \tag{53}$$

Let the control $[v]_n^* \in V_n$ is the arbitrary solution of the problem (11)-(15). $P_n\left([v]_n^*\right) \in V$ from Lemma 2 and the following inequality is satisfied:

$$\left| J\left(P_n\left([v]_n^*\right)\right) - I_n\left([v]_n^*\right) \right| \le c_8 \sqrt{\beta_{h_1 h_2 \tau}},\ n = 1,2,...$$

Hence the following inequality is valid:

$$J_* \le J\left(P_n\left([v]_n^*\right)\right) \le I_n\left([v]_n^*\right) + c_8 \sqrt{\beta_{h_1 h_2 \tau}} = I_{n*} + c_8 \sqrt{\beta_{h_1 h_2 \tau}},\ n = 1,2,...$$

Using the last inequality, we can write the following inequality:

$$I_{n*} - J_* \ge -c_8 \sqrt{\beta_{h_1 h_2 \tau}},\ n = 1,2,... \tag{54}$$

Hence we get the following from (53) and (54):

$$\left| I_{n*} - J_* \right| \le c_8 \sqrt{\beta_{h_1 h_2 \tau}},\ n = 1,2,...$$

Considering $\tau = \tau_n,\ h_1 = h_{1n},\ h_2 = h_{2n}$ and $\lim_{n\to\infty} \tau_n = 0$, $\lim_{n\to\infty} h_{1n} = 0$, $\lim_{n\to\infty} h_{2n} = 0$ we obtain $\lim_{n\to\infty} \beta_{h_{1n} h_{2n} \tau_n} = 0$ for $\beta_{h_1 h_2 \tau}$. Using these in the estimate (52), we have $\lim_{n\to\infty} I_{n*} = J_*$ for $n \to \infty$.

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