

Research Article

Geometric Voting Vector Analysis of Strategic Candidate Nomination and Voter Retaliation in Positional Voting

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Abstract

Positional voting fails both the Independence of Irrelevant Alternatives and the Independence of Clones criteria. It is therefore vulnerable to the strategic nomination of insincere candidates that may adversely affect election outcomes. By introducing identical clones, other candidates may be either promoted or demoted in the resultant collective ranking. The analysis of the ramifications of cloning is restricted here to 'geometric voting' vectors where preference weightings form a geometric progression. With the common ratio of the weightings as the sole variable, a full continuous spectrum of vectors from plurality through the Borda count to anti-plurality can be simultaneously analyzed. This universal vector may handle any given number of candidates. Starting with an example three-candidate election, the effect of varying the common ratio and the margin between the clone and non-clone candidates is analyzed. It is then extended to include additional such candidates. Cloning maps that display all possible election outcomes for the example elections are introduced. They exhibit regions where teaming attempts either succeed or fail; the latter due to vote-splitting. The geometric voting vector that represents all those from the Borda count to anti-plurality is highly vulnerable to teaming while the one equivalent to plurality is instead vulnerable to vote-splitting. The intermediate vector with a common ratio of one-half – called the consecutively halved positional voting vector – is a balanced one with no net bias towards either polarizing or consensus candidates. The research here establishes that this vector exactly counterbalances the possibilities for teaming against those for vote-splitting when a non-cloned candidate and a pre-cloned one are tied. Also, it inherently thwarts teaming attempts where identical clones are undifferentiated, and is the most consensual vector to exhibit this property. Further, it enables opposition voters to retaliate successfully using a tit-for-tat clone-reversal strategy even when one clone is consistently promoted by clone supporters over the others. Unlike the Borda count, it possesses such disincentives to clone candidates. This balanced vector hence offers an optimal compromise between maximizing the consensus index of a geometric voting vector and minimizing its vulnerability to teaming attempts through strategic nominations.

Keywords

Positional Voting, Geometric Voting, Borda Count, Plurality, Strategic Nomination, Cloning, Teaming, Vote-Splitting

1. Introduction

In positional voting (PV), a standard mathematical sequence of preference weightings may be employed in an election vector. The Borda count uses an arithmetic

progression while the Dowdall method employs a harmonic sequence instead. Saari and many others have studied such PV systems [1-6]. A third option is to use a geometric progression

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of weightings. This option has had some specific but insufficient individual attention [7-9]. In our first article, my colleague Hal M. Switkay and I referred to such a vector as a geometric voting (GV) one [7]. In it, we demonstrated that, by using the common ratio of the sequence of weightings as a variable, a full continuous spectrum of PV vectors from plurality to the Borda count is smoothly traversed.

To extend this PV spectrum beyond the Borda count to anti-plurality, we employ vectors with negative weightings called anti-vectors. The weighting magnitudes again form a geometric progression albeit in reverse order from last to first preference. Using such anti-vectors, where differing values are awarded against rather than for the various candidates, an entire range of PV vectors that span from the extreme polarization of plurality to the maximal consensus of anti-plurality is encompassed. This trajectory is independent of the number of preferences; an important analytical property.

For investigating all PV vectors along a complete spectrum simultaneously, the use of a universal GV vector as an analytical tool yields significant insights. For three preferences, all possible vectors across the PV spectrum may be represented using normalized weightings. For four or more preferences, some, but not all, PV vectors may be represented and incorporated within a continuous spectrum. Nevertheless, this complete spectrum can range from plurality to anti-plurality by just varying the common ratio of the GV vector.

In our previous paper, we suggested that an investigation of strategic nominations in single-winner GV elections would be worthy of further research [7]. This article addresses this topic in some depth and seeks to identify GV vector common ratios with key threshold properties that deter, counteract or thwart manipulation through insincere candidate nomination. GV elections are studied where initially two candidates compete but where one of them decides to add a clone of themselves prior to the close of nominations. The effect of the difference in support for a non-clone candidate and the clone set is studied, as is the number of clones in the set and the number of non-clones contesting the election.

Cloning maps that display all possible election profiles for a specific GV election are introduced. A map is typically divided into three regions; one where the non-clone candidate wins through vote-splitting and two where one of two clones wins through teaming. The vector analysis here indicates that a successful teaming attempt is more likely to result when one clone is strongly promoted by its supporters over others in the set; otherwise without this differentiation vote-splitting may well occur. As the relative level of support for the non-clone over the leading clone increases or when the number of non-clones rises, teaming attempts are less likely to be successful. Where every voter ranks the clone set in either 'forward' or 'reverse' order, the number of clones in the set is generally not significant for GV vectors; but for anti-vectors, the effect of teaming is intensified when the set is enlarged.

All GV anti-vectors including anti-plurality are inherently vulnerable to teaming attempts. The Borda count vector is also vulnerable to teaming but not to vote-splitting while for the plurality one this vulnerability is reversed. The two GV vectors that represent these two extremes have common ratios of virtually one and zero respectively [7]. As the common ratio is steadily varied between these two values, susceptibility to one vulnerability increases while the other correspondingly declines. With the intermediate common ratio of one-half, these two vulnerabilities, each with an even chance of occurring, thereby exactly counterbalance one another.

We called a GV vector with this ratio a consecutively halved positional voting one in our previous article as we identified it as a special case of GV with important properties and a system bias balanced mid-way between polarization and consensus [7]. The research below shows that this vector thwarts teaming attempts without any response being required by clone opponents when the identical clones in the illustrative elections are undifferentiated. Of those vectors with this property, this one is the most consensual. However, when they are differentiated, this vector enables clone opponents to retaliate successfully by activating a tit-for-tat clone-reversal strategy. Such features present disincentives to the cloning of candidates.

1.1. Positional Voting Definitions and Nomenclature

The definitions and nomenclature used in this article are consistent with those used in our original paper and are summarized below [7]. In positional voting, voters express their preferences for the various candidates or options by casting a ranked ballot. Each voter expresses a preference (P) for each candidate in strict descending rank order so no truncated or tied rankings are permitted. For a PV election with a choice of $N \geq 2$ preferences, i defines the rank position from the top rank $i = 1$ for the highest preference P_1 down to the bottom rank $i = N$ for the lowest preference P_N .

Each preference P_i is given a weighting of value v_i according to its rank position i within the range $1 \leq i \leq N$. The vector $\mathbf{v} = (v_1, v_2, \dots, v_{N-1}, v_N)$ represents the N weightings associated with each ballot. Any vector must satisfy two criteria [1]. Firstly, the weightings are non-increasing; namely, $v_i \geq v_{i+1}$ for all i such that $1 \leq i < N$. Secondly, the first preference must be worth more than the last preference; namely, $v_1 > v_N$.

The total number of valid ballots cast in a PV election is V votes. Each candidate is awarded a total of V preferences from the voters. The V weightings associated with these specific preferences are then totaled to become the tally (T_C) for that candidate (C). These tallies are then rank ordered from highest to lowest score. As the focus throughout is on single-winner elections, the candidate with the highest tally

wins. Two or more candidates may tie in the resultant ranking. For practical elections, ties for first place may be resolved by employing a random tie-break.

Positive affine transformations of the weightings do not change candidate rank orders. That is, if $m > 0$ and k is arbitrary, then $T_A > T_B$ implies $m(T_A + Vk) > m(T_B + Vk)$; as confirmed by Saari [1]. Therefore, different vectors may generate identical candidate rankings under all election profiles. Such vectors are hence affine equivalent. It is useful to employ just one 'normalized' vector $w = (w_1, \dots, w_N)$ to represent all those vectors that are affine equivalent to it. The normalized vector used here is $w = (1, \dots, 0)$ where $w_1 = 1$ and $w_N = 0$. To normalize vector v , the components of vector w are determined using the rule $w_i = (v_i - v_N)/(v_1 - v_N)$ for affine equivalence; that is, $w \equiv v$.

The sum of the components of a normalized vector w is $\Sigma = w_1 + \dots + w_N$. The bias of this vector is defined by either its polarization index (PI) or the complementary consensus index (CI) where $PI = 1/\Sigma$, $CI = (\Sigma - 1)/\Sigma$ and $PI + CI = 1$ [7]. To every normalized vector $w = (w_1, w_2, \dots, w_{N-1}, w_N)$ its corresponding anti-vector w^A and conjugate vector w^C is defined by:

$$w^A = (-w_N, -w_{N-1}, \dots, -w_2, -w_1)$$

$$w^C = (1 - w_N, 1 - w_{N-1}, \dots, 1 - w_2, 1 - w_1)$$

Note that the anti-vector and conjugate vector are affine equivalent as they differ by the invalid indifference vector $(1, \dots, 1)$ but that only the conjugate vector is normalized as $w_1 = 1$ and $w_N = 0$.

A $GV(r)$ vector employs a sequence of weightings that form a geometric progression where its common ratio r must be less than one but not less than zero. Hence, the standard $GV(0 \leq r < 1)$ vector $v = (1, r, \dots, r^{N-2}, r^{N-1})$ where the first preference is unity. The normalized $GV(0 \leq r < 1)$ vector w is therefore:

$$\frac{1}{1-r^{N-1}}(1 - r^{N-1}, r - r^{N-1}, \dots, r^{N-2} - r^{N-1}, 0)$$

Since Σ is the sum of all the normalized preference weightings, then [7]:

$$\Sigma = \frac{1}{1-r} - \frac{(N-1)r^{N-1}}{1-r^{N-1}}$$

As N increases without bound, Σ approaches $1/(1-r)$ and $CI = 1 - 1/\Sigma$ approaches r .

1.2. Equivalent Positional and Geometric Voting Vectors

It was established in our original article that a full range of vectors from plurality (PL) to the Borda count (BC) can be

smoothly interpolated by using the common ratio of a $GV(r)$ vector as the sole variable regardless of the number of candidates standing [7]. Also, a full range of anti-vectors from anti-plurality (A-PL) to the anti-Borda count (A-BC) in conjugate format can be similarly traversed [7]. GV anti-vectors are henceforth referred to as A- $GV(r)$ ones. So, $GV(r)$ and A- $GV(r)$ vectors are useful tools to analyze a whole spectrum of PV systems simultaneously.

Our first article also established the equivalence of the following PV vectors [7]:

$GV(0)$ vector $(1, 0, \dots, 0) = PL$ vector $(1, 0, \dots, 0)$

$GV(\rightarrow 1)$ vector $(1, 1 - \delta, 1 - 2\delta, \dots)$ where $\delta \rightarrow 0 \equiv BC$ vector $(1, 1 - d, 1 - 2d, \dots)$

BC vector $(1, 1 - d, 1 - 2d, \dots) \equiv A-BC$ vector $(0, -d, -2d, \dots)$

Conjugate vector $(1, 1 - \delta, 1 - 2\delta, \dots)$ where $\delta \rightarrow 0 \equiv A-GV(\rightarrow 1)$ vector $(0, -\delta, -2\delta, \dots)$ where $\delta \rightarrow 0 \equiv A-BC$ vector $(0, -d, -2d, \dots)$

Conjugate vector $(1, \dots, 1, 0) \equiv A-GV(0)$ vector $(0, \dots, 0, -1) = A-PL$ vector $(0, \dots, 0, -1)$

Figure 1 illustrates a complete and continuous spectrum of PV vectors from plurality to anti-plurality as Σ varies from its minimum value of 1 to its maximum of $N - 1$. Note that $r = 0$ for both plurality and anti-plurality, and that $r \rightarrow 1$ for both the Borda count and the anti-Borda count. Progressing rightwards along the spectrum, the vectors become less polarized and more consensual as Σ rises. Plurality is wholly polarized as its $PI = 1/\Sigma = 1$ while anti-plurality is maximally-consensual with a $PI = 1/\Sigma = 1/(N - 1)$.



Figure 1. Positional voting vector spectrum spanning the range $1 \leq \Sigma \leq N - 1$.

2. Geometric Voting Vectors and Anti-Vectors

2.1. Three-Preference Vectors

For PV elections with just three candidates standing, the equilateral-triangular map shown in Figure 2 can represent any possible vector u , with the three preferences $P_1 = (1, 0, 0)$, $P_2 = (0, 1, 0)$, and $P_3 = (0, 0, 1)$; whether valid or not [7]. A barycentric coordinate system is used for measuring perpendicular distances from the respective sides of the triangle, with coordinates summing to one. A general vector projects onto this plane using the rule $u = (u_1, u_2, u_3)/U$, where $U = u_1 + u_2 + u_3$. The $GV(r)$ vector u is represented by $(1, r, r^2)/(1 + r + r^2)$.

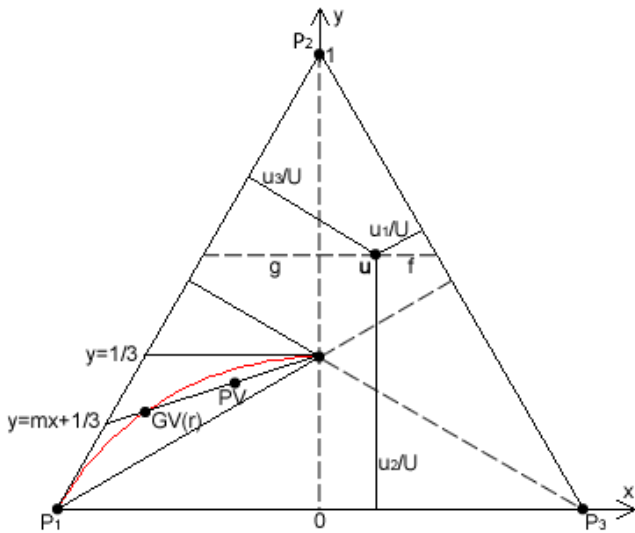


Figure 2. Three-preference vector map.

A rectangular coordinate system is superimposed onto Figure 2. The x-axis is the directed line through P_1 and P_3 . The y-axis perpendicularly bisects the edge connecting P_1 and P_3 , and is directed towards P_2 . The origin is the midpoint of the edge connecting P_1 and P_3 .

Next, g is defined as the directed horizontal distance measured leftward from u to the edge connecting P_1 and P_2 , and f is the directed horizontal distance measured rightward from u to the edge connecting P_2 and P_3 . The cartesian coordinates for any vector u are given below:

$$y = \frac{u_2}{U}$$

$$x = \frac{g-f}{2} = \frac{1}{2} \left(\frac{2u_3}{U\sqrt{3}} - \frac{2u_1}{U\sqrt{3}} \right) = \frac{u_3-u_1}{U\sqrt{3}}$$

Of the six congruent ranking regions within the map representing the six possible orderings of the three barycentric coordinates, only the one containing the point P_V - corresponding to a specific P_V vector - represents valid vectors, since only here is $u_1 \geq u_2 \geq u_3$. The map's center where $u_1 = u_2 = u_3$ represents the invalid indifference vector, and is not considered part of any ranking region.

In terms of collective candidate rankings, Saari has shown that any valid P_V vector is equivalent to the vector $u = (1-y, y, 0)$ in the lower half-edge connecting P_1 and P_2 , where $0 \leq y \leq 1/2$ [1]. This half-edge is also one of the three boundaries of the valid ranking region. When plotted on the map in Figure 2 all the P_V vectors equivalent to u are located at points along a line between the map's edge at $(1-y, y, 0)$ and its center at $(1/3, 1/3, 1/3)$; see the line $y = mx + 1/3$ on the map.

Further, every valid P_V vector is located on such a line. As the vertex P_1 represents the PL vector $(1,0,0)$, all points along the line from here to the map center are PL equivalent, as they do not distinguish between second and third place

rankings. Similarly, all points along the line to the map's center from $(1/2, 1/2, 0)$ are equivalent to the anti-plurality conjugate vector $(1,1,0)$, as they do not distinguish between first and second place rankings. These two lines are the other boundaries of the valid ranking region. The horizontal straight line $y = 1/3$ represents all the BC-equivalent vectors $(y+d, y, y-d)$ where $0 < d \leq 1/3$.

With non-negative weightings, $GV(r)$ vectors may also be plotted on a three-preference map. Using cartesian coordinates and the equation for a circle, our original paper proved that the circular arc of radius $2/3$ and centered at $(x, y) = (0, -1/3)$ or $(2/3, -1/3, 2/3)$ in barycentric coordinates as displayed in red on the map is indeed the locus of $GV(0 \leq r < 1)$ vectors [7].

For P_V vectors formed as a non-negative linear combination of PL and BC, any positive-slope line in the valid ranking region from the map's edge to its center only intersects the arc of $GV(r)$ vectors at one unique point. Therefore, the $GV(r)$ vector represented by this point is equivalent to all the other P_V vectors identified by this line. The common ratio here is derived as follows from the definition of the P_V vector u weightings:

$$u = \frac{(u_1, u_2, u_3)}{u_1+u_2+u_3} = \frac{(1, r, r^2)}{1+r+r^2}$$

$$\frac{u_1-u_2}{U} = \frac{1-r}{1+r+r^2}$$

$$\frac{u_2-u_3}{U} = \frac{r(1-r)}{1+r+r^2}$$

$$r = \frac{u_2-u_3}{u_1-u_2}$$

Therefore, for any three-preference P_V vector that is a non-negative linear combination of PL and BC, there is an equivalent $GV(r)$ vector as defined below, provided $2u_2 < u_1 + u_3$ so that $r < 1$:

$$PV(u_1, u_2, u_3) \equiv GV\left(r = \frac{u_2-u_3}{u_1-u_2}\right)$$

With only three preferences, just a single variable (y or r) is sufficient to allow an equivalent vector to represent one somewhere between PL and BC.

2.2. Three-Preference Anti-Vectors

The single variable used by Saari (y here) can not only interpolate between PL where $y = 0$ and BC where $y = 1/3$ but it is also able to range further without any discontinuity to A-PL where $y = 1/2$. However, unlike the common ratio r , the variable y is restricted to three preferences only. Nevertheless, as demonstrated in section 1.2, A- $GV(r)$ vectors can be employed in this extended region between A-BC and A-PL. As any such anti-vector has non-positive weightings, its normalized conjugate equivalent is plotted on the map

instead; see Figure 3. Within the valid ranking region, all $GV(r)$ vectors are located on or below the horizontal line $y = 1/3$ while all A- $GV(r)$ vectors are plotted on or above this line.

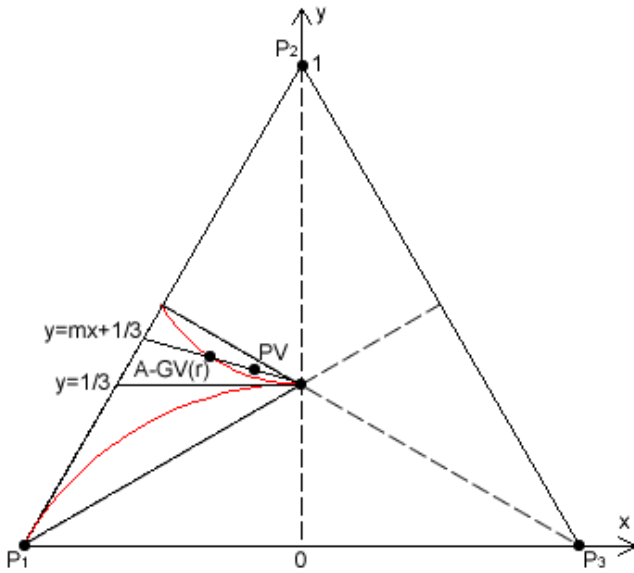


Figure 3. Three-preference vector and anti-vector map.

Using a similar approach to that of plotting $GV(r)$ vectors, another arc - this time for A- $GV(r)$ vectors - can be drawn on the map. The upper red arc in Figure 3 is part of a circle with radius $1/3$ and centered at $(x, y) = (0, 2/3)$ or $(1/6, 2/3, 1/6)$ in barycentric coordinates; see Appendix I. Where a straight line from the map's edge to its center intersects with this upper red arc the conjugate vector for A- $GV(r)$ at this intersection is equivalent to all the PV vectors represented by this line. Provided $2u_2 > u_1 + u_3$ so that $r < 1$, this vector equivalence as derived in Appendix I is stated below:

$$PV(u_1, u_2, u_3) \equiv GV\left(r = \frac{u_1 - u_2}{u_2 - u_3}\right)^A$$

The $GV (\rightarrow 1)$ and A- $GV (\rightarrow 1)$ vectors tend asymptotically toward the horizontal line $y = 1/3$ that represents both BC and A-BC equivalent vectors. The line between the midpoint of the edge connecting P_1 and P_2 and the map's center represents A- $GV(0)$ and thus all A-PL equivalent vectors.

For three-candidate PV-equivalent GV elections, vectors are valid and anti-vectors invalid when u_2 is closer to u_3 than to u_1 . The reverse is true when u_2 is closer to u_1 than to u_3 . Note the reciprocal relationship between vector and anti-vector common ratios.

3. Examples of Strategic Nomination in a Geometric Voting Election

Geometric voting is a sub-category of positional voting that is in turn one type of preferential voting. Arrow's Impossibility Theorem confirms that no such ranked-based systems are free from being insincerely manipulated through the strategic inclusion or withdrawal of one or more candidates at the nomination stage of an election [10]. All ranked voting systems fail to satisfy the condition known as the Independence of Irrelevant Alternatives (IIA) criterion [11, 12]. In positional voting, election profiles exist whereby candidate A would win if candidate C stood but, if not, then candidate B would win instead [11]. Candidate C loses in both scenarios and is hence an irrelevant alternative yet C 's presence or absence determines the winner. This lack of independence allows preferential-voting elections to be manipulated.

The cloning of a candidate may introduce an irrelevant alternative into an election that thereby affects the result. As later analysis will show, positional voting outcomes are vulnerable to the addition or deletion of clones. PV thus also fails to satisfy the Independence of Clones (IoC) criterion [13]. A clone candidate is one that is very similar to - indeed nominally identical to - an existing one (say C_1) and its entry is intended to elevate the rank of C_1 . One or more clones ($C_2, \dots, C_K$) may be added alongside C_1 in the election. The K candidates in the clone set ($C_1, \dots, C_K$) are only considered to be clones if they are adjacently ranked within a block by every voter. On each ballot, they may be ranked in any order within the block and starting at any rank position but no non-clone candidate is ranked within the clone set. This restrictive definition of a clone set enables manageable mathematical analyses of various strategic nomination scenarios to be undertaken.

As a positional voting method, geometric voting equally fails both the IIA and IoC criteria and so is inherently vulnerable to strategic nomination. The following analysis investigates how a GV election can be manipulated through different cloning strategies, whether some GV vectors are less vulnerable than others and whether some counter strategies exist to combat the effects of cloning.

3.1. Cloning within a GV Election

Prior to cloning, consider the case of just two completing candidates A and B in a GV election. The likely optimum situation for a clone candidate to alter the outcome here is when there is a tie between A and B . With each of the two candidates having half of all first and second preferences, it would only require the introduction of a clone candidate to 'push' one of these preferences down into third place to break the tie. The effect of cloning in non-tied GV elections is addressed in section 4.

Suppose A decides to nominate a clone of itself in the hope

of breaking the tie in its own favor. The two candidates A_1 and A_2 then compete against B . Of the V_A voters supporting A over B , a proportion p rank A_1 above A_2 while the remainder $(1 - p)$ choose the reverse order. Similarly, of the V_B voters preferring B to A , a proportion q rank A_1 above A_2 while the remainder $(1 - q)$ again reverses this ranking. The resulting election profile is displayed in Table 1.

Table 1. $GV(r)$ election profile.

Votes	Rank Position (i) /Weighting (v_i)		
	1 st / 1	2 nd / r	3 rd / r^2
pV_A	A_1	A_2	B
$(1 - p)V_A$	A_2	A_1	B
qV_B	B	A_1	A_2
$(1 - q)V_B$	B	A_2	A_1

For one candidate to be ranked higher than another one when V_A is tied with V_B , the following conditions as derived in Appendix II apply.

For candidate A_1 to be preferred over B :

$$p > 1 - rq$$

For candidate A_2 to be preferred over B :

$$p < r - rq$$

For candidate A_1 to be preferred over A_2 :

$$p > \frac{1+r}{2} - rq$$

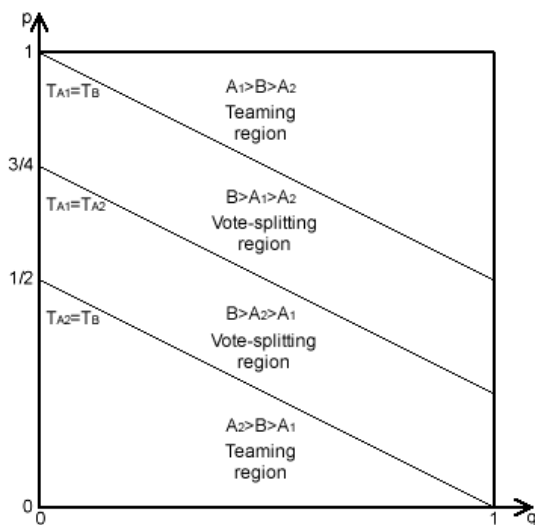


Figure 4. Cloning map for the $GV(1/2) = \text{CHPV}$ vector.

When any two of the three candidates achieve the same tally, the relevant boundary condition may be represented by a straight line on a graph of p against q where $0 \leq p \leq 1$ and $0 \leq q \leq 1$; see the cloning map example in figure 4. Notice that as the negative slope of each line has the same magnitude r , then all three lines are parallel to each other. Also, note that the p -axis intersection of the $T_{A1} = T_{A2}$ clone-equality boundary is equidistant between those for the other two boundaries.

The example map in Figure 4 uses the central intermediate value of $r = 1/2$ for the $GV(r)$ vector. It is also known as the Consecutively Halved Positional Voting (CHPV) vector since it was identified in our first paper as a special and unique case of GV [7]. Later analysis will further highlight its other important properties.

The original tie between A and B is only maintained at points along the $T_{A1} = T_B$ and $T_{A2} = T_B$ thresholds on the map. Between these two lines is the vote-splitting region where B wins. Such splitting of votes between the two clones is an act of self-harm since the cloning of A was intended to promote one of the clones relative to B 's tally. Along the $T_{A1} = T_{A2}$ line, both clones have the same tally but it is smaller than A 's original one. This scenario is the worst for attempting to beat B as it is furthest from the two remaining regions on the map where either A_1 or A_2 wins. These two teaming regions are where the clone set has unfairly yet successfully acted as a 'team' to defeat B . In the corner of the map where either $(p, q) = (0, 0)$ or $(p, q) = (1, 1)$, one clone has the maximum tally and the other has the minimum share. Being furthest from the vote-splitting region is the best strategy for beating B . The lines separating teaming regions from other areas are hereafter referred to as teaming thresholds.

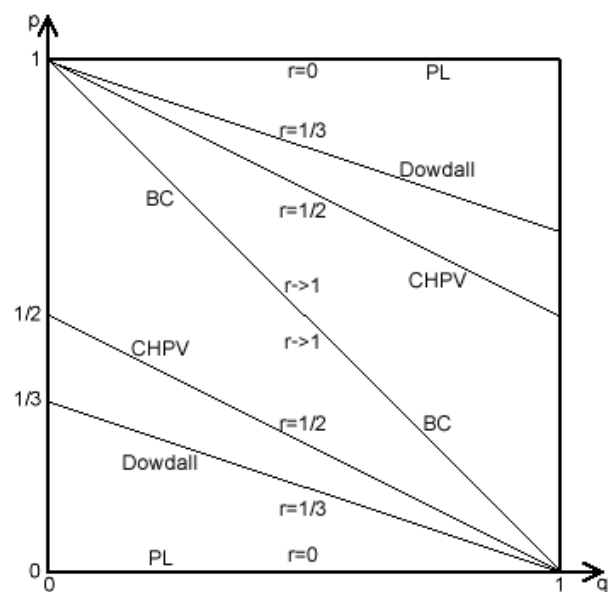


Figure 5. Examples of $GV(r)$ teaming thresholds.

The two teaming thresholds for other important values of r in $GV(r)$ elections are shown superimposed onto the one cloning map displayed in Figure 5. The vectors shown are for $PL = GV(0)$, Dowdall, $CHPV = GV(1/2)$ and $BC \equiv GV(\rightarrow 1)$. The Dowdall positional voting system is employed in Nauru's parliamentary elections and with three preferences its standard vector $v = (1, 1/2, 1/3)$ [14, 15]. Its equivalent vector $GV(r = \frac{(1/2-1/3)}{(1-1/2)} = 1/3)$ is shown here instead.

The $T_{A1} = T_B$ teaming threshold rotates clockwise pivoting about the $(p, q) = (1, 0)$ corner of the map from horizontal for PL to the diagonal line for BC as r increases. Similarly, the $T_{A2} = T_B$ one rotates clockwise about the $(p, q) = (0, 1)$ corner from horizontal for PL to the same diagonal for BC as r increases. For PL, the vote-splitting region covers the whole of the map while, for BC, it is the two teaming regions that cover the entire map. Hence, PL is the vector most susceptible to vote-splitting but it is invulnerable to teaming. Conversely, the BC vector is the most susceptible to teaming yet it is invulnerable to vote-splitting. These well-known features of PL and BC are therefore confirmed in this example election.

The $T_{A2} = T_B$ teaming threshold intersects with the p -axis at $p = r$ so the proportion of the map covered by the two triangular teaming regions is $2(1 \times r)/2 = r$. Let the Teaming Index (TI) be equal to the proportion of the map that results in successful teaming attempts. Therefore, $TI = r$ for this election. Where the range of possibilities for vote-splitting and for teaming are equally matched, $TI = r = 1/2$. So, the CHPV vector is the sole $GV(r)$ one that exactly counterbalances these two vulnerabilities.

3.2. Cloning within an A-GV Election

The example GV election profile in section 3.1 employed $GV(r)$ vectors ranging from $PL = GV(0)$ to $BC \equiv GV(\rightarrow 1)$. This election profile is continued here but with anti-vectors ranging from $A-BC \equiv A-GV(\rightarrow 1)$ to $A-PL = A-GV(0)$ instead. The entire spectrum of vectors from PL to A-PL is thereby encompassed.

Table 2. A- $GV(r)$ election profile.

Votes	Rank Position (i) /Weighting (v_i)		
	1 st / $1 - r^2$	2 nd / $1 - r$	3 rd / 0
pV_A	A_1	A_2	B
$(1 - p)V_A$	A_2	A_1	B
qV_B	B	A_1	A_2
$(1 - q)V_B$	B	A_2	A_1

For the standard $GV(r)$ vector $(1, r, r^2)$, its conjugate is $(1 - r^2, 1 - r, 1 - 1)$. Recall that the affine equivalent anti-vector is $(-r^2, -r, -1)$. Table 2 reflects the same election profile as in Table 1 but with the conjugate weightings used instead.

For one candidate to be ranked higher than another one when V_A is tied with V_B , the following conditions as derived in Appendix II apply.

For candidate A_1 to be preferred over B :

$$p > 1 - \frac{q}{r}$$

For candidate A_2 to be preferred over B :

$$p < \frac{1}{r} - \frac{q}{r}$$

For candidate A_1 to be preferred over A_2 :

$$p > \frac{1+r}{2r} - \frac{q}{r}$$

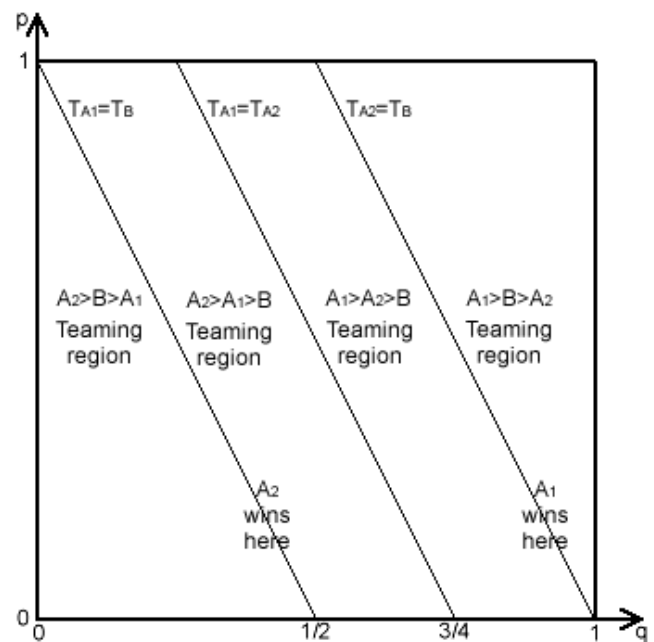


Figure 6. Cloning map for the A- $GV(1/2) = A-CHPV$ vector.

All three of the boundary conditions that follow from the above equations can again be represented by straight lines on a cloning map; see the example in Figure 6. The central intermediate value of $r = 1/2$ is retained so that a visual comparison between $GV(r)$ and A- $GV(r)$ vector thresholds can be made. For the $GV(1/2) = CHPV$ vector, its anti-vector may then be called anti-CHPV or A-CHPV, with standard weightings $(-1/4, -1/2, -1)$. As with the previous $GV(r)$ election example, the two teaming thresholds for an A- $GV(r)$ vector are also parallel to the clone-equality threshold and equidistant from it; irrespective of the value of

r .

The teaming region where A_1 beats B now extends to three-quarters of the cloning map; as does the other teaming region where A_2 beats B . In the central half of the map where these two regions overlap, B is ‘pushed’ into third place. Candidate B never wins in any scenario on the map so either one or the other clone of A does, or they tie in first place.

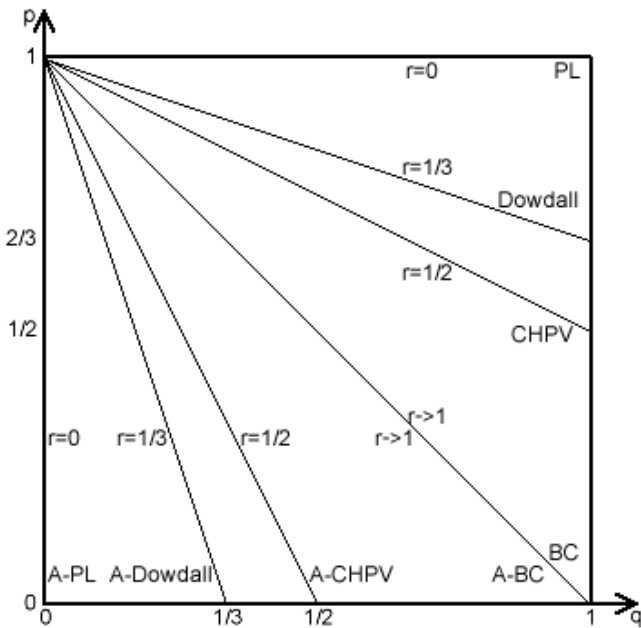


Figure 7. Examples of $T_{A1} = T_B$ teaming thresholds.

Figure 7 displays the teaming thresholds for all the $GV(r)$ vectors highlighted in the cloning map in Figure 5 and their corresponding $A-GV(r)$ ones. For clarity, only the $T_{A1} = T_B$ threshold for each vector is shown. It rotates about the $(p, q) = (1, 0)$ corner from horizontal for PL to the diagonal for BC as r increases from zero towards one. From the same diagonal - now for A-BC - it continues to rotate clockwise to the vertical line for A-PL as r now decreases back to zero. Note that this threshold intersects the q -axis at $q = r$. Likewise, the $T_{A2} = T_B$ teaming threshold (not shown) rotates clockwise about the $(p, q) = (0, 1)$ corner from horizontal for PL to vertical for A-PL. The clone-equality threshold is always precisely intermediate between and parallel to the two teaming thresholds. So, this threshold rotates clockwise pivoting about the map's center at $(p, q) = (1/2, 1/2)$ from horizontal for PL to vertical for A-PL.

The diagonal from $(p, q) = (1, 0)$ to $(p, q) = (0, 1)$ representing both BC and A-BC is the only part of a cloning map where there is a three-way tie between all the candidates. Every part of a cloning map for an $A-GV(0 \leq r < 1)$ vector is covered by at least one teaming region. Hence, there are no vote-splitting regions on such a map and thus $TI = 1$ for all $A-GV(r)$ vectors. It therefore remains the case that CHPV is

the only GV vector that counterbalances the range of opportunities for vote-splitting with those for teaming.

3.3. Combating Cloning

In the context of cloning, strategic nomination is the introduction of a set of clone candidates with the objective of demoting one or more opposing candidates and promoting one clone in the set, ideally to victory. As witnessed in sections 3.1 and 3.2, cloning can easily break a tie between two competing candidates depending on how preferences are shared by the clones. Cloning may fail in an act of self-harm if preferences are shared too evenly between the clones. In contrast, it may succeed when one clone has a substantial lead over the other clone. The likelihood of either outcome is heavily dependent on what common ratio is chosen for the GV election.

Figure 8 shows the teaming index of $GV(r)$ and $A-GV(r)$ vectors across the complete PV spectrum. For $GV(r)$ vectors $TI = r$ while for all $A-GV(r)$ ones $TI = 1$. Using BC or any $A-GV(r)$ vector in the election profiled in sections 3.1 and 3.2, teaming is inevitably successful and cannot be combatted. Even Jean-Charles de Borda himself recognized that his BC voting system was “for honest men only” [16]. Using PL, teaming attempts are pointless but it is the vector that is most vulnerable to vote-splitting. What $GV(r)$ vectors can be employed that render teaming attempts ineffective, counterproductive or allow effective countermeasures to be taken? Although CHPV is the sole vector that counterbalances the opportunities for teaming versus vote-splitting, can it or another intermediate value of the common ratio enable teaming attempts to be thwarted?

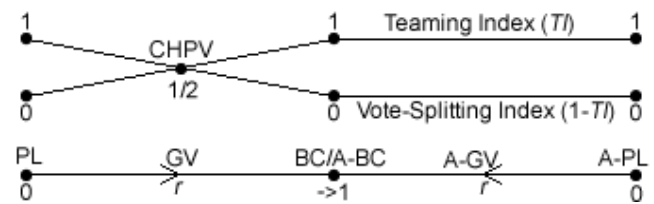


Figure 8. Teaming index across the PV spectrum.

Consider first the central intermediate common ratio of one-half as used in the CHPV vector. Its cloning map for the election example is shown in Figure 9. Prior to cloning, there were two tied candidates A and B . Adding the clone A_2 of candidate A - now called A_1 - then resulted in three candidates. The initial two-candidate tied scenario S_1 is represented on the map by the $T_{A1} = T_B$ teaming threshold.

If no guidance was given by A or B to their own supporters, and being unable to distinguish between truly identical clones, then both the V_A and V_B voters are just as likely to rank A_1 above A_2 as to prefer A_2 over A_1 . This default random clone-preference scenario S_2 where $(p, q) =$

$(1/2, 1/2)$ generates the worst outcome for the teaming attempt by A as it is in the middle of the vote-splitting region that is furthest from either teaming one. The best strategy for promoting A is for its supporters to rank one clone (say A_1) consistently above the other. This teaming scenario S_3 where $(p, q) = (1, 1/2)$ is the optimally successful one for team A . The only successful retaliation strategy for B to pursue is to rank the two clones in reverse order to that of the A -supporters. This fourth scenario S_4 where $(p, q) = (1, 0)$ not only thwarts the teaming attempt by team A but it also re-establishes the original two-way tie between A and B .

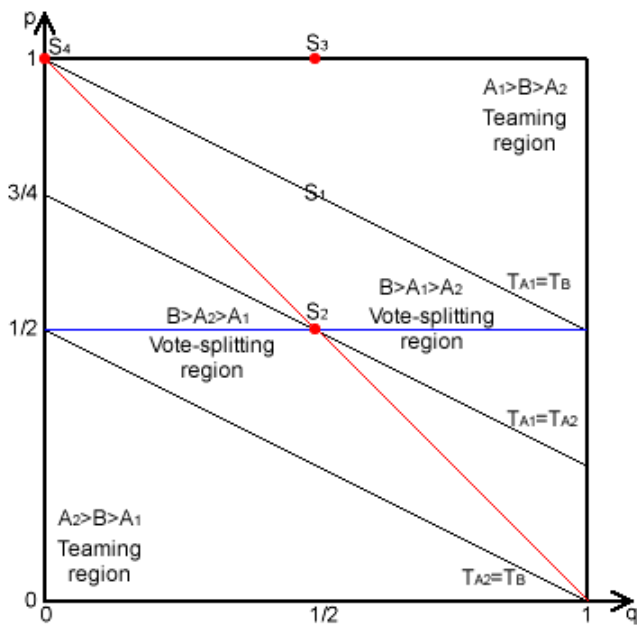


Figure 9. CHPV cloning map and scenarios.

It depends on the circumstances of the election as to whether B can predict or establish the behavior of the A -supporters in time to advise its own supporters on how to vote. Its optimum strategy is to try and match the number of A 's proportion p voters with its own proportion $1 - q$ ones. The red diagonal in Figure 9 where $p + q = 1$ is wholly outside either teaming region so A 's attempt at teaming will always fail if this retaliation condition is met. For the same reason, this optimum strategy not only applies to the CHPV vector it also applies to all other $GV(r > 0)$ vectors; except BC where a three-way tie ensues.

For a $GV(0 < r \leq 1/2)$ vector, any value of q defeats a teaming attempt when A fails to differentiate its identical clones; see the blue line in Figure 9 at $p = 1/2$ that extends horizontally between the two teaming thresholds. However, for $GV(1/2 < r < 1)$ vectors, B -supporters must respond with a suitable q value to avoid B losing since the length of the blue line now recedes as r increases beyond one-half and no longer stretches across the full width of the map. Hence, the full range of $0 \leq q \leq 1$ to thwart teaming is no longer

available and vanishes entirely for the $BC \equiv GV(\rightarrow 1)$ vector.

For this example GV election, the choice of a suitable common ratio is important when considering the effects of a strategic nomination. Despite being the most polarized vector, if concerns about teaming are paramount, then PL offers the best solution as it is invulnerable to it. At the other extreme, BC and every A - $GV(r)$ vector is inherently vulnerable as no strategy can prevent teaming attempts being successful. Even in the tied BC election, 'team' A improves its chances of winning a random tie-break through having two rivals to B ; not one.

As the common ratio rises, the vector becomes ever more consensual but also ever more prone to teaming. $CHPV$ offers a balance between these two conflicting features. Given its $TI = r = 1/2$, $CHPV$ counterbalances the likelihood of vote-splitting against that of teaming. Also, it minimises the prospect of vote-splitting for vectors in the range $0 \leq r \leq 1/2$ that require no retaliation ($0 \leq q \leq 1$) against teaming when identical clones are not differentiated ($p = 1/2$) and it is the most consensual vector within this range. Any higher common ratio here may require q to be closer to $1/2$ to avoid successful teaming. When one clone is strongly promoted over the other, a successful tit-for-tat retaliation ($q = 1 - p$) also ensures that teaming fails. Unlike the more consensual BC vector, $CHPV$ has properties that act as disincentives to cloning.

4. Teaming Thresholds in a Non-Tied Election

Prior to cloning, the election profile investigated in section 3 features a tie between candidates A and B . So, with an even chance, either could win a random tie-break. It is postulated that this scenario is the optimum one for teaming to succeed. Where V_B voters even slightly outnumber V_A ones, A is guaranteed to lose unless team A attempts to win through cloning. There is a clear incentive here for A to do so. How close behind B must A be to be able to leapfrog into the lead through teaming? How does B respond to any such cloning and does this response depend on the common ratio of the GV vector employed in the election?

The election profile as defined in Tables 1 and 2 is again used here. To illustrate the effects of a difference in support for A and B , first consider the case where B receives 20% more first preferences than A ; namely, $V_B = 1.2V_A$. Margins smaller and larger than this are also addressed below. The three threshold conditions for a $GV(r)$ vector and an A - $GV(r)$ one where $V_B \neq V_A$ are derived in Appendix II. To confirm any following statement involving these thresholds reference should be made to this appendix.

Starting again with the central intermediate $GV(r)$ common ratio of one-half, the $CHPV$ cloning map in Figure 9 is repeated in Figure 10 but with the addition of the three revised thresholds for this non-tied election shown in green. Each of

the two vote-splitting regions is now larger than before but teaming by A might still succeed if enough V_A voters prefer A_1 over A_2 , or vice versa. If scenario S_3 where $(p, q) = (1, 1/2)$ results, then there is again a tie between A_1 and B . If the proportion q of V_B voters is any higher then A_1 wins through teaming. For this election, the teaming index $TI = 2(0.3 \times 0.5)/2 = 0.15$. The $T_{A1} > T_B$ teaming region shrinks as its threshold moves further from the map's center with increasing V_B/V_A . This region reduces to zero when $V_B = 1.5V_A$. The chances of successful teaming are maximized when there is an initial tie between the original two competing candidates but it becomes increasingly less likely - if not impossible - as the share of support for B rises.

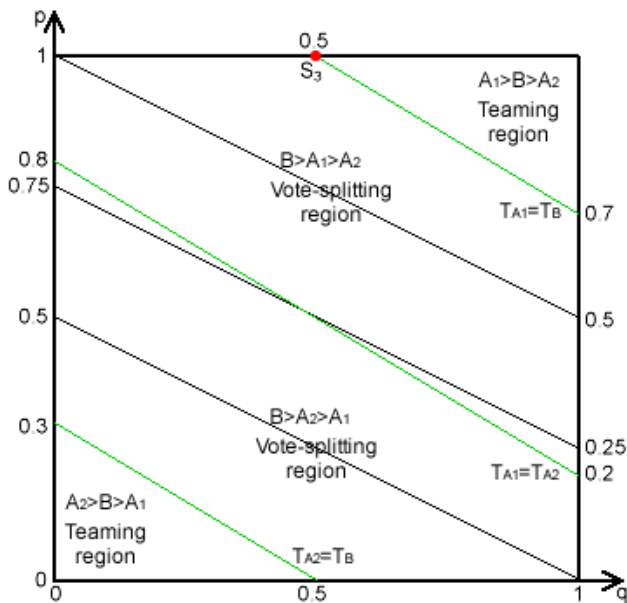


Figure 10. CHPV cloning map for $V_B = 1.2V_A$.

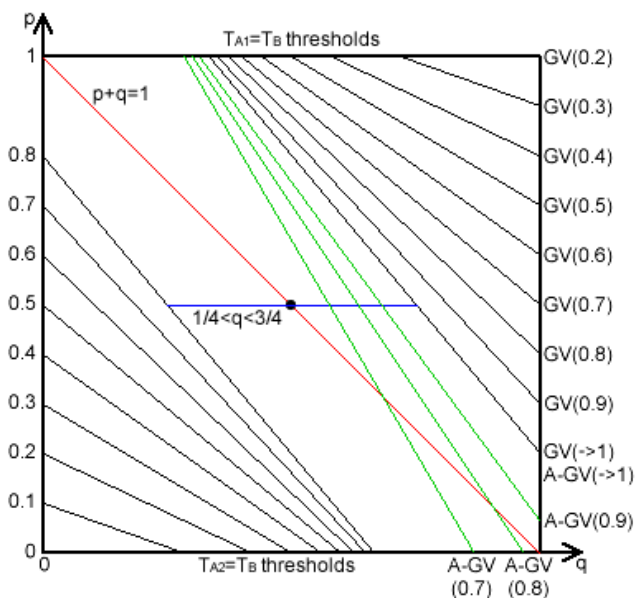


Figure 11. Teaming thresholds for $V_B = 1.2V_A$ and selected r .

Figure 11 shows the two teaming thresholds for each vector superimposed onto one map for a selection of $GV(r)$ common ratios in this non-tied election example. For common ratios of 0.2 or lower, there are no teaming regions on the map. For the BC vector, the teaming index is now $TI = 2(0.8 \times 2/3)/2 = 0.533$ whereas it was unity when $V_A = V_B$. Teaming is no longer guaranteed to succeed but may do so depending on the response of the B -supporters. Provided the proportion q of them is between one-quarter and three-quarters, then the teaming attempt will fail; see the blue line in Figure 11. The $T_{A1} > T_B$ teaming region for the BC vector shrinks to zero when $V_B = 2V_A$. The likelihood of successful teaming increases as the $GV(r)$ common ratio rises while the opportunity and incentive for a teaming attempt diminishes - if not vanishes - as, for any r , V_B/V_A increases.

Also displayed in Figure 11 in green are the $T_{A1} = T_B$ teaming thresholds for three A- $GV(r)$ vectors with common ratios of 0.9, 0.8 and 0.7. For clarity, other examples are omitted. Nevertheless, with decreasing values of r , this threshold rotates clockwise until, for A-PL = A- $GV(0)$, it is represented by a vertical line at $q = 1/6$. For A- $GV(r \cong 0.571)$ its two parallel teaming thresholds merge and intersect with the center of the map such that its $TI = 1$. For lower common ratios, teaming is therefore guaranteed to succeed. For A- $GV(r > 0.571)$ and all $GV(r)$ vectors, some scenarios exist that thwart teaming.

For all $GV(r)$ vectors, the optimum strategy for B to defeat teaming attempts by A remains the one where the proportion $(1 - q)$ of V_B tries to match the proportion p of V_A voters; see the red diagonal for $p + q = 1$ in Figure 11. If this matching condition is met, then teaming fails. Note that neither teaming threshold intersects with this red line so it exists wholly within the central vote-splitting region.

For many of the A- $GV(r > 0.571)$ vectors such intersections do occur. Hence, this matching strategy - even if met - cannot ensure the defeat of teaming attempts. For all A- $GV(r < 0.571)$ vectors, there are no strategies to thwart teaming as their $TI = 1$. With the original tied result, all A- $GV(r)$ vectors are inherently vulnerable to strategic nominations and especially so for A-PL and those with low common ratios. To discourage or prevent teaming attempts, a suitable $GV(r)$ vector should be chosen for the election in preference to an A- $GV(r)$ one.

Imagine the case where A believes that B will win but, in reality, B would lose as $V_B < V_A$. If A unnecessarily introduces the two-clone set to beat B , then such cloning could backfire. This is most likely when the two identical clones are undifferentiated. The two teaming thresholds for the $GV(r)$ vector now move closer together as the V_B/V_A ratio declines towards zero. Consequently, the teaming index for the vector increases towards one until both thresholds merge with the clone-equality one. For CHPV, its $TI = 1$ for $V_B/V_A < 0.8$. Below this value, teaming will always yield the genuine victory for A but, above it, the unnecessary attempt to win through teaming may instead cause A to lose. When

cloning is potentially counterproductive, there is a strong incentive against it.

5. Enlargement of the Clone Set

In sections 3 and 4, the effect of introducing clones into an otherwise two-candidate GV election is explored regarding both vector selection and the relative support for these two candidates. Here, two clones A_1 and A_2 are substituted for the original A candidate. Are two clones sufficient to win through teaming or are more clones of A likely to achieve a better result? How does enlargement of the clone set affect the outcome of a GV election?

The ongoing GV(r) election example with a now expanded clone set is defined in table 3. The number of candidates is N where $3 \leq N \leq K + 1$ and the number of clones is K where $K \geq 2$. All voters rank the clone set in strict forward order A_1 through to A_K or in strict reverse order A_K through to A_1 . Of the V_A voters supporting A , proportion p where $0 \leq p \leq 1$ rank A_1 first. Similarly, of the V_B voters supporting B , proportion q where $0 \leq q \leq 1$ rank A_K last. Either A_1 or A_K will have the highest tally within the clone set so no intermediate clone can win using a GV(r) vector; see later for justification.

Table 3. GV(r) election profile with an expanded clone set.

Votes	Rank Position (i) / Weighting (v_i)				
	1 st / v_1	2 nd / v_2	→	N-1 th / v_{N-1}	N th / v_N
pV_A	A_1	→	→	A_K	B
$(1-p)V_A$	A_K	→	→	A_1	B
qV_B	B	A_1	→	→	A_K
$(1-q)V_B$	B	A_K	→	→	A_1

For the standard GV(r) vector, $v_1 = 1$, $v_2 = r$, $v_{N-1} = r^{N-2}$ and $v_N = r^{N-1}$. The following three threshold conditions for this vector are derived in Appendix II.

For candidate A_1 to be preferred over A_K :

$$p > \frac{1}{2} + \frac{rV_B}{2V_A} - \left(\frac{rV_B}{V_A}\right)q$$

Note that the straight-line clone-equality threshold is independent of N and K . Therefore, its position on a cloning map is unaffected by the addition or deletion of any number of clones. Its slope and p -axis intersection are determined solely by V_B/V_A and r . Also, note that when $q = 1/2$, $p = 1/2$ regardless of these two variables so this threshold always intersects with any map's center.

For candidate A_1 to be preferred over B :

$$p > \frac{(1-r^{N-1})V_B}{(1-r^{N-2})V_A} - \frac{r^{N-2}(1-r)}{(1-r^{N-2})V_A} - \left(\frac{rV_B}{V_A}\right)q$$

For candidate A_K to be preferred over B :

$$p < \frac{(1-r^{N-1})}{(1-r^{N-2})} - \frac{(1-r)V_B}{(1-r^{N-2})V_A} - \left(\frac{rV_B}{V_A}\right)q$$

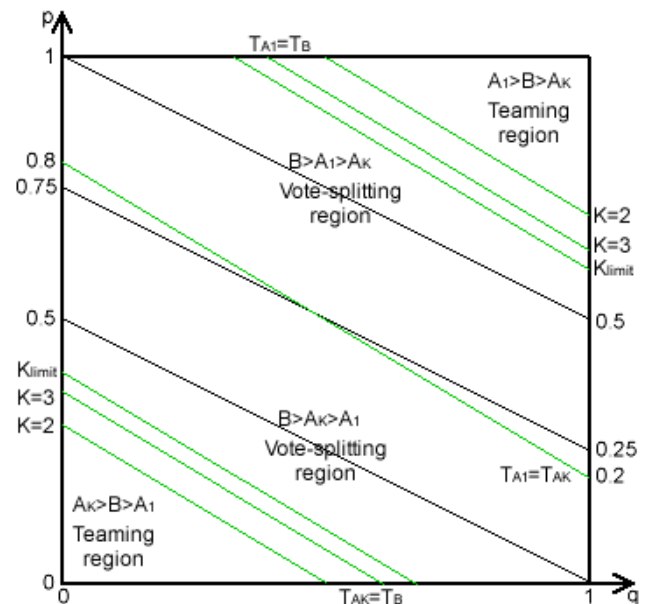


Figure 12. CHPV teaming thresholds for $V_B = 1.2V_A$ and selected K .

These two straight-line teaming thresholds have the same slope as the clone-equality one so all three are parallel to each

other on any map. Further, the two teaming thresholds are also equidistant from the clone-equality boundary; see Appendix II.

Figure 12 displays the cloning map for the ongoing CHPV example election. For the original tied outcome where $V_B = V_A$, the three threshold conditions simplify to $p > (1+r)/2 - rq$ for the $T_{A1} = T_{AK}$ threshold, $p > 1 - rq$ for the $T_{A1} = T_B$ teaming threshold and $p < r - rq$ for the $T_{AK} = T_B$ one. All three conditions are independent of K so each threshold (shown in black) is the same as on the CHPV map in Figure 4. The addition or deletion of any number of clones neither helps or hinders teaming attempts by A when $V_B = V_A$ despite this tied scenario being the most vulnerable to such attempts.

In the non-tied election example, support for B is 20% higher than for A ; namely, $V_B = 1.2V_A$. Encompassing the extended clone set, revised thresholds for this scenario are shown in green in Figure 12. This cloning map is consistent with the one in Figure 10 for the earlier $K = 2$ case when only two clones are nominated. As more clones are added to the set, each teaming threshold edges closer to the fixed clone-equality threshold; see the pair for the case where $K = 3$ in Figure 12. In the limit as K increases without bound, each teaming threshold reaches the boundary threshold labelled K_{limit} on the map. Here, the lowest two preference weightings tend to zero so the $T_{A1} = T_B$ threshold condition simplifies to:

$$p > \frac{V_B}{V_A} - \left(\frac{rV_B}{V_A} \right) q$$

When the ratio V_B/V_A rises from one, this and the other teaming region shrinks in size so that the teaming index TI falls and eventually reaches zero. For a given value of this ratio beyond unity, each teaming region - if there is one - gradually increases as more clones are added beyond $K = 2$. The addition of clones may then slightly enhance the chances of teaming being successful. Nevertheless, each teaming region remains smaller when $V_B > V_A$ than when $V_B = V_A$. Therefore, teaming attempts may be successful but this likelihood diminishes as the relative support for B strengthens. For the CHPV vector, its teaming index never exceeds one-half so most possibilities result in vote-splitting.

For the standard A-GV(r) vector in conjugate format, $v_1 = 1 - r^{N-1}$, $v_2 = 1 - r^{N-2}$, $v_{N-1} = 1 - r$ and $v_N = 1 - 1 = 0$ in Table 3. Unfortunately, for anti-vectors, one of the two end clones A_1 and A_K in the set is no longer guaranteed to be awarded the highest clone tally. Consider the case where $K = 3$. Of the V_A voters where $p = 1/2$, A_1 and A_3 will each receive a first and third preference weighting for every two votes cast. In contrast, A_2 will get two second preferences worth $2v_2$. For the BC/A-BC vector, $v_1 + v_3 = 2v_2$ due to the common difference between adjacent ranks.

For all GV ($r < 1$) vectors, $v_1 + v_3 > 2v_2$ but, for all A-GV ($r < 1$) ones, $v_1 + v_3 < 2v_2$. Hence, for an anti-vector, A_2 has a higher tally than either end clone A_1 or A_3 . For cases where $K > 3$, an intermediate clone will again have a lower tally than an end clone when using a GV(r) vector but a higher tally when employing an A-GV(r) one. So, teaming regions for anti-vectors are larger than that implied by the $T_{A1} = T_B$ and $T_{AK} = T_B$ thresholds. There is therefore a strong incentive to clone more candidates when employing A-GV(r) vectors. The A-PL = A-GV(0) vector is the extreme case as all intermediate clones tie for first place while B and both end clones lose as they are the only ones to be ranked last on some ballots; that is, voted against.

6. Examples of Multiple Non-Clone Candidate Elections

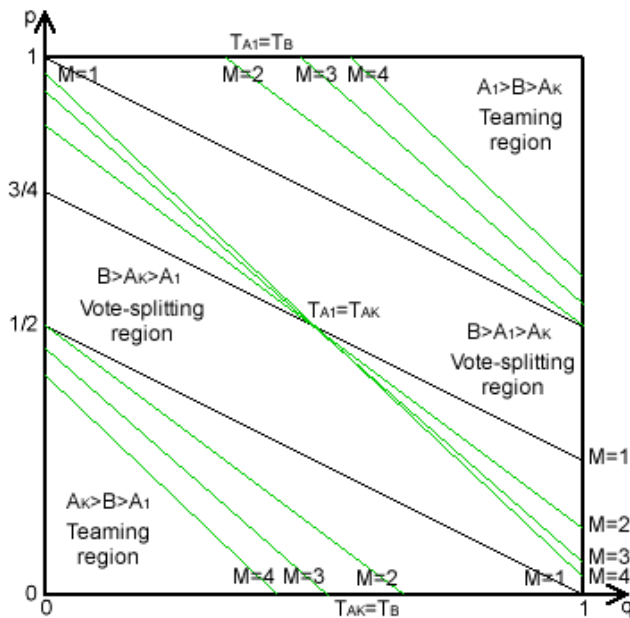
Typically, elections will have more than just two sincere candidates A and B standing. With three or more competing, is teaming likely to be more successful or less? Next, consider a GV(r) election with $M \geq 1$ non-clone candidates completing against each other as well as the clone set such that $N = M + K$. As witnessed earlier, teaming is most likely to be successful when a non-clone candidate is initially faced with a tied rival but who is replaced by their clone set. In this further extended election example, all M candidates are initially tied with each other and candidate A prior to cloning; so, $T_B = T_C = T_D = T_E$ where such non-clones are nominated. Later, the case where $T_B > T_C$ is assessed. As before, A is cloned into the clone set A_1 through to A_K with every voter ranking it in strict forward or reverse order.

The case where $M = 1$ is analyzed in section 5 and for $M = 2$, $M = 3$ and $M = 4$ the derivation of the three threshold conditions for each profile is addressed in Appendix III. All threshold equations continue to be represented on a cloning map by straight lines. For each of these four cases, the p -axis intersection and slope of the straight line are given in Table 4. Note that as all non-clone candidates are tied here, all their pairs of teaming thresholds are co-incident with those for $T_{A1} = T_B$ and $T_{AK} = T_B$.

The three thresholds for $M = 1$ appear on the CHPV cloning map in Figure 4 and are repeated in black on the map in Figure 13. Using the data in Table 4, the same thresholds for $M = 2$, $M = 3$ and $M = 4$ are plotted in green on this same cloning map. Note that the clone-equality boundary rotates clockwise about the map's center as M rises. As M increases without bound, this boundary approaches the $p + q = 1$ diagonal of the CHPV map. For the PL = GV(0) vector, this boundary is a horizontal line while, for a BC $\equiv$ GV($\rightarrow 1$) vector, it approaches a vertical line.

Table 4. Threshold conditions for the M multiple non-clone candidates.

M	Threshold intersection with p -axis			Negative slope common to each threshold
	$T_{A1} = T_B$	$T_{A1} = T_{AK}$	$T_{AK} = T_B$	
1	1	$\frac{1+r}{2}$	r	r
2	$\frac{2+r}{2}$	$\frac{1+r+r^2}{2}$	$\frac{r+2r^2}{2}$	$r + r^2$
3	$\frac{3+2r+r^2}{3}$	$\frac{1+r+r^2+r^3}{2}$	$\frac{r+2r^2+3r^3}{3}$	$r + r^2 + r^3$
4	$\frac{4+3r+2r^2+r^3}{4}$	$\frac{1+r+r^2+r^3+r^4}{2}$	$\frac{r+2r^2+3r^3+4r^4}{4}$	$r + r^2 + r^3 + r^4$

**Figure 13.** CHPV cloning map for selected M .

In this specific election, all the M non-clone candidates are acting in concert against the clone set, so as M increases, the size of each teaming region recedes towards zero. Therefore, for a $GV(r)$ election with these M candidates, opportunities for successful teaming become more restricted as M rises. Note that the thresholds are independent of the number of clones in the clone set; so, adding or subtracting a clone makes no difference to the election outcome.

For the $PL = GV(0)$ vector, the two teaming thresholds are co-incident with the top and bottom sides of the cloning map according to the data in Table 4 so teaming attempts always fail. As r increases for a $GV(0 < r < 1)$ vector, the size of each teaming region grows for a given M but this area is always smaller as M increases, for any given r . For the $BC \equiv GV(\rightarrow 1)$ vector, the two teaming thresholds are again co-incident with the clone-equality boundary according to the

data in Table 4 so teaming always succeeds as there is no vote-splitting region.

In practice, a multiple-tied result is most unlikely. Where a leading non-clone candidate (say B) completes closely with a pre-cloned one (A), the other candidates remain uncompetitive alternatives by not cloning themselves. Consider the case where $T_B > T_C$, $T_C > 0$ and, prior to cloning, $T_A = T_B$. Therefore, $T_A < T_B + T_C$. With both B and C acting against any teaming attempt by A , cloning becomes increasingly less likely to succeed as T_C rises as a proportion of the combined tallies. Where $T_C = 0$ or $T_C = T_B$, then $M = 1$ or $M = 2$ respectively. Hence, the cloning map where $0 < T_C < T_B$ will be intermediate between the ones above for $M = 1$ and $M = 2$; see Figure 13. As the total tally for all non-clones increases, the chances of successful teaming by the clone set are steadily reduced even given the initial tie between A and B .

7. Conclusions

Using vector analysis, a continuous spectrum of PV vectors from PL via BC to A-PL is spanned using equivalent GV vectors and anti-vectors with the common ratio being the sole variable. This study has not been restricted to just three preferences as such analysis can handle multiple candidates while still simultaneously representing all possible vectors on a full PV spectrum. $GV(r)/A-GV(r)$ vectors with four or more candidates smoothly span such a continuous spectrum as r is varied but only those vectors that can be represented by a geometric progression of weightings occur along it. The Dowdall vector employed in elections to Nauru's Parliament cannot be represented as a $GV(r)$ one where more than three candidates compete so its performance in relation to strategic nomination is not addressed here [14, 15].

With N candidates, there are $N!$ different types of ranked ballot and each type will vary in the degree of support it receives from voters. This vast multi-dimensional space is a "curse" according to Saari and is "the primary cause of all voting paradoxes" [17]. He exhorts researchers to "Find

appropriate properties and relationships to eliminate subjectivity in selecting a rule” [17]. In the context of a GV election, this search is for a suitable common ratio. The investigation here is focused on how an election is affected by strategic candidate nominations and how voters may effectively respond. To limit the scope of this investigation to a manageable size, the initial primary focus is on a contest between two leading and closely-competing candidates where one of them is minded to clone itself.

If two identical clones are introduced to compete with the non-clone candidate, then both clone supporters and opponents may rank either clone above the other one. For many GV vectors, this is likely to split the vote between them and allow the non-clone to win. Alternatively, if clone supporters are advised to support one leading clone in preference to the trailing one, then the non-clone is instead likely to lose. Here, the non-clone supporters may retaliate by preferring the trailing clone over the leading one. This tit-for-tat reversal is likely to produce a very similar, if not identical, result to that without cloning.

A cloning map can be employed to display all possible election outcomes for this single-winner election profile. The vertical axis of this graph is the proportion of clone supporters who rank a particular clone higher than the other one and the horizontal axis is the proportion of clone opponents who do likewise. Three thresholds may be plotted on the map and they are always straight lines. The clone-equality threshold is where both clones are awarded the same tally. This line pivots about the center of the map and rotates clockwise as r is raised. It hence divides the map into two equal regions regardless of the common ratio.

The threshold where one of the two clones and the non-clone share the same tally is called a teaming threshold. The two teaming thresholds are consistently parallel to and equidistant from the clone-equality one. For a GV(r) vector, the two teaming thresholds each rotate clockwise and move closer together as r rises and eventually merge with the clone-equality threshold for the BC = GV($\rightarrow 1$) vector. Between the two teaming thresholds, the outcome of the election is a win for the non-clone candidate through the self-harming act of vote-splitting. The remaining two corner regions are where teaming occurs such that one or other clone wins. Here, the clone ‘team’ has been successful in unfairly manipulating the outcome in its own favor through the deceitful act of cloning. The worst teaming strategy is to match the clone-equality threshold by not differentiating between the two clones. The best one is to be furthest from this threshold through clone supporters consistently preferring one clone over the other. The proportion of the entire map that represents a successful attempt at teaming is called the teaming index of the election profile.

For an A-GV(r) vector, the two teaming thresholds continue to rotate clockwise as r reverts towards zero such that the now overlapping teaming regions ensure that both clones are preferred to the non-clone candidate. For this

election, there are no vote-spitting regions and all A-GV(r) vectors have a teaming index of one.

An attempt to win through teaming is most likely to succeed when there is an initial tie between the pre-cloned candidate and the non-clone one. As the relative support for the non-clone over the pre-cloned one rises, the chances of teaming being successful diminishes and eventually vanishes. With GV(r) vectors, the best strategy for defeating teaming attempts remains the tit-for-tat clone-reversal one. If the matching condition is met, then teaming can be successfully thwarted. With A-GV(r) vectors, no such guarantee is available. If the pre-cloned candidate, despite being ahead of the non-clone, decides to clone itself, then this decision could backfire and grant the non-clone victory; especially when it is unclear which of the two clones is the leading one.

The two clones may be replaced by a larger clone set in the hope that the trailing clone ‘pushes’ the non-clone further down in rank such that the leading clone consequently wins. However, for GV(r) vectors and a pre-cloning tie between the two candidates, this strategy is ineffective if the non-clone supporters retaliate in a tit-for-tat fashion by ‘pushing’ the leading clone down in rank to compensate. These two strategies cancel each other out such that adding or subtracting clones has no effect on the election outcome. The teaming attempt may similarly be defeated when the pre-cloned candidate lags the non-clone. Here, the teaming index may increase slightly as more clones are added but it will always be smaller than when there is no lag.

For A-GV(r) vectors, enlarging the clone set significantly enhances the chances of teaming being successful even when this lag is quite large. Indeed, unlike for GV(r) vectors, an intermediate clone within the set may win the election. At the extreme, the A-PL = A-GV(0) vector awards a tied victory to all the intermediate clones when only the end clones in the set and the non-clones are voted against by being ranked last.

Where a clone set is closely competing with more than one non-clone candidate in a CHPV election, then the opportunity for a teaming attempt being successful steadily recedes as these candidates increase in number regardless of how many clones comprise the clone set. For a PL = GV(0) vector, it remains invulnerable to teaming while a BC $\equiv$ GV($\rightarrow 1$) one remains inherently susceptible to it.

Factors such as the initial support for each candidate, the number of them, the number of clones introduced and their share of the preferences within the clone set are all significant in how strategic nominations affect the outcome of a GV election. Given this knowledge, the GV common ratio is the remaining variable that determines how elections may be manipulated by candidates and how voters may respond to such challenges. This key variable is under the control of those organizing the election rather than those participating in it.

By introducing undifferentiated clones, the chances of vote-splitting are enhanced. This self-inflicted prospect of harm should act as a deterrent to cloning. On the other hand,

by adding clones with one promoted as leader within the set, the chances of leapfrogging to victory through teaming are boosted instead. This act, if successful, deprives voters of their genuine preferred winner. On this basis, a common ratio that promotes or allows teaming is therefore to be avoided. All A-GV($0 \leq r < 1$) vectors ranging from A-PL to A-BC have a teaming index of one when a pre-cloned candidate and a non-clone are initially tied. Indeed, there is a strong incentive to clone candidates when any GV anti-vector is employed. The BC $\equiv$ GV($\rightarrow 1$) vector is also inherently vulnerable to teaming attempts and affords no opportunities for opposition voters to combat such attempts.

If invulnerability to teaming is of paramount importance, then PL is the optimum vector as its teaming index for any election profile is always zero. However, the corollary is that it is highly vulnerable to vote-splitting instead. In practice, clones are never truly identical but may be similar enough in attracting and splitting votes from like-minded voters. PL = GV(0) is hence prone to tactical voting where many voters feel compelled to vote for a competitive ‘lesser-of-two-evils’ candidate rather than their sincere preference whom they expect to lose anyway. By distorting sincere preferences, a polarizing candidate is more likely to emerge victorious than a consensus one in a PL election. Saari highlights the desire “for incentives and strategy-proof mechanisms, which encourage sincere reactions” [18].

Midway between the consensus BC and polarized PL vectors is the CHPV one that is exactly intermediate within the range of GV($0 < r < 1$) vectors with its common ratio of one-half. For an initial tie between a pre-cloned candidate and a non-clone one, the post-cloning teaming index is also one-half. For this scenario, CHPV precisely counterbalances the prospect of vote-splitting against the threat of teaming. This feature acts as a disincentive to clone candidates since it is unclear whether the outcome will help or hinder any attempt to win through cloning. Also, CHPV represents the upper boundary for GV ($0 < r \leq 1/2$) vectors that require no retaliation by clone opponents to defeat teaming attempts with undifferentiated clones. Of these vectors, CHPV is the most consensual one. Further, even with a strongly differentiated clone set, opposition voters in a CHPV election may successfully retaliate with a tit-for-tat clone-reversal strategy to nullify the effects of cloning.

With its polarization and consensus indices both approaching one-half, CHPV is a nominally balanced vector favoring neither extreme [7]. One preference at any rank is worth two at the adjacent lower rank so neither strong support for one candidate nor broad support for a competitor is allowed to dominate the other. CHPV is the only GV(r) vector that becomes ever more balanced as more candidates compete in an election so adding clones further reinforces its balanced system bias [7]. In addition to its other beneficial features highlighted in our original article, CHPV enables voters to thwart teaming attempts so deterring candidates from engaging in such attempts [7].

As the common ratio of a GV(r) vector rises, its system bias becomes increasingly less polarized and ever more consensual. Unfortunately, it also becomes ever more vulnerable to strategic nominations. If election organizers seek an optimal balance between maximizing the consensus index of a GV(r) vector and minimizing the possibility of a successful teaming attempt, then CHPV offers a viable and practical alternative to both plurality and the Borda count.

Appendix

Appendix I: Equivalence of Positional and Geometric Voting Anti-Vectors

For A-GV(r) vectors, their equivalent conjugate vectors having non-negative weightings may also be plotted on a three-preference map; see Figure 3 in section 2.2. Within the valid ranking region, GV(r) vectors are associated with points on or below the horizontal line $y = 1/3$ that represents BC-equivalent vectors. A-GV(r) vectors are however associated with points on or above this horizontal line that simultaneously also represents A-BC equivalent vectors.

When plotting any vector on the triangular map, the three preferences must sum to unity. The GV(r) vector u as defined below is employed here.

$$u = \frac{(u_1, u_2, u_3)}{u_1 + u_2 + u_3} = \frac{(u_1, u_2, u_3)}{U} = \frac{(1, r, r^2)}{1 + r + r^2} = \frac{(1, r, r^2)}{R} \text{ where } R = 1 + r + r^2$$

A three-preference conjugate vector u^C is hence $(1 - r^2/R, 1 - r/R, 1 - 1/R)/2$ as the sum of the bracketed components is $3 - (1 + r + r^2)/R = 2$. The scaling to ensure that the mapped components sum to unity does not affect the equivalence of any vectors. Hence, the corresponding preference coordinates for the conjugate vector u^C that sum to unity are:

$$u^C = \frac{\left(1 - \frac{r^2}{R}, 1 - \frac{r}{R}, 1 - \frac{1}{R}\right)}{\left(3 - \frac{1 + r + r^2}{R}\right)} = \frac{(R - r^2, R - r, R - 1)}{2R} = \frac{(1 + r, 1 + r^2, r + r^2)}{2(1 + r + r^2)}$$

Using the coordinate transform equations derived in section 2.1, the cartesian coordinates (x, y) for vector u^C are:

$$\left(\frac{u_3 - u_1}{U\sqrt{3}}, \frac{u_2}{U}\right) = \left(\frac{(r + r^2) - (1 + r)}{2(1 + r + r^2)\sqrt{3}}, \frac{1 + r^2}{2(1 + r + r^2)}\right) = \left(\frac{r^2 - 1}{2(1 + r + r^2)\sqrt{3}}, \frac{1 + r^2}{2(1 + r + r^2)}\right)$$

The locus for conjugate vectors also forms a circular arc as shown by the upper red curve on the map in Figure 3 of section 2.2. This arc has a radius of $1/3$ centered at $(x, y) = (0, 2/3)$, or $(1/6, 2/3, 1/6)$ in barycentric coordinates. The following proof uses the equation for a circle to confirm that the arc is valid:

$$\left(\frac{1}{3}\right)^2 = \left(\frac{r^2-1}{2(1+r+r^2)\sqrt{3}}\right)^2 + \left(\frac{1+r^2}{2(1+r+r^2)} - \frac{2}{3}\right)^2$$

$$\frac{1}{9} = \frac{1}{3} \left(\frac{r^2-1}{2(1+r+r^2)}\right)^2 + \left(\frac{3+3r^2-4-4r-4r^2}{2(1+r+r^2)\sqrt{3}}\right)^2$$

$$1 = 3 \left(\frac{r^2-1}{2(1+r+r^2)}\right)^2 + \left(\frac{-r^2-4r-1}{2(1+r+r^2)}\right)^2$$

$$4(1+r+r^2)^2 = 3(r^2-1)^2 + (r^2+4r+1)^2$$

$$(4+4r+4r^2)(1+r+r^2) = (3r^2-3)(r^2-1) + (r^2+4r+1)(r^2+4r+1)$$

$$4+4r+4r^2+4r+4r^2+4r^3+4r^2+4r^3+4r^4 = 3r^4-3r^2-3r^2+3+r^4+4r^3+r^2+4r^3+16r^2+4r+r^2+4r+1$$

$$4+8r+12r^2+8r^3+4r^4 = 4+8r+12r^2+8r^3+4r^4$$

As the equation is demonstrated, the upper circular arc displayed in red in Figure 3 is indeed the locus of conjugate vectors that represent A-GV($0 \leq r < 1$) vectors.

For a PV vector that is a non-negative linear combination of A-PL and A-BC, any negative-slope line in the valid ranking region from the map's edge to its center only intersects the arc of conjugate vectors at one unique point. Therefore, the conjugate vector represented by this point is equivalent to all the other PV vectors associated with this line. The common ratio here is derived as follows from the conjugate weightings definition:

$$u^C = \frac{(u_1, u_2, u_3)}{U} = \frac{(1+r, 1+r^2, r+r^2)}{2R}$$

$$\frac{u_2-u_3}{U} = \frac{1-r}{2R}$$

$$\frac{u_1-u_2}{U} = \frac{r(1-r)}{2R}$$

$$r = \frac{u_1-u_2}{u_2-u_3}$$

Therefore, for any three-preference PV vector that is a non-negative linear combination of A-PL and A-BC, there is an equivalent A-GV(r) vector as defined below, provided $2u_2 > u_1 + u_3$ so that $r < 1$:

$$PV(u_1, u_2, u_3) \equiv GV\left(r = \frac{u_1-u_2}{u_2-u_3}\right)^A$$

Appendix II: Threshold Conditions for Multiple Clone Candidate GV Elections

The election profile analyzed here is defined by Table 3 in section 5. The number of candidates is N where $N = K + 1$ and the number of clones is K where $K \geq 2$. All voters rank

the clone set in strict forward order A_1 through to A_K or in strict reverse order A_K through to A_1 . Of the V_A voters supporting A , proportion p where $0 \leq p \leq 1$ rank A_1 first. Similarly, of the V_B voters supporting B , proportion q where $0 \leq q \leq 1$ rank A_K last. For the GV election examples in sections 3 and 4, $N = 3$, $K = 2$ and $A_K = A_2$.

From Table 3, the tally for each candidate is then:

$$T_{A1} = v_1 p V_A + v_2 q V_B + v_{N-1}(1-p)V_A + v_N(1-q)V_B$$

$$T_{AK} = v_1(1-p)V_A + v_2(1-q)V_B + v_{N-1}pV_A + v_NqV_B$$

$$T_B = v_1 V_B + v_N V_A$$

For one candidate to rank higher than another one, the following conditions apply.

For candidate A_1 to be preferred over B :

$$T_{A1} > T_B$$

$$v_1 p V_A + v_2 q V_B + v_{N-1}(1-p)V_A + v_N(1-q)V_B > v_1 V_B + v_N V_A$$

$$v_1 p V_A + v_2 q V_B + v_{N-1} V_A - v_{N-1} p V_A + v_N V_B - v_N q V_B > v_1 V_B + v_N V_A$$

$$p(v_1 - v_{N-1})V_A > (v_1 - v_N)V_B + (v_N - v_{N-1})V_A - q(v_2 - v_N)V_B$$

$$p > \frac{(v_1 - v_N)V_B}{(v_1 - v_{N-1})V_A} + \frac{(v_N - v_{N-1})}{(v_1 - v_{N-1})} - q \frac{(v_2 - v_N)V_B}{(v_1 - v_{N-1})V_A}$$

For a standard GV(r) vector, $v_1 = 1$, $v_2 = r$, $v_{N-1} = r^{N-2}$ and $v_N = r(r^{N-2}) = r^{N-1}$ hence:

$$p > \frac{(1-r^{N-1})V_B}{(1-r^{N-2})V_A} - \frac{(r^{N-2}-r^{N-1})}{(1-r^{N-2})} - q \frac{r(1-r^{N-2})V_B}{(1-r^{N-2})V_A}$$

$$p > \frac{(1-r^{N-1})V_B}{(1-r^{N-2})V_A} - \frac{r^{N-2}(1-r)}{(1-r^{N-2})} - \left(\frac{rV_B}{V_A}\right)q$$

Where $V_B = V_A$ then:

$$p > 1 - rq$$

For the conjugate format of a standard A-GV(r) vector, $v_1 = 1 - r^{N-1}$, $v_2 = 1 - r^{N-2}$, $v_{N-1} = 1 - r$ and $v_N = 1 - 1 = 0$ hence:

$$p > \frac{(1-r^{N-1}-0)V_B}{(1-r^{N-1}-1+r)V_A} + \frac{(0-1+r)}{(1-r^{N-1}-1+r)} - q \frac{(1-r^{N-2}-0)V_B}{(1-r^{N-1}-1+r)V_A}$$

$$p > \frac{(1-r^{N-1})V_B}{r(1-r^{N-2})V_A} - \frac{(1-r)}{r(1-r^{N-2})} - \left(\frac{V_B}{rV_A}\right)q$$

Where $V_B = V_A$ then:

$$p > 1 - \frac{q}{r}$$

For candidate A_K to be preferred over B :

$$T_{AK} > T_B$$

$$v_1(1-p)V_A + v_2(1-q)V_B + v_{N-1}pV_A + v_NqV_B > v_1V_B + v_NV_A$$

$$v_1V_A - v_1pV_A + v_2V_B - v_2qV_B + v_{N-1}pV_A + v_NqV_B > v_1V_B + v_NV_A$$

$$p(v_{N-1} - v_1)V_A > (v_N - v_1)V_A + (v_1 - v_2)V_B - q(v_N - v_2)V_B$$

$$p(v_1 - v_{N-1})V_A < (v_1 - v_N)V_A + (v_2 - v_1)V_B - q(v_2 - v_N)V_B$$

$$p < \frac{(v_1 - v_N)}{(v_1 - v_{N-1})} - \frac{(v_1 - v_2)V_B}{(v_1 - v_{N-1})V_A} - q \frac{(v_2 - v_N)V_B}{(v_1 - v_{N-1})V_A}$$

For a standard GV(r) vector:

$$p < \frac{(1-r^{N-1})}{(1-r^{N-2})} - \frac{(1-r)V_B}{(1-r^{N-2})V_A} - q \frac{r(1-r^{N-2})V_B}{(1-r^{N-2})V_A}$$

$$p < \frac{(1-r^{N-1})}{(1-r^{N-2})} - \frac{(1-r)V_B}{(1-r^{N-2})V_A} - \left(\frac{rV_B}{V_A}\right)q$$

Where $V_B = V_A$ then:

$$p < r - rq$$

For the conjugate format of a standard A-GV(r) vector:

$$p < \frac{(1-r^{N-1}-0)}{(1-r^{N-1}-1+r)} - \frac{(1-r^{N-1}-1+r^{N-2})V_B}{(1-r^{N-1}-1+r)V_A} - q \frac{(1-r^{N-2}-0)V_B}{(1-r^{N-1}-1+r)V_A}$$

$$p < \frac{(1-r^{N-1})}{r(1-r^{N-2})} - \frac{r^{N-2}(1-r)V_B}{r(1-r^{N-2})V_A} - \left(\frac{V_B}{rV_A}\right)q$$

Where $V_B = V_A$ then:

$$p < \frac{1}{r} - \frac{q}{r}$$

For candidate A_1 to be preferred over A_K :

$$T_{A1} > T_{AK}$$

$$v_1pV_A + v_2qV_B + v_{N-1}(1-p)V_A + v_N(1-q)V_B > v_1(1-p)V_A + v_2(1-q)V_B + v_{N-1}pV_A + v_NqV_B$$

$$v_1pV_A + v_2qV_B + v_{N-1}V_A - v_{N-1}pV_A + v_NV_B - v_NqV_B > v_1V_A - v_1pV_A + v_2V_B - v_2qV_B + v_{N-1}pV_A + v_NqV_B$$

$$2p(v_1 - v_{N-1})V_A > (v_1 - v_{N-1})V_A + (v_2 - v_N)V_B - 2q(v_2 - v_N)V_B$$

$$p > \frac{1}{2} + \frac{(v_2 - v_N)V_B}{2(v_1 - v_{N-1})V_A} - q \frac{(v_2 - v_N)V_B}{(v_1 - v_{N-1})V_A}$$

For a standard GV(r) vector:

$$p > \frac{1}{2} + \frac{(r-r^{N-1})V_B}{2(1-r^{N-2})V_A} - q \frac{(r-r^{N-1})V_B}{(1-r^{N-2})V_A}$$

$$p > \frac{1}{2} + \frac{rV_B}{2V_A} - \left(\frac{rV_B}{V_A}\right)q$$

Note that on the $T_{A1} = T_{AK}$ threshold at $q = 1/2$, $p = 1/2$ regardless of V_B/V_A or r .

Where $V_B = V_A$ then:

$$p > \frac{1+r}{2} - rq$$

For the conjugate format of a standard A-GV(r) vector

$$p > \frac{1}{2} + \frac{(1-r^{N-2}-0)V_B}{2(1-r^{N-1}-1+r)V_A} - q \frac{(1-r^{N-2}-0)V_B}{(1-r^{N-1}-1+r)V_A}$$

$$p > \frac{1}{2} + \frac{(1-r^{N-2})V_B}{2r(1-r^{N-2})V_A} - q \frac{(1-r^{N-2})V_B}{r(1-r^{N-2})V_A}$$

$$p > \frac{1}{2} + \frac{V_B}{2rV_A} - \left(\frac{V_B}{rV_A}\right)q$$

Again, note that on the $T_{A1} = T_{AK}$ threshold at $q = 1/2$, $p = 1/2$ regardless of V_B/V_A or r .

Where $V_B = V_A$ then:

$$p > \frac{1+r}{2r} - \frac{q}{r}$$

For a GV(r) vector or an A-GV(r) one, the three thresholds are all independent of N where $V_B = V_A$. Hence, adding or subtracting clones in the clone set has no impact on the teaming regions of the cloning map when the result without any cloning is a tie.

Note that the negative slope of each of the three straight-line thresholds has the same magnitude; namely rV_B/V_A for a GV(r) vector and V_B/rV_A for an A-GV(r) one. Therefore, all three thresholds for a given vector are parallel to each other.

Also note that the intersection with the p -axis for the $T_{A1} = T_{AK}$ threshold is exactly intermediate between those for the two teaming thresholds; see following proof. The intersection of the straight-line thresholds with the p -axis occur when $q = 0$. For the above statement to be true, then the $T_{A1} = T_{AK}$ threshold intersection should be equal to the average of the two teaming threshold ones; namely:

$$p(\text{for } T_{A1} = T_{AK}) = \frac{1}{2}(p(\text{for } T_{A1} = T_B) + p(\text{for } T_{AK} = T_B))$$

For the GV(r) vector:

$$\begin{aligned} \frac{1}{2} + \frac{rV_B}{2V_A} &= \\ \frac{1}{2} \left(\frac{(1-r^{N-1})V_B}{(1-r^{N-2})V_A} - \frac{r^{N-2}(1-r)}{(1-r^{N-2})} + \frac{(1-r^{N-1})}{(1-r^{N-2})} - \frac{(1-r)V_B}{(1-r^{N-2})V_A} \right) \\ 1 + \frac{rV_B}{V_A} &= \frac{(1-r^{N-1}-1+r)V_B}{(1-r^{N-2})V_A} + \frac{(1-r^{N-1}-r^{N-2}+r^{N-1})}{(1-r^{N-2})} \\ 1 + \frac{rV_B}{V_A} &= \frac{r(1-r^{N-2})V_B}{(1-r^{N-2})V_A} + \frac{(1-r^{N-2})}{(1-r^{N-2})} \\ 1 + \frac{rV_B}{V_A} &= \frac{rV_B}{V_A} + 1 \end{aligned}$$

For the conjugate format of the A-GV(r) vector:

$$\begin{aligned} \frac{1}{2} + \frac{V_B}{2rV_A} &= \frac{1}{2} \left(\frac{(1-r^{N-1})V_B}{r(1-r^{N-2})V_A} - \frac{(1-r)}{r(1-r^{N-2})} + \frac{(1-r^{N-1})}{r(1-r^{N-2})} - \frac{r^{N-2}(1-r)V_B}{r(1-r^{N-2})V_A} \right) \\ 1 + \frac{V_B}{rV_A} &= \frac{(1-r^{N-1}-r^{N-2}+r^{N-1})V_B}{r(1-r^{N-2})V_A} + \frac{(1-r^{N-1}-1+r)}{r(1-r^{N-2})} \\ 1 + \frac{V_B}{rV_A} &= \frac{(1-r^{N-2})V_B}{r(1-r^{N-2})V_A} + \frac{r(1-r^{N-2})}{r(1-r^{N-2})} \\ 1 + \frac{V_B}{rV_A} &= \frac{V_B}{rV_A} + 1 \end{aligned}$$

Reversing the steps from the final identity, the original equation for each vector is established. Hence, each teaming threshold is parallel with and equidistant from the $T_{A1} = T_{AK}$ threshold in all cases. Recall from above that this threshold intersects with the center of the map $(p, q) = (1/2, 1/2)$ in all cases regardless of V_B/V_A or r . Further notice that this threshold rotates clockwise about the map's center from horizontal for PL via diagonal for BC to vertical for A-PL as r rises for a GV(r) vector and then falls back for an A-GV(r) one.

For a three-way tie to occur when $V_B \neq V_A$, all three thresholds must be co-incident. Hence, $T_{A1} = T_B$ and $T_{AK} = T_B$ at the same p -axis intersection where $q = 0$.

For the GV(r) vector:

$$p(\text{for } T_{A1} = T_B) = p(\text{for } T_{AK} = T_B)$$

$$\begin{aligned} \frac{(1-r^{N-1})V_B}{(1-r^{N-2})V_A} - \frac{r^{N-2}(1-r)}{(1-r^{N-2})} &= \frac{(1-r^{N-1})}{(1-r^{N-2})} - \frac{(1-r)V_B}{(1-r^{N-2})V_A} \\ \frac{V_B}{V_A} (1 - r^{N-1} + 1 - r) &= 1 - r^{N-1} + r^{N-2} - r^{N-1} \\ \frac{V_B}{V_A} &= \frac{1+r^{N-2}-2r^{N-1}}{2-r-r^{N-1}} \end{aligned}$$

For the conjugate format of the A-GV(r) vector:

$$\begin{aligned} p(\text{for } T_{A1} = T_B) &= p(\text{for } T_{AK} = T_B) \\ \frac{(1-r^{N-1})V_B}{r(1-r^{N-2})V_A} - \frac{(1-r)}{r(1-r^{N-2})} &= \frac{(1-r^{N-1})}{r(1-r^{N-2})} - \frac{r^{N-2}(1-r)V_B}{r(1-r^{N-2})V_A} \\ \frac{V_B}{V_A} (1 - r^{N-1} + r^{N-2} - r^{N-1}) &= 1 - r^{N-1} + 1 - r \\ \frac{V_B}{V_A} &= \frac{2-r-r^{N-1}}{1+r^{N-2}-2r^{N-1}} \end{aligned}$$

Note the reciprocal nature of the V_B/V_A ratio for a GV(r) vector and its corresponding A-GV(r) one.

Appendix III: Threshold Conditions for Multiple Non-Clone Candidate GV Elections

The GV election profile in section 5 is defined in Table 3. Here, there are N candidates of which K are clones of A and one non-clone B . Let the number of non-clone candidates be M where $N = K + M$, $K \geq 2$ and $M \geq 1$. The threshold conditions for the case where $M = 1$ are derived in section 5. For $M = 2$, the election profile is given in Table 5. As successful teaming is most likely when a pre-cloned candidate is tied with a non-clone one, this scenario is reflected in Table 5. As any teaming attempt would involve trying to beat both B and C , then in this analysis B and C are also tied. For three candidates A , B and C , there are six possible rank orderings and here each one receives one-sixth of the V votes cast; hence producing the initial three-way tie between A , B and C . As the clone set may be ranked in either forward or reverse order, there are then twelve different types of ranked ballots following the cloning of A . Candidates B and C remain tied but they may lead or trail A depending upon the voter preferences for the clone set.

Table 5. GV(r) election profile with two non-clone candidates.

Votes	Rank Position (i) / Weighting (v_i)						
	1 st / v_1	2 nd / v_2	3 rd / v_3	→	N-2 _{th} / v_{N-2}	N-1 _{th} / v_{N-1}	N _{th} / v_N
$pV/6$	A_1	→	→	→	A_K	B	C
$(1-p)V/6$	A_K	→	→	→	A_1	B	C

Votes	Rank Position (<i>i</i>) / Weighting (<i>v_i</i>)						
	1 st / <i>v₁</i>	2 nd / <i>v₂</i>	3 rd / <i>v₃</i>	→	N-2 th / <i>v_{N-2}</i>	N-1 th / <i>v_{N-1}</i>	N th / <i>v_N</i>
<i>pV</i> /6	<i>A₁</i>	→	→	→	<i>A_K</i>	<i>C</i>	<i>B</i>
$(1-p)V/6$	<i>A_K</i>	→	→	→	<i>A₁</i>	<i>C</i>	<i>B</i>
<i>qV</i> /6	<i>B</i>	<i>A₁</i>	→	→	→	<i>A_K</i>	<i>C</i>
$(1-q)V/6$	<i>B</i>	<i>A_K</i>	→	→	→	<i>A₁</i>	<i>C</i>
<i>qV</i> /6	<i>B</i>	<i>C</i>	<i>A₁</i>	→	→	→	<i>A_K</i>
$(1-q)V/6$	<i>B</i>	<i>C</i>	<i>A_K</i>	→	→	→	<i>A₁</i>
<i>qV</i> /6	<i>C</i>	<i>A₁</i>	→	→	→	<i>A_K</i>	<i>B</i>
$(1-q)V/6$	<i>C</i>	<i>A_K</i>	→	→	→	<i>A₁</i>	<i>B</i>
<i>qV</i> /6	<i>C</i>	<i>B</i>	<i>A₁</i>	→	→	→	<i>A_K</i>
$(1-q)V/6$	<i>C</i>	<i>B</i>	<i>A_K</i>	→	→	→	<i>A₁</i>

From Table 5, the tally for each potentially leading candidate is then:

$$T_{A_1} = \frac{V}{6}(2pv_1 + 2qv_2 + 2qv_3 + 2(1-p)v_{N-2} + 2(1-q)v_{N-1} + 2(1-q)v_N)$$

$$T_{A_K} = \frac{V}{6}(2(1-p)v_1 + 2(1-q)v_2 + 2(1-q)v_3 + 2pv_{N-2} + 2qv_{N-1} + 2qv_N)$$

$$T_B = T_C = \frac{V}{6}(2v_1 + v_2 + v_{N-1} + 2v_N)$$

For one candidate to rank higher than another one, the following conditions apply.

For candidate *A₁* to be preferred over *B* and *C*:

$$T_{A_1} > T_B \text{ and } T_{A_1} > T_C$$

$$pv_1 + qv_2 + qv_3 + (1-p)v_{N-2} + (1-q)v_{N-1} + (1-q)v_N > v_1 + v_2/2 + v_{N-1}/2 + v_N$$

$$pv_1 + qv_2 + qv_3 + v_{N-2} - pv_{N-2} + v_{N-1} - qv_{N-1} + v_N - qv_N > v_1 + v_2/2 + v_{N-1}/2 + v_N$$

$$p(v_1 - v_{N-2}) > v_1 + v_2/2 - v_{N-2} - v_{N-1}/2 - q(v_2 + v_3 - v_{N-1} - v_N)$$

$$p > \frac{(v_1 - v_{N-2}) + (v_2 - v_{N-1})/2}{(v_1 - v_{N-2})} - q \frac{(v_2 - v_{N-1}) + (v_3 - v_N)}{(v_1 - v_{N-2})}$$

For a standard GV(*r*) vector, $v_1 = 1$, $v_2 = r$, $v_3 = r^2$, $v_{N-2} = r^{N-3}$, $v_{N-1} = r^{N-2}$ and $v_N = r^{N-1}$ hence:

$$p > \frac{(v_1 - v_{N-2}) + r(v_1 - v_{N-2})/2}{(v_1 - v_{N-2})} - q \frac{r(v_1 - v_{N-2}) + r^2(v_1 - v_{N-2})}{(v_1 - v_{N-2})}$$

$$p > \frac{2+r}{2} - (r + r^2)q$$

For candidate *A_K* to be preferred over *B* and *C*:

$$T_{A_K} > T_B \text{ and } T_{A_K} > T_C$$

$$(1-p)v_1 + (1-q)v_2 + (1-q)v_3 + pv_{N-2} + qv_{N-1} + qv_N > v_1 + v_2/2 + v_{N-1}/2 + v_N$$

$$v_1 - pv_1 + v_2 - qv_2 + v_3 - qv_3 + pv_{N-2} + qv_{N-1} + qv_N > v_1 + v_2/2 + v_{N-1}/2 + v_N$$

$$p(v_{N-2} - v_1) > -v_2/2 - v_3 + v_{N-1}/2 + v_N + q(v_2 + v_3 - v_{N-1} - v_N)$$

$$p(v_1 - v_{N-2}) < v_2/2 + v_3 - v_{N-1}/2 - v_N - q(v_2 + v_3 - v_{N-1} - v_N)$$

$$p < \frac{(v_2 - v_{N-1})/2 + (v_3 - v_N)}{(v_1 - v_{N-2})} - q \frac{(v_2 - v_{N-1}) + (v_3 - v_N)}{(v_1 - v_{N-2})}$$

For a standard GV(*r*) vector, then:

$$p < \frac{r(v_1 - v_{N-2})/2 + r^2(v_1 - v_{N-2})}{(v_1 - v_{N-2})} - q \frac{r(v_1 - v_{N-2}) + r^2(v_1 - v_{N-2})}{(v_1 - v_{N-2})}$$

$$p < \frac{r+2r^2}{2} - (r + r^2)q$$

For candidate *A₁* to be preferred over *A_K*:

$$T_{A_1} > T_{A_K}$$

$$pv_1 + qv_2 + qv_3 + (1-p)v_{N-2} + (1-q)v_{N-1} + (1-q)v_N > (1-p)v_1 + (1-q)v_2 + (1-q)v_3 + pv_{N-2} + qv_{N-1} + qv_N$$

$$pv_1 + qv_2 + qv_3 + v_{N-2} - pv_{N-2} + v_{N-1} - qv_{N-1} + v_N - qv_N > v_1 - pv_1 + v_2 - qv_2 + v_3 - qv_3 + pv_{N-2} + qv_{N-1} + qv_N$$

$$2p > \frac{(v_1 - v_{N-2}) + r(v_1 - v_{N-2}) + r^2(v_1 - v_{N-2})}{(v_1 - v_{N-2})} - 2q \frac{r(v_1 - v_{N-2}) + r^2(v_1 - v_{N-2})}{(v_1 - v_{N-2})}$$

$$2p(v_1 - v_{N-2}) > v_1 + v_2 + v_3 - v_{N-2} - v_{N-1} - v_N - 2q(v_2 + v_3 - v_{N-1} - v_N)$$

$$2p > \frac{(v_1 - v_{N-2}) + (v_2 - v_{N-1}) + (v_3 - v_N)}{(v_1 - v_{N-2})} - 2q \frac{(v_2 - v_{N-1}) + (v_3 - v_N)}{(v_1 - v_{N-2})}$$

For a standard GV(r) vector, then:

$$p > \frac{1+r+r^2}{2} - (r+r^2)q$$

For cases where $M > 2$, the threshold conditions are derived in a similar manner to that for $M = 1$ in section 5 and for $M = 2$ above but with $2 \times (M + 1)!$ types of ranked ballots. The threshold conditions for $M = 3$ and $M = 4$ are given in Table 4 in section 6.

Abbreviations

A-BC	Anti-Borda Count
A-CHPV	Anti-CHPV Vector
A-Dowdall	Anti-Dowdall Vector
A-GV(r)	Anti-Geometric Voting Vector with Common Ratio r
A-PL	Anti-Plurality
BC	Borda Count
CHPV	Consecutively Halved Positional Voting
GV	Geometric Voting
GV(r)	Geometric Voting Vector with Common Ratio r
IIA	Independence of Irrelevant Alternatives Criterion
IoC	Independence of Clones Criterion
PL	Plurality
PV	Positional Voting

Nomenclature

CI	Consensus Index of vector (where $0 \leq CI < 1$)
d	Common difference of the arithmetic progression
i	Rank position (where $1 \leq i \leq N$)
K	Number of clone candidates
M	Number of non-clone candidates
N	Number of preferences / candidates / options
p	Proportion of voters preferring candidate A over B
P	Preference
P_i	Preference associated with specified rank position
PI	Polarization Index of vector (where $0 < PI \leq 1$)
q	Proportion of voters preferring candidate B over A
S	Scenario (as specified by p, q coordinates)
Σ	Sum of the components of a normalized vector w or its conjugate w^C (where $1 \leq \Sigma \leq N - 1$)
T	Tally (where subscript specifies the candidate)
TI	Teaming Index (proportion of a cloning map resulting in a successful teaming attempt)
r	Common ratio of the geometric progression (where $0 \leq r < 1$)
R	Sum of the components of a GV(r) vector v
u	Vector – normalized version with components summing to unity
u^C	Conjugate of vector u
U	Sum of the components of vector u
u_i	Value of vector u weighting associated with specified rank position
v	Vector – standard version as used in election
V	Total number of valid votes cast in an election
V_A	Number of voters supporting A over B

V_B	Number of voters supporting B over A
v_i	Value of vector v weighting associated with specified rank position
w	Vector – normalized version with first and last preferences worth one and zero respectively
w^A	Anti-vector – normalized version
w^C	Conjugate vector – normalized version
w_i	Value of vector w weighting associated with specified rank position
x, y	Cartesian coordinates

Author Contributions

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Conflicts of Interest

The author declares no conflicts of interest.

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