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Constructions of Implications Satisfying the Order Property on a Complete Lattice

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Abstract: In this paper, we further investigate the constructions of fuzzy connectives on a complete lattice. We firstly illustrate the concepts of left (right) semi-uninorms and implications satisfying the order property by means of some examples. Then we give out the formulas for calculating the upper and lower approximation implications, which satisfy the order property, of a binary operation.

Keywords: Fuzzy Logic, Fuzzy Connective, Implication, Order Property

1. Introduction

In fuzzy logic systems (see [1-2]), connectives "and", "or" and "not" are usually modeled by t-norms, t-conorms, and strong negations on $[0,1]$ (see [3]), respectively. Based on these logical operators on $[0,1]$, the three fundamental classes of fuzzy implications on $[0,1]$, i.e., R-, S-, and QL-implications on $[0,1]$, were defined and extensively studied (see [4-8]). But, as was pointed out by Fodor and Keresztfalvi [9], sometimes there is no need of the commutativity or associativity for the connectives "and" and "or". Thus, many authors investigated implications based on some other operators like weak t-norms [10], pseudo t-norms [11], pseudo-uninorms [12], left and right uninorms [13], semi-uninorms [14], aggregation operators [15] and so on.

Uninorms, introduced by Yager and Rybalov [16] and studied by Fodor et al. [17], are special aggregation operators that have proven useful in many fields like fuzzy logic, expert systems, neural networks, aggregation, and fuzzy system modeling. This kind of operation is an important generalization of both t-norms and t-conorms and a special combination of t-norms and t-conorms [17]. However, there are real-life situations when truth functions cannot be associative or commutative. By throwing away the commutativity from the axioms of uninorms, Mas et al. introduced the concepts of left and right uninorms on [0, 1] in [18] and later in a finite chain in [19], Wang and Fang [13, 20] studied the left and right uninorms on a complete lattice. By removing the associativity and commutativity from the axioms of uninorms, Liu [14] introduced the concept of semi-uninorms and Su et al. [21] discussed the notion of left and right semi-uninorms on a complete lattice. On the other hand, it is well known that a uninorm (semi-uninorm, left and right uninorms) U can be conjunctive or disjunctive whenever $U(0,1)=0$ or 1, respectively. This fact allows to use uninorms in defining fuzzy implications [13-14, 22-23].

Constructing fuzzy connectives is an interesting topic. Recently, Wang [24] laid bare the formulas for calculating the smallest pseudo-t-norm that is stronger than a binary operation and the largest implication that is weaker than a binary operation, Su and Wang [25] investigated the constructions of implications and coimplications on a complete lattice and Wang et al. [26-28] studied the relations among implications, coimplications and left (right) semi-uninorms on a complete lattice. Moreover, Wang et al. [27, 29-30] investigated the constructions of implications and coimplications satisfying the neutrality principle.

In this paper, based on [24-30], we study the constructions of implications satisfying the order property on a complete lattice. After recalling some necessary definitions and examples about the left (right) semi-uninorms and implications on a complete lattice in Section 2, we give out the formulas for calculating the upper and lower approximation implications, which satisfy the order property, of a binary operation in Section 3.

The knowledge about lattices required in this paper can be found in [31].

Throughout this paper, unless otherwise stated, L always represents any given complete lattice with maximal element 1 and minimal element 0; J stands for any index set.

2. Left (Right) Semi-Uninorms and Implications

In this section, we recall some necessary definitions and examples about the left (right) semi-uninorms and implications on a complete lattice.

Definition 2.1 (Su et al. [21]). *A binary operation U on L is called a left (right) semi-uninorm if it satisfies the following two conditions:*

(U1) *there exists a left (right) neutral element, i.e., an element $e_L \in L$ ($e_R \in L$) satisfying $U(e_L, x) = x$ ($U(x, e_R) = x$ for all $x \in L$,*

(U2) *U is non-decreasing in each variable.*

If a left (right) semi-uninorm U is associative, then U is the left (right) uninorm [13] on L.

If a left (right) semi-uninorm U with the left (right) neutral element $e_L \in L$ ($e_R \in L$) has a right (left) neutral element $e_R \in L$ ($e_L \in L$), then $e_L = U(e_L, e_R) = e_R$. Let $e = e_L = e_R$. Here, U is the semi-uninorm [14].

For any left (right) semi-uninorm U on L, U is said to be left-conjunctive and right-conjunctive if $U(0,1) = 0$ and $U(1,0) = 0$, respectively. U is said to be conjunctive if both $U(1,0) = 0$ and $U(1,0) = 0$ since it satisfies the classical boundary conditions of AND.

U is said to be strict left-conjunctive and strict right-conjunctive if U is conjunctive and for any $x \in L, U(x,1) = 0 \Leftrightarrow x = 0$ and $U(1,x) = 0 \Leftrightarrow x = 0$, respectively.

Definition 2.2 (Wang and Fang [13]). *A binary operation U on L is called left (right) arbitrary $\vee$-distributive if*

$$U(\vee_{j\in J} x_j, y) = \vee_{j\in J} U(x_j, y) \quad \forall x_j, y \in L$$

$$\left(U(x, \vee_{j\in J} y_j) = \vee_{j\in J} U(x, y_j) \quad \forall x, y_j \in L\right); \tag{1}$$

left (right) arbitrary $\wedge$-distributive if

$$U(\wedge_{j\in J} x_j, y) = \wedge_{j\in J} U(x_j, y) \quad \forall x_j, y \in L$$

$$\left(U(x, \wedge_{j\in J} y_j) = \wedge_{j\in J} U(x, y_j) \quad \forall x, y_j \in L\right). \tag{2}$$

If a binary operation U is left arbitrary $\vee$-distributive ($\wedge$- distributive) and also right arbitrary $\vee$-distributive ($\wedge$-distributive), then U is said to be arbitrary $\vee$-distributive ($\wedge$-distributive).

Noting that the least upper bound of the empty set is 0 and the greatest lower bound of the empty set is 1, we have

$$U(0, y) = U(\vee_{j\in\Phi} x_j, y) = \vee_{j\in\Phi} U(x_j, y) = 0$$

$$\left(U(x, 0) = U(x, \vee_{j\in\Phi} y_j) = \vee_{j\in\Phi} U(x, y_j) = 0\right) \tag{3}$$

for any $x, y \in L$ when U is left (right) arbitrary $\vee$-distributive,

$$U(1, y) = U(\wedge_{j\in\Phi} x_j, y) = \wedge_{j\in\Phi} U(x_j, y) = 1$$

$$\left(U(x, 1) = U(x, \wedge_{j\in\Phi} y_j) = \wedge_{j\in\Phi} U(x, y_j) = 1\right) \tag{4}$$

for any $x, y \in L$ when U is left (right) arbitrary $\wedge$-distributive.

For the sake of convenience, we introduce the following symbols:

$U_s^{e_L}(L)$: the set of all left semi-uninorms with the left neutral element e_L on L;

$U_s^{e_R}(L)$: the set of all right semi-uninorms with the right neutral element e_R on L;

$U_{cs}^{se_L}(L)$: the set of all strict left-conjunctive left semi-uninorms with the left neutral element e_L on L;

$U_{cs}^{e_R s}(L)$: the set of all strict right-conjunctive right semi-uninorms with the right neutral element e_R on L;

$U_{\vee cs}^{se_L}(L)$: the set of all strict left-conjunctive left arbitrary $\vee$-distributive left semi-uninorms with the left neutral element e_L on L;

$U_{cs\vee}^{e_R s}(L)$: the set of all strict right-conjunctive right arbitrary $\vee$-distributive right semi-uninorms with the right neutral element e_R on L.

Example 2.1 (Su et al. [21]). Let $e_L \in L$,

$$U_{sW}^{e_L}(x, y) = \begin{cases} y \text{ if } x \geq e_L, \\ 0 \text{ otherwise,} \end{cases} \quad U_{sM}^{e_L}(x, y) = \begin{cases} y \text{ if } x \geq e_L, \\ 1 \text{ otherwise,} \end{cases}$$

$$U_{csM}^{e_L}(x, y) = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0, \\ y & \text{if } 0 < x \leq e_L, y \neq 0, \\ 1 & \text{otherwise,} \end{cases}$$

where x and y are elements of L. Then $U_{sW}^{e_L}$ and $U_{sM}^{e_L}$ are, respectively, the smallest and greatest elements of

$U_s^{e_L}(L)$. By Example 2 and Theorem 8 in [26], we see that $U_{cs}^{se_L}(L)$ and $U_{\vee cs}^{se_L}(L)$ are two join-semilattices with the greatest element $U_{csM}^{e_L}$.

Example 2.2. Let $e_L \in L$,

$$U_{csW}^{se_L}(x, y) = \begin{cases} y & \text{if } x \geq e_L, \\ \wedge\{a \in L \mid a \neq 0\} & \text{if } 0 < x \text{ not} \geq e_L, y = 1, \\ 0 & \text{otherwise.} \end{cases}$$

When $e_L \neq 0$ and $\wedge\{a \in L \mid a \neq 0\} \neq 0$, it is straightforward to verify that $U_{csW}^{se_L}$ is a strict left-conjunctive left semi-uninorm with the left neutral element e_L. If $U \in U_{cs}^{se_L}(L)$, then

$$U(x, y) \geq \begin{cases} U(e_L, y) = y & \text{when } x \geq e_L, \\ \wedge\{a \in L \mid a \neq 0\} & \text{when } 0 < x \text{ not} \geq e_L, y = 1, \\ 0 & \text{otherwise,} \end{cases}$$

i.e., $U \geq U_{csW}^{se_L}$. Thus, $U_{csW}^{se_L}$ is the smallest element of $U_{cs}^{se_L}(L)$.

Moreover, assume that $\vee\{a \in L \mid a \text{ not} \geq e_L\} \text{ not} \geq e_L$. For any $x_j \in L\ (j \in J)$, if $\vee_{j\in J}\, x_j \geq e_L$, then there exists $j_0 \in J$ such that $x_{j_0} \geq e_L$,

$$\begin{aligned} U_{csW}^{se_L}(\vee_{j\in J} x_j, y) &= y = U_{csW}^{se_L}(x_{j_0}, y) \\ &= \vee_{j\in J}\, U_{csW}^{se_L}(x_j, y) \quad \forall y \in L; \end{aligned} \tag{5}$$

if $0 < \vee_{j\in J} x_j \text{ not} \geq e_L$, then $x_j \text{ not} \geq e_L$ for any $j \in J$ and there exists $j_0 \in J$ such that $0 < x_{j_0} \text{ not} \geq e_L$,

$$\begin{aligned} U_{csW}^{se_L}(\vee_{j\in J} x_j, 1) &= \wedge\{a \mid a \neq 0\} = U_{csW}^{se_L}(x_{j_0}, 1) \\ &= \vee_{j\in J}\, U_{csW}^{se_L}(x_j, 1); \end{aligned} \tag{6}$$

$$\begin{aligned} U_{csW}^{se_L}(\vee_{j\in J} x_j, y) &= 0 = U_{csW}^{se_L}(x_{j_0}, y) \\ &= \vee_{j\in J}\, U_{csW}^{se_L}(x_j, y) \quad y \neq 1; \end{aligned} \tag{7}$$

if $\vee_{j\in J} x_j = 0$, then $x_j = 0$ for any $j \in J$,

$$U_{csW}^{se_L}(\vee_{j\in J} x_j, y) = 0 = \vee_{j\in J}\, U_{csW}^{se_L}(x_j, y) \quad \forall y \in L. \tag{8}$$

Therefore, $U_{csW}^{se_L}$ is left arbitrary $\vee$-distributive and the smallest element of $U_{\vee cs}^{se_L}(L)$.

Example 2.3. Let $e_R \in L$,

$$U_{sW}^{e_R}(x, y) = \begin{cases} x \text{ if } y \geq e_R, \\ 0 \text{ otherwise,} \end{cases} \quad U_{sM}^{e_R}(x, y) = \begin{cases} x \text{ if } y \geq e_R, \\ 1 \text{ otherwise,} \end{cases}$$

$$U_{csM}^{e_R}(x, y) = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0, \\ x & \text{if } 0 < y \leq e_R, x \neq 0, \\ 1 & \text{otherwise,} \end{cases}$$

$$U_{csW}^{e_R s}(x, y) = \begin{cases} x & \text{if } y \geq e_R, \\ \wedge\{a \in L \mid a \neq 0\} & \text{if } 0 < y \text{ not} \geq e_R, x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

where x and y are elements of L. By Example 2.6 in [26], we know that $U_{sW}^{e_R}$ and $U_{sM}^{e_R}$ are, respectively, the smallest and greatest elements of $U_s^{e_R}(L)$. By Example 3 and Theorem 8 in [26], we see that $U_{cs}^{e_R s}(L)$ and $U_{cs\vee}^{e_R s}(L)$ are two join-semilattices with the greatest element $U_{csM}^{e_R}$.

Similarly, When $e_R \neq 0$ and $\wedge\{a \in L \mid a \neq 0\} \neq 0$, $U_{csW}^{e_R s}$ is the smallest element of $U_{cs}^{e_R s}(L)$. Moreover, if $\vee\{a \in L \mid a \text{ not} \geq e_R\} \text{ not} \geq e_R$, then $U_{csW}^{e_R s}$ is the smallest element of $U_{cs\vee}^{e_R s}(L)$.

Definition 2.3 (Fodor and Roubens [1], Baczynski and Jayaram [4], Bustince et al. [6], De Baets and Fodor [22]). *An implication I on L is a hybrid monotonous (with decreasing first and increasing second partial mappings) binary operation that satisfies the corner conditions $I(0, 0) = I(1, 1) = 1$ and $I(1, 0) = 0$.*

An implication I is said to satisfy the order property with respect to e (w. r. t. e, for short) when $x \leq y$ if and only if $I(x, y) \geq e$ for any $x, y \in L$.

Implications are extensions of the Boolean implication $\rightarrow$ ($P \rightarrow Q$ meaning that P is sufficient for Q).

Note that for any implication I on L, due to the monotonicity, the absorption principle holds, i.e., $I(0, x) = I(x, 1) = 1$ for any $x \in L$.

For the sake of convenience, we introduce the following symbols:

$I(L)$: the set of all implications on L;

$I_\wedge(L)$: the set of all right arbitrary $\wedge$-distributive implications on L;

$I^{ope}(L)$: the set of all implications which satisfy the order property w. r. t. *e* on L;

$I_\wedge^{ope}(L)$: the set of all right arbitrary $\wedge$-distributive implications which satisfy the order property w.r.t. *e* on L.

Clearly, $I(L)$, $I_\wedge(L)$, $I^{ope}(L)$ and $I_\wedge^{ope}(L)$ are all meet-semilattices. By Example 2.4 in [25], we know that $I_\wedge(L)$ is not a join-semilattice.

Definition 2.3. Let U be a binary operation on L. Define $I_U^L, I_U^R \in L^{L\times L}$ as follows:

$$I_U^L(x, y) = \vee\{z \in L \mid U(z, x) \leq y\} \quad \forall x, y \in L, \tag{9}$$

$$I_U^R(x, y) = \vee\{z \in L \mid U(x, z) \leq y\} \quad \forall x, y \in L. \tag{10}$$

Here, I_U^L and I_U^R are, respectively, called the left and right residuum of the binary operation U.

By virtue of Theorems 4.4 and 4.5 in [13], we know that U and I_U^R satisfy the following right residual principle:

$$U(x,z)\le y \Leftrightarrow z\le I_U^R(x,y) \quad \forall x,y,z\in L \tag{11}$$

when a binary operation U is right arbitrary $\vee$-distributive; U and I_U^L satisfy the following left residual principle:

$$U(z,x)\le y \Leftrightarrow z\le I_U^L(x,y) \quad \forall x,y,z\in L \tag{12}$$

when U is left arbitrary $\vee$-distributive.

When U is non-decreasing in each variable, it is easy to see that I_U^L and I_U^R are all decreasing in the first variable and increasing in the second one by Definition 2.3.

Example 2.4. For some left and right semi-uninorms in Examples 2.1-2.3, a simple computation shows that

$$I^L_{U^{se_L}_{csW}}(x,y)=\begin{cases} 0 & \text{if } x=1 \text{ and } y=0,\\ 1 & \text{if } x\le y,\\ \vee\{a\in L\mid a \text{ not}\ge e_L\} & \text{otherwise,}\end{cases}$$

$$I^L_{U^{e_L}_{csM}}(x,y)=\begin{cases} 1 & \text{if } x=0 \text{ or } y=1,\\ e_L & \text{if } 0<x\le y<1,\\ 0 & \text{otherwise,}\end{cases}$$

$$I^R_{U^{e_{R}s}_{csW}}(x,y)=\begin{cases} 0 & \text{if } x=1 \text{ and } y=0,\\ 1 & \text{if } x\le y,\\ \vee\{a\in L\mid a \text{ not}\ge e_R\} & \text{otherwise,}\end{cases}$$

$$I^R_{U^{e_R}_{csM}}(x,y)=\begin{cases} 1 & \text{if } x=0 \text{ or } y=1,\\ e_R & \text{if } 0<x\le y<1,\\ 0 & \text{otherwise,}\end{cases}$$

where x and y are elements of L. By the virtue of Theorem 8 in [26], we see that $I^L_{U^{e_L}_{csM}}$ is the smallest element of both $I^{ope_L}(L)$ and $I^{ope_L}_{\wedge}(L)$.

When $e_L\ne 0$ and $\vee\{a\in L\mid a \text{ not}\ge e_L\}\text{not}\ge e_L$, it is easy to see that $I^L_{U^{se_L}_{csW}}$ is the greatest element of $I^{ope_L}(L)$.

Moreover, assume that $\wedge\{a\in L\mid a\ne 0\}\ne 0$. For any $y_j\in L\,(j\in J)$, if $\wedge_{j\in J}y_j=0$, then there exists $j_0\in J$ such that $y_{j_0}=0$,

$$\begin{aligned} &I^L_{U^{se_L}_{csW}}(x,\wedge_{j\in J}y_j)=I^L_{U^{se_L}_{csW}}(x,0)=I^L_{U^{se_L}_{csW}}(x,y_{j_0})\\ &=\wedge_{j\in J} I^L_{U^{se_L}_{csW}}(x,y_j)\quad \forall x\in L;\end{aligned} \tag{13}$$

if $x\le\wedge_{j\in J}y_j$, then $x\le y_j$ for any $j\in J$,

$$I^L_{U^{se_L}_{csW}}(x,\wedge_{j\in J}y_j)=1=\wedge_{j\in J} I^L_{U^{se_L}_{csW}}(x,y_j)\quad \forall x\in L; \tag{14}$$

if $0<\wedge_{j\in J}y_j \text{ not}\ge x$, then $0<y_j$ for any $j\in J$ and there exists $j_0\in J$ such that $0<y_{j_0}\text{ not}\ge x$,

$$I^L_{U^{se_L}_{csW}}(x,y_j)\ge\vee\{a\in L\mid a \text{ not}\ge e_L\}; \tag{15}$$

$$\begin{aligned} &I^L_{U^{se_L}_{csW}}(x,\wedge_{j\in J}y_j)=\vee\{a\in L\mid a \text{ not}\ge e_L\}\\ &=I^L_{U^{se_L}_{csW}}(x,y_{j_0})=\wedge_{j\in J} I^L_{U^{se_L}_{csW}}(x,y_j).\end{aligned} \tag{16}$$

Therefore, $I^L_{U^{se_L}_{csW}}$ is the greatest element of $I^{ope_L}_{\wedge}(L)$.

$I^L_{U^{e_L}_{csM}}$ Similar conclusions hold for $I^{ope_R}(L)$ and $I^{ope_R}_{\wedge}(L)$.

3. Constructing the Implications Satisfying the Order Property

Recently, Su and Wang [25] have studied the constructions of implications and coimplications and Wang et al. [27, 29-30] further investigated the constructions of implications and coimplications satisfying the neutrality principle on a complete lattice.

This section is a continuation of [25, 27, 29-30]. We will study the constructions of the upper and lower approximation implications which satisfy the order property.

It is easy to verify that if $J\ne\Phi$, then

$$I_j\in I^{ope_L}(L)\,\forall j\in J \quad\Rightarrow\quad \wedge_{j\in J} I_j\in I^{ope_L}(L). \tag{17}$$

When $e_L\ne 0$ and $\vee\{a\in L\mid a \text{ not}\ge e_L\}\text{not}\ge e_L$, we see that $I^{ope_L}(L)$ is also a complete lattice with the smallest element and greatest element $I^L_{U^{se_L}_{csW}}$ by Example 2.4. Thus, for a binary operation A on L, if there exists $I\in I^{ope_L}(L)$ such that $A\le I$, then

$$\wedge\{I\mid A\le I, I\in I^{ope_L}(L)\} \tag{18}$$

is the smallest implication that is stronger than A and satisfies the order property w. r. t. e_L on L. Here, we call it the upper approximation implication, which satisfies the order property w. r. t. e_L, of A and write as $[A)^{ope_L}_I$. Similarly, if there exists $I\in I^{ope_L}(L)$ such that $I\le A$, then

$$\vee\{I\mid I\le A, I\in I^{ope_L}(L)\} \tag{19}$$

is the largest implication that is weaker than A and satisfies the order property w. r. t. e_L on L. Here, we call it the lower approximation implication, which satisfies the order property w. r. t. e_L, of A and write as $(A]^{ope_L}_I$.

Likewise, for a binary operation A on L, we may introduce the following symbols:

$[A)_I^{ope_R}$: the upper approximation implication, which satisfies the order property w. r. t. e_R, of A;

$(A]_I^{ope_R}$: the lower approximation implication, which satisfies the order property w. r. t. e_R, of A;

$[A)_I^{ope_L\wedge}$ ($[A)_I^{ope_R\wedge}$): the upper approximation right arbitrary $\wedge$-distributive implication, which satisfies the order property w. r. t. e_L (e_R), of A;

$(A]_I^{ope_L\wedge}$ ($(A]_I^{ope_R\wedge}$): the lower approximation right arbitrary $\wedge$-distributive implication, which satisfies the order property w. r. t. e_L (e_R), of A.

Definition 3.1 (see Su and Wang [25]). *Let A be a binary operation on L. Define the upper approximation implicator A_{ui} and the lower approximation implicator A_{li} of A as follows:*

$$A_{ui}(x, y) = \vee\{A(u, v) \mid u \geq x, v \leq y\} \quad \forall x, y \in L, \tag{20}$$

$$A_{li}(x, y) = \wedge\{A(u, v) \mid u \leq x, v \geq y\} \quad \forall x, y \in L. \tag{21}$$

Theorem 3.1 (see Su and Wang [25]). *Let $A, B \in L^{L\times L}$. Then the following statements hold:*

$$A_{li} \leq A \leq A_{ui}. \tag{22}$$

$(A \vee B)_{ui} = A_{ui} \vee B_{ui}$ *and*

$$(A \wedge B)_{li} = A_{li} \wedge B_{li}. \tag{23}$$

A_{ui} and A_{li} are hybrid monotonous.

If A is are hybrid monotonous, then $A_{ui} = A_{li} = A$.

Theorem 3.2. Let $A \in L^{L\times L}$.

(1) If A is right arbitrary $\vee$-distributive, then A_{ui} is also right arbitrary $\vee$-distributive,

$$(I_A^R)_{li} = I_{A_{ua}}^R, (I_A^R)_{ui} \leq I_{A_{la}}^R, \tag{24}$$

$$A_{ua}(x, (I_A^R)_{li}(x, y)) \leq y \quad \forall x, y \in L. \tag{25}$$

(2) If A is right arbitrary $\wedge$-distributive, then A_{li} is also right arbitrary $\wedge$-distributive.

(3) If A is left arbitrary $\vee$-distributive, then,

$$(I_A^L)_{li} = I_{A_{ua}}^L, (I_A^L)_{ui} \leq I_{A_{la}}^L, \tag{26}$$

$$A_{ua}((I_A^L)_{li}(x, y), x) \leq y \quad \forall x, y \in L. \tag{27}$$

Proof. We only prove that statement (1) holds.

Assume that A is a right arbitrary $\vee$-distributive binary operation on L. Clearly, A_{ua} is also right arbitrary $\vee$-distributive. By Definition 3.1, the monotonicity of A and I_A^R, and the right residual principle, we have that

$$\begin{aligned} I_{A_{ua}}^R(x, y) &= \vee\{z \in L \mid A_{ua}(x, z) \leq y\} \\ &= \vee\{z \in L \mid \vee\{A(u, v) \mid u \leq x, v \leq z\} \leq y\} \\ &= \vee\{z \in L \mid \vee\{A(u, z) \mid u \leq x\} \leq y\} \\ &= \vee\{z \in L \mid A(u, z) \leq y \ \forall u \leq x\} \\ &= \vee\{z \in L \mid z \leq I_A^R(u, y) \ \forall u \leq x\} \\ &= \vee\{z \in L \mid z \leq \wedge_{u \leq x} I_A^R(u, y)\} \\ &= \wedge_{u \leq x} I_A^R(u, y) \ \forall x, y \in L, \end{aligned} \tag{28}$$

$$\begin{aligned} &(I_A^R)_{li}(x, y) = \wedge\{I_A^R(u, v) \mid u \leq x, v \geq y\} \\ &= \wedge\{I_A^R(u, y) \mid u \leq x\} = \wedge_{u\leq x} I_A^R(u, y) \ \forall x, y \in L. \end{aligned} \tag{29}$$

Thus, $(I_A^R)_{li} = I_{A_{ua}}^R$. Similarly, we have that

$$(I_A^R)_{ui}(x, y) = \vee\{I_A^R(u, y) \mid u \geq x\} \ \forall x, y \in L, \tag{30}$$

$$\begin{aligned} A_{la}(x, z) &= \wedge\{A(u_1, v) \mid u_1 \geq x, v \geq z\} \\ &= \wedge\{A(u_1, v) \mid u_1 \geq x\} \ \forall x, z \in L, \end{aligned} \tag{31}$$

$$\begin{aligned} &(I_{A_{la}}^R)(x, y) \\ &= \vee\{z \in L \mid \wedge\{A(u_1, z) \mid u_1 \geq x\} \leq y\} \ \forall x, y \in L. \end{aligned} \tag{32}$$

If $u \geq x$, let $z = I_A^R(u, y)$, then

$$\begin{aligned} &A(u, z) = A(u, \vee\{c \in L \mid A(u, c) \leq y\}) \\ &= \vee\{A(u, c) \mid A(u, c) \leq y\} \leq y, \\ &\wedge\{A(u_1, z) \mid u_1 \geq x\} \leq A(u, z) \leq y. \end{aligned} \tag{33}$$

So, $(I_A^R)_{ui}(x, y) \leq (I_{A_{la}}^R)(x, y)$ for any $x, y \in L$, i.e., $(I_A^R)_{ui} \leq I_{A_{la}}^R$.

Moreover, we know that A_{ua} is right arbitrary $\vee$-distributive and hence

$$\begin{aligned} &A_{ua}(x, (I_A^R)_{li}(x, y)) = A_{ua}(x, I_{A_{ua}}^R(x, y)) \\ &= A_{ua}(x, \vee\{z \in L \mid A_{ua}(x, z) \leq y\}) \\ &= \vee\{A_{ua}(x, z) \mid A_{ua}(x, z) \leq y\} \leq y \ \forall x, y \in L. \end{aligned} \tag{34}$$

The theorem is proved.

Below, we give out the formulas for calculating the upper and lower approximation implications which satisfy the order property.

Theorem 3.3. Suppose that $A \in L^{L\times L}$, $e_L \neq 0$ and $\vee\{a \in L \mid a \text{ not} \geq e_L\}\text{not} \geq e_L$.

(1) If $A \leq I_{U_{csW}^{se_L}}^L$, then $[A)_I^{ope_L} = I_{U_{csM}^{e_L}}^L \vee A_{ui}$;

if $A \geq I_{U_{csM}^{e_L}}^L$, then $(A]_I^{ope_L} = I_{U_{csW}^{se_L}}^L \wedge A_{li}$.

(2) If $\wedge\{a \in L \mid a \neq 0\} \neq 0$, $A \geq I_{U_{csM}^{e_L}}^L$ and A is right

arbitrary $\wedge$-distributive, then

$$(A]_I^{ope_L \wedge} = I^L_{U^{se_L}_{csW}} \wedge A_{li}. \tag{35}$$

Moreover, if A is non-decreasing in its first variable, then $(A]_I^{ope_L \wedge} = I^L_{U^{se_L}_{csW}} \wedge A$.

Proof. Assume that $\vee\{a \in L \mid a \text{ not} \geq e_L\} \text{not} \geq e_L$ and $e_L \neq 0$. Then $I^L_{U^{e_L}_{csM}}$ and $I^L_{U^{se_L}_{csW}}$ are, respectively, the smallest and greatest elements of $I^{ope_L}(L)$ by Example 2.4.

(1) If $A \leq I^L_{U^{se_L}_{csW}}$, let $I_1 = I^L_{U^{e_L}_{csM}} \vee A_{ui}$, then $A \leq I_1$ and

$$I^L_{U^{e_L}_{csM}} \leq I_1 \leq I^L_{U^{se_L}_{csW}} \tag{36}$$

Thus, $I_1(0,0) = I_1(1,1) = 1$ and $I_1(1,0) = 0$. If $x \leq y$, then $I_1(x,y) \geq I^L_{U^{e_L}_{csM}}(x,y) \geq e_L$; if $I_1(x,y) \geq e_L$, then $I^L_{U^{se_L}_{csW}}(x,y) \geq I_1(x,y) \geq e_L$ and so $x \leq y$, i.e., I_1 satisfies the order property w. r. t. e_L. By Theorem 3.1 (3) and the hybrid monotonicity of $I^L_{U^{e_L}_{csM}}$, we know that I_1 is hybrid monotonous. So, $I_1 \in I^{ope_L}(L)$. If $A \leq I$ and $I \in I^{ope_L}(L)$, then $A_{ui} \leq I_{ui} = I$ and $I_1 = I^L_{U^{se_L}_{csM}} \vee A_{ui} \leq I$. Therefore,

$$[A)_I^{ope_L} = I^L_{U^{se_L}_{csM}} \vee A_{ui}. \tag{37}$$

If $A \geq I^L_{U^{e_L}_{csM}}$, let $I_2 = I^L_{U^{se_L}_{csW}} \wedge A_{li}$, then $I_2 \leq A$,

$$A_{li} \geq (I^L_{U^{e_L}_{csM}})_{li} = I^L_{U^{e_L}_{csM}}, I^L_{U^{e_L}_{csM}} \leq I_2 \leq I^L_{U^{se_L}_{csW}}. \tag{38}$$

Thus, we can prove in an analogous way that $I_2 \in I^{ope_L}(L)$ and $(A]_I^{ope_L} = I^L_{U^{se_L}_{csW}} \wedge A_{li}$.

(2) When $\wedge\{a \in L \mid a \neq 0\} \neq 0$, $I^L_{U^{e_L}_{csM}}$ and $I^L_{U^{se_L}_{csW}}$ are, respectively, the smallest and greatest elements of $I_\wedge^{\ ope_L}(L)$ by Example 2.4. Let $I_3 = I^L_{U^{se_L}_{csW}} \wedge A_{li}$. If $A \geq I^L_{U^{e_L}_{csM}}$, then $I_3 \in I^{ope_L}(L)$ by statement (1). Noting that A is right arbitrary $\wedge$-distributive, we can see that A_{li} is also right arbitrary $\wedge$-distributive by Theorem 3.2 (1). So, I_3 is right arbitrary $\wedge$-distributive, i.e., $I_3 \in I_\wedge^{\ ope_L}(L)$. By the proof of statement (1), we know that $(A]_I^{ope_L \wedge} = I^L_{U^{se_L}_{csW}} \wedge A$.

Moreover, if A is non-decreasing in its first variable, then $A_{li} = A$ by Theorem 3.1 (4) and so

$$(A]_I^{ope_L \wedge} = I^L_{U^{se_L}_{csW}} \wedge A. \tag{39}$$

The theorem is proved.

Analogous to Theorem 3.3, we have the following theorem.

Theorem 3.4. Suppose that $A \in L^{L \times L}$, $e_R \neq 0$ *and* $\vee\{a \in L \mid a \text{ not} \geq e_R\} \text{not} \geq e_R$.

(1) If $A \leq I^R_{U^{e_{Rs}}_{csW}}$*, then* $[A)_I^{ope_R} = I^R_{U^{e_R}_{csM}} \vee A_{ui}$;

if $A \geq I^R_{U^{e_R}_{csM}}$*, then* $(A]_I^{ope_R} = I^R_{U^{e_{Rs}}_{csW}} \wedge A_{li}$.

(2) If $\wedge\{a \in L \mid a \neq 0\} \neq 0$, $A \geq I^R_{U^{e_R}_{csM}}$ *and* A *is right arbitrary* $\wedge$*-distributive, then*

$$(A]_I^{ope_R \wedge} = I^R_{U^{e_{Rs}}_{csW}} \wedge A_{li}. \tag{40}$$

Moreover, if A is non-decreasing in its first variable, then $(A]_I^{ope_R \wedge} = I^R_{U^{e_{Rs}}_{csW}} \wedge A$.

4. Conclusions and Future Works

Constructing fuzzy connectives is an interesting topic. Recently, Wang et al. [24-25, 27, 29-30] investigated the constructions of implications and coimplications on a complete lattice. In this paper, motivated by these works, we give out the formulas for calculating the upper and lower approximation implications, which satisfy the order property, of a binary operation.

In a forthcoming paper, we will investigate the relationships between left (right) semi-uninorms and implications on a complete lattice.

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