

Research Article

The Importance of Variational Thinking in the Training of Engineers Case TecNM

J. Agustín Flores Avila^{1,*} , Georgina Flores Garduño¹,
Gabriela Amyra Flores Garduño², Ricardo Flores Garduño², Anbal Flores Garduño³

¹Department of Basic Sciences, Technological Institute of the Lagoon, Torreón City, Mexico

²Department of Basic Sciences, Technological University of Durango, Lerdo City, Mexico

³Department of Basic Sciences, University of Guadalajara, Guadalajara, Mexico

Abstract

In the teaching of the various branches of mathematics, emphasis is placed on algorithms, to such an extent that it is common to characterize someone who is “good at doing math” as being good at math. A look is taken at the applications based on routine procedures and the conceptualization and, most importantly, the training as the final objective of the Construction of Mathematical Knowledge are left aside. Some branches of mathematics encourage the construction of a characteristic thought that, in the case of Calculus, is Variational Thinking. These thoughts imply a change in the way of understanding oneself and one’s environment, which leads to the construction of a certain neuronal arrangement that determines the difficulty of the particular branch. This work highlights the importance of Variational Thinking in the training of engineers in the Technological Education System embodied in the National Technological Institute of Mexico (TecNM). Variational Thinking allows to successfully attack the problems of change, which are characteristic of engineering, focusing its resolution not on “what formula to use” (static thinking) but on what mathematical model enables its resolution (variational thinking). It is identified as an epistemological problem, the activities to be developed to achieve the objective are described and the change of strategy is shown through the resolution of problems in which Variational Thinking is decisive. A paradigm shift occurs when trying to solve a new problem.

Keywords

Algorithms, Worldview, Variational Thinking, Problems of Change, Mathematical Model

1. Introduction

One of the obstacles that the teacher faces in the classroom in the process of constructing Mathematical Knowledge is the so-called Epistemological Problems, which are those that humanity faced, at a given time, in the construction of knowledge. These are problems that, due to their characteris-

tics, could not be solved with the cognitive baggage of the time (worldview -Weltanschauung-) and, therefore, the construction had to be deferred until adequate progress was achieved. The one that interests us is the construction of Calculus as a mathematical tool to study curves (Leibniz:

*Corresponding author: cidde2010@gmail.com (J. Agustín Flores Avila)

Received: 8 May 2025; **Accepted:** 23 June 2025; **Published:** 30 July 2025



Copyright: © The Author(s), 2025. Published by Science Publishing Group. This is an **Open Access** article, distributed under the terms of the Creative Commons Attribution 4.0 License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution and reproduction in any medium, provided the original work is properly cited.

static situation) or study the movement of the planets (Newton: dynamic situation). Leibniz's Calculus algebraically manipulated infinitely small quantities (hence its initial name of Infinitesimal Calculus) that were then eliminated and then not and he constructed it in a geometric context; he referred to it as a new method to rectify curves. The book by the Marquis de L'Hopital, which is credited with being the first book written to teach Calculus, is called: *L'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* ("Analysis of the infinitely small for the understanding of curved lines") and it has precisely that purpose: to understand curves. The context of this Calculus is geometry and algebra, static elements that do not change or vary, so their manipulation was easy. However, since infinitely small quantities were sometimes eliminated, sometimes not, it was harshly questioned, but since it solved problems, it could not be discarded and, therefore, it was used for almost two centuries despite its mathematical informality. Until the end of the first half of the last century, Calculus continued to be taught via infinitesimals until the Bourbaki school changed this approach. Anfossi and Granville were compulsory books in his teaching and, since it was a static thought, there was no problem in its algebraic manipulation in geometric contexts. The derivation through the four steps, still used in some high schools, was characteristic of this Calculus. Do we want our students to develop the ability to do derivatives and solve application problems whose statement indicates the procedure to follow? Let's teach Infinitesimal Calculus!

For its part, the Calculus built by Newton is given in a dynamic context from the moment it was the tool that allowed him to describe the movement of, in this case, the moon around the earth and which he later extended to the movement of the planets around the sun. His concepts of fluents (variables), fluxions (instantaneous rate of change) and first and last ratios (limits) are related to the change that occurs with respect to time [14]. We assume that Newton had developed variational thinking, the causes of which we cannot guess and which allowed him to be known as the great genius of science. Calculus studies the problems of change, of variation, and the thought of the time was static, which is why Newton's Calculus was not understood. Variational thinking was not present at that time and almost two centuries had to pass for humanity to mature and develop it. Thus, Variational Thinking defines an Epistemological Problem in the construction of Calculus; it was not until the 19th century that, via Cauchy and Weierstrass, it acquired a letter of naturalization with Variational Thinking contained in the limits as a cornerstone. Limits are a concept that gives the idea of a dynamic and continuous situation while infinitesimals contain a static and discrete idea. It took humanity two centuries to overcome this obstacle and in the classroom we have one semester to achieve it, hence the saying that Calculus is the most "difficult" branch of mathematics since it involves the construction of a mental structure via neural connections and defines a neuro-physiological modification.

The concept of Variational Thinking underlies the idea of the ability to model problems in which change is present in order to describe the phenomenon under study and predict its future behavior. [19, 5-14]. Knowing a system implies having the necessary information to be able to describe it, that is, knowing its behavior based on the response to certain input signals under controlled conditions. In this way, we can predict its future behavior and, if necessary, take preventive actions at the appropriate times to avoid further damage. The work of the engineer is the study of systems that vary over time, whether hydraulic, thermal, chemical, electrical processes, etc. Our surroundings are full of engineering and, therefore, mathematics. To carry out such studies, it is essential to have the mathematical model of the system, so modeling is a determining factor in the training of the engineer. Normally, when approaching the resolution of a problem, the strategy involves knowing and remembering the formula that enables its resolution. It is a problem and one or more formulas and this strategy is characteristic of static-discrete thinking. In variational thinking, the problem is conceived as a complete, integrated, comprehensive and consistent whole (whole vision); its parameters and the ways in which they vary are identified as well as the modes of variation that are determined by the laws that support the resolution and this leads us to the mathematical model as a strategy to solve the problem. The Records of Semiotic Representations [16] are fundamental in this process. In science and engineering it is said that a problem is solved when the mathematical expression that models it is available. For example, in the courses of Electric Circuits the response in time is attacked through algebra and the response to frequency through the phasor method; Currents or voltage drops in specific elements are calculated by breaking down the circuit using the concept of impedance and equivalent impedance, treating the system under static conditions. A mathematical expression that models it is not obtained. If we use variational thinking, we would have as a mathematical model a differential equation or integral differential equation or a system of equations of this kind in which the change is made present through the characteristic derivatives of these models. That the engineer builds variational thinking is decisive to increase his capacity to solve problems that involve change, which, as we have already pointed out, are typical of all engineering. Do we want our engineers to develop the capacity to solve problems via mathematical modeling in order to understand their behavior and be able to take preventive measures, when appropriate, to avoid unwanted responses? Let us teach them Calculus in its formative aspect that enables the construction of variational thinking!

2. Theoretical or Conceptual Elements

Calculus is the branch of mathematics that allows us to study those natural, social, economic and other events in which change and variation are present and it is a tool that

favors modeling and knowledge through the description of such events by delving into the relationships of the variables involved (Palafox, Grijalva and Ávila op cit). The authors also mention that Kaput (1992), (cited in Artigue, 1995, p. 119) states that “learning Calculus [has] to be based on the study of quantifiable change and accumulation and on the relationship between the two.” They add that unfortunately, in its teaching, an algorithmic approach is favored to develop the ability to obtain derivatives and their use in routine applications that, rather than inventiveness and innovation, resort to repeating a strategy studied in class, so that everything is reduced to memorizing routine procedures in the classroom [18]. [19] proposes to decisively promote the change from static mathematics to dynamic mathematics, from the thought of eternal and immutable mathematical truths to Variational Thinking, and from the traditional idea of applying mathematics to the mathematization and modeling of reality to build new mathematics or reconstruct old ones.

For their part, [6] point out that the notion of change denotes “the modification of state, appearance, behavior or condition of a body, a system or an object, while we are understanding variation as a quantification of change.”

On the other hand, [19] points out that:

Variational thinking can be described approximately as a dynamic way of thinking, which attempts to mentally produce systems that relate their internal variables in such a way that they covary in a manner similar to the patterns of covariation of quantities of the same or different magnitudes in the sub-processes cut out from reality.

He later added that “The object of variational thinking is therefore the capture and modelling of the covariation between quantities of magnitude, mainly - but not exclusively - the variations in time.”

Thus, the concept of Variational Thinking is based on the idea of the ability to model problems in which change is present in order to describe the phenomenon under study and predict its future behavior [2, 5, 14-19], a capacity that can only be achieved through teaching calculus in its formative aspect, that is, with the ultimate purpose of constructing Variational Thinking.

In some works such as those by [1, 2, 5, 19] activities are proposed that aim to develop variational thinking through the study and analysis of situations in which change is present. In all cases, promising results are mentioned, so it is recommended to continue with this line of work and research.

3. Methodological Elements

In this work, the interest is focused on carrying out a series of activities whose purpose and objective is to build/develop Variational Thinking in the engineering students of The Lagoon Campus of the National Technological Institute of Mexico (TecNM), specifically in the specialties of Electrical and Mechatronics, which are the ones with which we have applied it.

The TecNM is the Higher Technological Education System of Mexico and its mission is to “integrally train competitive professionals in science, technology and other areas of knowledge, committed to the economic, social, cultural development and sustainability of the country”, with a presence throughout the national territory.

The antecedents of the TecNM are the Technological Institutes that emerged in Mexico in 1948, when those of Durango and Chihuahua were created. Shortly after, those of Saltillo (1951) and Ciudad Madero (1954) were founded. By 1955, these first four Technological Institutes served a student population of 1,795 students, of which 1,688 were men and only 107 women. Today there are 266 institutions, of which 126 are federal, 134 decentralized, four Crodes, one Ciidet and one Cenidet, serving 521,105 students at the undergraduate and graduate levels.

The TecNM La Laguna Campus began operations in September 1966 offering the Industrial Engineering degree with options in Production and Electrical Engineering; currently it offers 9 engineering degrees, one undergraduate degree, four graduate degrees, one doctorate and has more than 6,000 students.

The activities we propose to develop (some of which we have already implemented with encouraging results) are described below and based on the consideration that:

1. Calculus is that branch of mathematics that is used in the study of the phenomena of change and variation, given that its development/construction is driven by this kind of events. Epistemological support.
2. However, from our perspective we distinguish between the “static” change of Leibniz and the dynamic change -variation- of Newton. Static change -which we will refer to simply as change- is related to algebra and geometry, in which a situation is in state “A” and then changes to state “B” without any perception of the change, and which we measure with the characteristics of each state: if both states are equal, then there was no change, while in dynamic change -which we will call variation-, the modification from state “A” to state “B” occurs continuously and imperceptibly and the notion of instantaneous variation appears (concept of derivative) which is the measure of variation. The first case determines the Calculus of Infinitely and the second the Calculus of Variation via limits. We can understand change as the accumulation of variation (Palafox, Grijalva and Ávila op cit).
3. For this reason, the formative character of Newtonian Calculus is conducive to the construction of Variational Thought. Here we point out that we use the concept of “construction” to characterize that set of activities that are carried out with a specific purpose or objective to differentiate it from the concept of “development” that we use to characterize those modifications that occur naturally, for example, the natural growth of plants; when grafts are used in the “development” of a plant,

then it is no longer “development” but “construction”.

4. Likewise, we consider that the concept of Variational Thought is based on the idea of mathematical modeling, which refers to the study of problems in which change and variation are present and consists of parameterization, identifying the modes of variation and relating the parameters as well as their mathematical expression using for this purpose the resources available in the Theory of Records of Semiotic Representations (Pecharroman op cit).
5. We hypothesize that the development of Variational Thinking involves the right hemisphere of the brain, since the left hemisphere is exclusive to the exact sciences, which adds another difficulty to its development.
6. We add that the purpose of modeling is the study of dynamic systems, characteristic of engineering sciences, for their understanding (analysis) through the description of their functioning (synthesis) in response to specific inputs in controlled situations in order to, if necessary, take preventive measures and thus avoid undesirable future behaviors (evaluation). Note that according to the approach we give to the concepts of development and construction, in this process we are building mathematical knowledge from activities specifically designed to achieve the objective and as a consequence we promote the development of the capacity for analysis, synthesis and evaluation; this development is the consequence of a construction process. The same is applicable to the development of Variational Thinking.
7. This is in line with the approaches made by [3, 4] and [6] about the four fundamental components in the construction of knowledge; its epistemological nature, its sociocultural dimension, the cognitive planes and the modes of transmission through teaching, which define this new branch of educational research called Socio-epistemology.
8. And, finally, we consider that in order to achieve competence in Variational Thinking, it is essential to modify the approach and content of the mathematics taught at TecNM. For example:
 - 1) In the Differential Calculus course, attention must be paid to continuous processes to distinguish them from discrete processes based on their similarities and differences, and this is done using the set of Real Numbers as a basis, the study of limits with the epsilon-delta model, the definition of the derivative as a limit and the optimization processes in which the modeling begins, and the variation is characteristic, must be studied in as much detail as possible;
 - 2) In Integral Calculus, which traditionally aims to develop the ability to solve an integral, the focus should be on the method for solving problems; remember that Integral Calculus is the method par excellence for solving problems in engineering and science.
 - 3) In Several Variables, the approach of Differential Calculus is repeated but extended to functions in two variables. In order to achieve the objective and given that it is a first approximation to this class of functions, $z = f(x, y)$ is sufficient, in addition to its graphical representation and the construction of models being accessible associated with functional expressions.
 - 4) Finally, as the last mathematics of the curriculum, in Differential Equations the degree of development of Variational Thinking can be detected since in this course modeling goes hand in hand with the resolution of a Differential Equation and as problems related to the specialty are studied, the motivation and interest of the students arise spontaneously.
 - 5) We will add that in this activity the use of computational resources via the corresponding package is essential. Numerical records, graphs and simulation are other ways of presenting information in a dynamic way [9]. Freehand and “in the air” graphing as a mathematical physiological resource is very useful. Do you remember when you counted with your fingers?. Good... Let's graph in the air. This resource is related to psychomotor and active knowledge that is not forgotten.
 - 6) It goes without saying that this forces us to write textbooks that have this approach based on the fact that we are promoting the construction of Mathematical Knowledge with the purpose of developing Variational Thinking in students who are being trained in the field of engineering to promote the industrial development of our country and thus be able to be generators of new knowledge.

4. Results

We have been applying this program since 2016, but it has been taking shape for two years prior, so we can say that it has been in the making for 10 years, and is currently at the right time for its birth through the realization of activities whose intention is its implementation. As results we can mention the following:

1. Three textbooks have been written with the orientation outlined here and with the corresponding registration in the National Institute of Copyright (Indautor); the books are: [8, 11, 12].
2. The practices for the Mathematics Laboratory in Math-Cad have been designed for the Differential Calculus, Integral Calculus and Differential Equations courses in line with the established objectives. These activities are guided by the Science, Technology, Engineering and Mathematics [S.T.E.M.] model with important presence and influence in this era of knowledge.
3. The feedback [7] in all cases has occurred naturally.
4. In [19] we present an example of the way in which we are working on this proposal for the Construction of

Mathematical Knowledge for engineering students at the TecNM The Lagoon Campus.

5. According to the comments of the teachers of the following mathematics courses, the students are proactive, and those of the specialty mention the better performance of this class of students, specifically in Control (Electrical) and in Vibrations (Mechatronics).
6. In surveys carried out by other teachers on their own initiative, they express the high degree of satisfaction with this way of teaching the subject.

5. Conclusions

The industrial development of countries is closely related to their ability to train high-quality professionals in the field of engineering. Thus, in this era of globalized technological competition in which the country with the most and best engineers has the greatest possibilities for development, China and India have a clear advantage. China graduates 2 million engineers annually, which exceeds the graduates of Japan, Germany and the USA combined. India graduates half a million and our country graduates 115,000 below Iran, Japan, Indonesia, the USA and South Korea, among others. Although Mexico occupied sixth place worldwide in 2023 (in 2022 it was in eighth place) in the number of engineers it has to promote industrial development, there is evidence that the number is not in line with the quality, so it is essential to address both the content and the purposes of the various engineering courses offered in our Higher Education System. [15].

In the training of highly qualified engineers to promote the industrial development of the country in accordance with the present times in which the S.T.E.M. model will prevail, the formative aspect of mathematics plays a fundamental role, since it encourages the development of basic skills for excellence in engineering. Variational Thinking is the main factor to develop the ability to model systems in engineering for their study, analysis and evaluation, so its presence as a basic characteristic in the graduation profile of our engineers, whose development is obtained from the formative aspect of mathematics, must be addressed as a priority. This final intention pursues the present proposal.

Abbreviations

TecNM	National Technological Institute of Mexico
S.T.E.M.	Science, Technology, Engineering and Mathematics

Conflicts of Interest

The authors declare no conflicts of interest.

References

- [1] Baez, A., Martínez, Y. Pérez, O. and Pérez, R. (2017) Task Proposals for the Development of Variational Thinking in Engineering Students. University Training. Vol. 10. No. 3. Unpaginated.
- [2] Caballero M. and Cantoral R. (n.d.) A Characterization of the Elements of Variational Thought and Language. Latin American Committee for Educational Mathematics, A. C. (CLAME) Chap. 3. Socioepistemological aspects in the analysis and re-design of school mathematics discourse. 1197-1205.
- [3] Camacho, A. (2006) Socioepistemology and social practices. Mathematics education. 18(1), 133-160.
- [4] Camacho, A. (2011) Socioepistemology and social practices. Towards a dynamic teaching of differential calculus. Ibero-American Journal of Higher Education. 3(2), 152-171.
- [5] Cantoral, R. and Farfán, R. (2014, March 29). Thought and variational language in the introduction to analysis. <https://www.researchgate.net/publication/242767113>
- [6] Cantoral, Molina and Sánchez. (2005) Socioepistemology of prediction. Acta latinoamericana de la Matemática Educativa. Vol. 18 462-468.
- [7] D'Amore, B. (2011). Conceptualization, records of semiotic representations and noetics: Constructivist interactions in the learning of mathematical concepts and hypotheses about some factors that inhibit feedback. Scientific Journal, 11, 150-164.
- [8] Flores A, J. A. (2018) Cálculo Vol. II México. Impreso en C.I.D.D.E.
- [9] Flores, A. (2021, November 20). What is mathematics? (video) You Tube: <https://www.youtube.com/watch?v=AubttONzUwg>
- [10] Flores, A. (2023, December 13). Didactic proposal for online teaching 4.0 (video) YouTube: <https://www.youtube.com/watch?v=O8KQzW2ng1c>
- [11] Flores A, J. A. & Flores G. G. (2023) Cálculo Diferencial para Estudiantes de Ingeniería con Enfoque en Competencias. México. Disponible en Amazon.
- [12] Flores A, J. A. (2023) Ecuaciones Diferenciales Ordinarias Para estudiantes de ingeniería con problemas de sistemas de segundo orden. México. Impreso en C.I.D.D.E.
- [13] Gómez, O. (2013, October) Development of variational thinking in ninth grade students. Scientific journal, special edition. 124-129.
- [14] Palafox, M., Grijalva, A. and Avila, R. (2023) The Development of Variational Thinking.
- [15] XVI Inter-American Conference on Mathematics Education. Lima, Peru. <https://xvi-ponencias.ciaem-iacme.org/index.php/xviciaem/xviciaem/paper/viewFile/1596/1101>

- [16] Pecharromán, C. (2014). Learning and understanding mathematical objects from an ontological perspective. *Mathematics Education*, 26, 111-133.
- [17] Quintero R., Ursini, S: From the tutorial approach to the constructivist use of the computer in the classroom; Research report; Cinvestav, Mexico. 1988.
- [18] Salas C. R. (2023) We have more engineers, but less prepared. What is the mistake? VOCETYS information portal. Available at:
<https://www.cetys.mx/noticias/tenemos-mas-ingenieros-pero-menos-preparados-cual-es-el-error/#:~:text=Para%20inicios%20de%202023%2C%20datos,sixth%20place%20at%20world%20level>
- [19] Vasco, C. (n.d.) Variational thinking and mathematical modeling.