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Stability analysis for finite difference scheme used for seismic imaging using amplitude and phase portrait

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Abstract: A finite difference scheme is produced when partial derivatives in the partial differential equation(s) governing a physical phenomenon like the propagation of seismic waves through real media are replaced by a finite difference approximation. The result is a single algebraic equation which, when solved, provide an approximation to the solution of the original partial differential equation at selected points of a solution grid. Stability of a numerical scheme like that of finite difference scheme in the solution of partial differential equations is crucial for correctness and validity and it means that the error caused by small perturbation in the numerical solution remains bound. This paper considers important concepts like the amplitude and phase portrait used to analyze the stability of finite difference scheme. Applying these concepts produces an amplification factor and celerity for the components of the numerical solution.

Keywords: Finite Difference Scheme, Stability, Seismic Waves, Phase Portrait, Amplitude Portrait, Amplification Factor, Celerity

1. Introduction

Seismic images are affected by amplitude and phase fluctuations of waves travelling through heterogeneous media, as such waves moves through elastic medium, they suffer from attenuation of seismic energy. The attenuation rate being the sum of the redistribution of seismic energy (scattering attenuation) and the conversion of seismic energy into heat (intrinsic attenuation). Coates and Schoeberge, (1995) and Frankel and Clayton, (1986) observed that scattering attenuation is the dominant factor for attenuation in the crust. Both scattering and intrinsic attenuation are frequency dependent, having its rate increasing for higher frequencies. Yan and Xie, (2010) however establishes that in addition to the frequency dependence of the two types of attenuation, they also depend on the medium parameter, i.e scattering of seismic waves causes amplitude attenuation and phase fluctuations.

Phase portrait in physics represents a space in which all possible states of a given system are represented. Each possible state of the system corresponds to one unique point. The attenuation of the earth media affects the amplitude and phase of propagating seismic waves, such effects if ignored, can be the source resulting in errors in forward modeling, imaging and inversion.

In seismology, reflection images are strongly affected by these two phenomena when target of investigation is located beneath strongly heterogeneous media representing the real earth composition. Saenger et al.,(2007) also reported that lateral variations in the phase and the amplitude of the primary field reducing the lateral coherence occur if the scattering angles are slightly larger. Numerical studies also shows that the variance of travel time fluctuations decreases with increasing offsets at small offsets (Yan and Xie,2012) while theoretical considerations showed that the variance of travel time fluctuations increases again after critical distance (Zeng et al.,2011). Therefore the approach used in the paper takes a look at these two phenomena from the basic principle and apply to a finite difference scheme used in modeling seismic wave propagation in heterogeneous medium.

2. Background Theory

Finite - difference modeling in 2D and 3D has now become a very useful tool for seismic application, in that it is used in forward modeling of seismic data from known or possible geological model. The generated synthetic data can then be used to test the effectiveness of specific acquisition geometry, processing and interpretation methods.

The elastic wave equation finite-difference solution in two dimensions (2D) requires not just a combination of 1D solutions along each axis, but also terms involving combinations of partial derivatives in two spatial directions. According to Manning (2008), a better approach is by the use of staggered grid with split of the calculations into two time stages namely the stress step and the velocity step. The staggered grid is essential to the success of the used method because it is more natural with Cartesian coordinates, and Virieux (1984, 1986) used this grid with stress/velocity splitting, and applied it to interesting exploration cases.

Not all FD schemes which may be generated for a given PDE offer a viable numerical method. To be useful for modeling, a FD scheme needs to be numerically stable. A PDE may be generally classified as an initial value or boundary value problems. Stability is much greater concern for initial value problems. Since equation of motion presents an initial value problem, discussions of numerical stability would be restricted to PDEs of that type.

For simplicity, if we consider a scalar PDE in one time and one space dimension. All the essential features of the more general problem of the stability of FD scheme for a vector PDE in multiple space dimensions are already found in the simplest instance of an initial value problem. Corresponding to the independent variable U(x,t) in the PDE, the dependent variable in the difference equation is

$$u_m^q = u(m\Delta x, q\Delta t) \tag{1}$$

Where Δx and Δt are the space and time step sizes, respectively, and m and q are the corresponding integer indices. The FD solution proceeds by calculating the values of the dependent variable U_m^q at the latest time step q from its values at earlier time steps. The FD scheme used to evolve the dependent variable maybe implicit or explicit. Given an implicit scheme, u_m^q at the latest time step q is found by solving a linear system of equations in which the u_m^q at different location m contains only the known values of the dependent variable at times earlier than q, and so u_m^q is given explicitly. Implicit FD schemes have the advantage of being unconditionally stable(given PDE with stable solutions), i.e the numerical solution does not grow without bounds with time, for all choices of Δx and Δt. While explicit schemes have the advantage of algorithmic simplicity, they are at best conditionally stable, being stable only for certain choices of Δx and Δt. In what follows, we use explicit schemes exclusively and therefore restrict our discussion of numerical stability to such schemes.

To solve a PDE using the method of FDs, the differential equation is approximated as a difference equation, and the latter is solved for the dependent variable(s) on a discrete grid, called the computational grid.

Applying the finite-difference scheme require consistency in the reasonable approximations of the derivatives.

Clearly, a variety of difference equations may be generated from the same differential equation, corresponding to different FD approximations. To systematize the different FD schemes, it is useful to introduce some fundamental difference operators. A FD scheme is generated from the PDE by approximating derivative operators in terms of difference operators. For time evolution PDEs (i.e hyperbolic and parabolic equations), the corresponding FD schemes may be classified in terms of the order of approximation used for the time and space derivatives, respectively, e.g., O(2,4) for a scheme second – order in time and fourth – order in space. Note, however, that this classification does not characterize a FD scheme uniquely.

Choice of any particular FD scheme based on reasonable approximations of the derivatives in addition to the consistency, must also satisfy conditions for convergence and stability as applicable to the analysis of partial differential equation. Nature of dispersion goes a long way in contributing to the stability of FD schemes. However, according to Thorbecke and Draganov (2011), finite-difference schemes are intrinsically dispersive and there is no fixed grid point per wavelength rule that can be given to avoid dispersion. The concept of dispersion hence can be better understood using the concept of phase portrait.

Phase portrait is a useful graphical tool to understand stability behavior of the equilibrium points of linear and non-linear systems.

Stability analysis for linear partial differential equation (PDE) and the FD numerical scheme produced two main results namely (i) an amplification factor and (ii) celerity, for each component of the numerical solution. Defining an amplification parameter R_1 as the ratio of the magnitudes of the numerical amplification factor to the true amplification factor (which happens to be 1.0), i.e., for the present case considered,

$$R_1 = \frac{\left|e^{(-I\beta_m\Delta t)}\right|}{1.0} = |1 - Irsin(\sigma_m\Delta x)| = \sqrt{1 + r^2\sin{(\sigma_m\Delta x)^2}} \tag{2}$$

where a r = aΔt/Δx, a = constant speed of advection = $\frac{\beta_m}{\sigma_m}$, $\beta_m = 2\pi/T_m$, $\sigma_m = 2\pi/L_m$ = wave number of the m-th component, and L_m - wavelength

A phase parameter, R_2, is defined as the ratio of the numerical celerity to that of the true (or analytical) celerity. Thus, for case considered,

$$R_2 = \frac{a_m}{a} = \frac{\operatorname{atan}(r\,sin(\sigma_m\Delta x))}{a\sigma_m\Delta t} \tag{3}$$

Plots of the parameters R_1 and R_2 versus the dimensionless parameter $\frac{L_m}{\Delta x} = \frac{2\pi}{\sigma_m\Delta x}$ are referred to as amplitude portrait and phase portrait, respectively. These portraits can be used to show graphically the stability, or lack thereof, of a numerical scheme. Typically, the portraits will show plots corresponding to the different values of r. While the parameter L_m represents the characteristic wavelength of the m-th component of the numerical solution, it can be taken to be length of the solution, i.e $0 < x < L_m$. The dimensionless parameter $L_s = L_m/\Delta x$ relates the length of the solution domain to the grid size. The smaller the grid size, Δx, the larger the value of L_s.

Considering the amplitude and phase portraits of the FD Scheme in use, first, the expression for R_1 can be written as a function of $L_s = \frac{L_m}{\Delta x} = \frac{2\pi}{\sigma_m \Delta x}$, by writing $\sigma_m \Delta x = \frac{2\pi}{L_s}$, with this result, the amplification parameter is given by

$$R_1 = \sqrt{1 + } \; r^2 sin\left(\frac{2\pi}{L_s}\right)^2 \tag{4}$$

Also, the phase portrait can be plotted by using $\sigma_m \Delta x = \frac{2\pi}{L}$, and $a\Delta t = r\Delta x$, so that

$$R_2 = \frac{atan\left(rsin\left(\frac{2\pi}{L_s}\right)\right)}{r2\pi/L_s} = \frac{L_s}{2\pi r}\, atan\left(rsin\left(\frac{2\pi}{L_s}\right)\right) \tag{5}$$

3. Results and Discussion

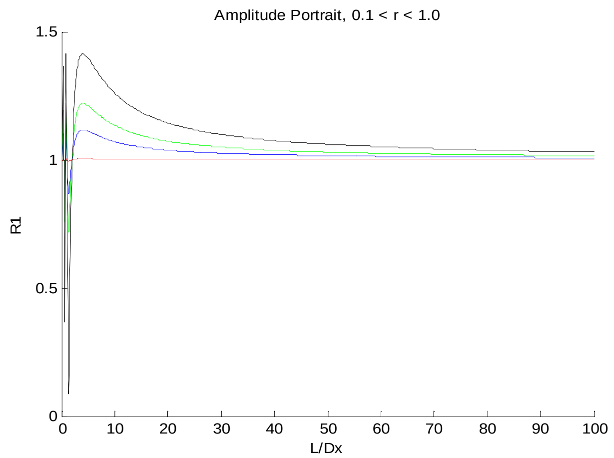

Fig 1. *Amplitude and phase portrait (R1) for 0.1< r < 1.0*

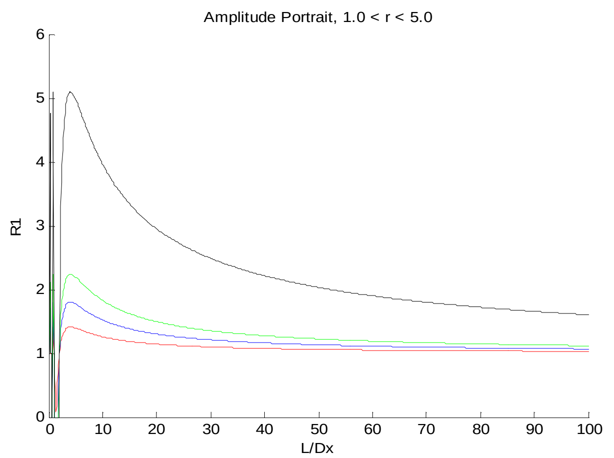

Fig 2. *Amplitude and phase portrait (R1) for 1.0 < r < 5.0*

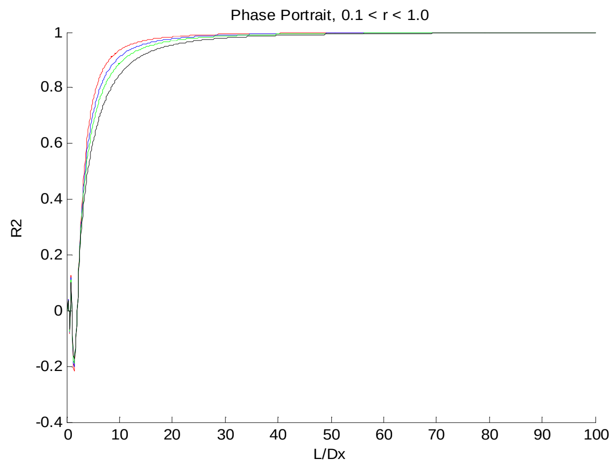

Fig 3. *Amplitude and phase portrait (R2) for 0.1 < r < 1.0*

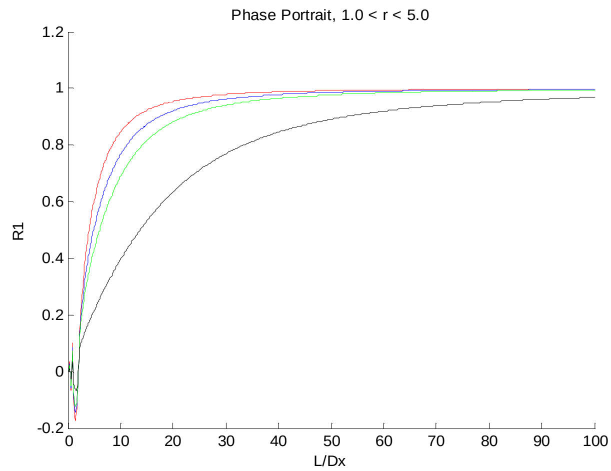

Fig 4. *Amplitude and phase portrait (R2) for 0.1 < r < 5.0*

Therefore, by letting L_s be between 0 and 2π, the amplitude and phase portraits for values of r = 0.1, 0.5 and 1.0 are plotted as shown in figures 1 to 4.

With reference to the ~~a~~ plots in figures 1-4, it would be noted that in most cases, the values of $L_s = \frac{L_m}{\Delta x}$ would be larger than 2π and therefore, more detailed amplitude and phase portrait for the FD scheme will include a larger range for L, e.g between 0 and 100, with corresponding phase portrait for r = 0.1, 0.5, and 1.0.

The amplification parameter peaks at about $L_s = 3$, and then decrease as L_s grows past the value $L_s = 3$. As the value of r grows larger than 1.0, the amplification parameter reaches larger values. Thus the FD scheme proposed herein will produce relatively large amplification parameters particularly for larger values of r and for small values of L_s.

4. Conclusion

The amplitude and phase fluctuations discussed influence greatly the reflectivity and the coherency of seismic images in heterogeneous media

The phase portrait shows the phase parameter as an oscillatory signal having its amplitude and corresponding wavelength increasing with increasing L_s. This is in agreement with earlier established fact that the celerity of the numerical solution produces numerical dispersion.

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