

Research Article

Nonlinear Inverse Heat Transfer Analysis in a Melting Furnace Using a High-order Modified Levenberg-Marquardt Method

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Abstract

The nonlinear inverse heat transfer problem is important in numerous scientific research and engineering applications. Considering the erosion-corrosion phenomenon of refractory brick walls in the melting furnace is a necessary research process for safe operation of the furnace. In this paper we present a nonlinear inverse heat transfer analysis approach for predicting wall erosion and time-varying thickness of the bank layer covering the inner surface of refractory brick walls of a melting furnace. The direct problem is a nonlinear one-dimensional mathematical model for the phase change process using the enthalpy method, which describes the concentrate melting process in the melting furnace. The numerical solution of this mathematical model uses finite difference method. In the nonlinear inverse heat transfer problem considered here, the time-varying heat flux and the thickness of the bank layer are unknown. We aim to determine these unknown parameters using temperature measurements obtained from the sensor. The inverse approach is based on the high-order modified Levenberg-Marquardt method (HMLMM) combined with the Broyden method. HMLMM combined with Broyden method can save a lot of Jacobian calculations and reduce the computational cost. Numerical results show that the proposed method is computationally more efficient than the inverse method in which the conventional Levenberg-Marquardt method (LMM) is used.

Keywords

Inverse Heat Transfer, Inverse Problem, High-order Modified Levenberg-Marquardt Method

1. Introduction

Melting furnaces such as electric furnaces ([Figure 1](#)) are used for material processing that requires high power and high temperature. The melting furnace consists of six electrodes, concentrate piles, a slag layer, a molten metal layer and an furnace body. Six electrodes and concentrate piles were partially immersed in the slag layer. Then the electrical resistance heat generated by the electrode in the slag layer is transferred to the immersed part of the concentrate piles to dissolve the concentrate.

The dissolved concentrate is separated into slag and molten metal. One of the interesting solid-liquid phase changes occurring in these melting furnaces is the formation of a rigid layer, called a bank, that covers the inner surface of the refractory brick wall. This bank plays an important role in these furnaces. The bank protects the brick walls from the molten bath that corrodes quickly. If the bank is too thick, the volume available for fusion is reduced and may affect the throughput

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of the furnace. Keeping a bank with optimum size is therefore necessary for safe and profitable operation of the melting furnace. It is very difficult to measure bank thickness. Furthermore, the formation of the bank is the complex process depending on the boundary conditions.

In recent years, the problem of bank formation inside high temperature melting furnaces has been tackled with various inverse heat transfer methods. The inverse heat transfer methods rest on various methods such as Levenberg-Marquardt method (LMM) [1, 9-11, 13, 18-21], Kalman filter method [12, 25], conjugate gradient method with conjugate equation [5, 14, 15, 17], Levenberg-Marquardt method combined gradient

projection method [20], regularization method [2, 3, 6, 7, 24], ANN method [16, 22, 23], and adaptive matrix inversion method [8]. Also, the inverse heat transfer methods for the heat transfer in non-stationary one-dimensional [1, 15], steady two-dimensional [14] and non-stationary three-dimensional [8-13] problems where the density, specific heat and thermal conductivity of the considered region depend on temperature were investigated. The finite difference method (FDM), Levenberg-Marquardt method (LMM) [1, 13], conjugate gradient method [14, 15], and finite-volume method (FVM) and adaptive matrix inversion method (AMIM) [8] were used here.

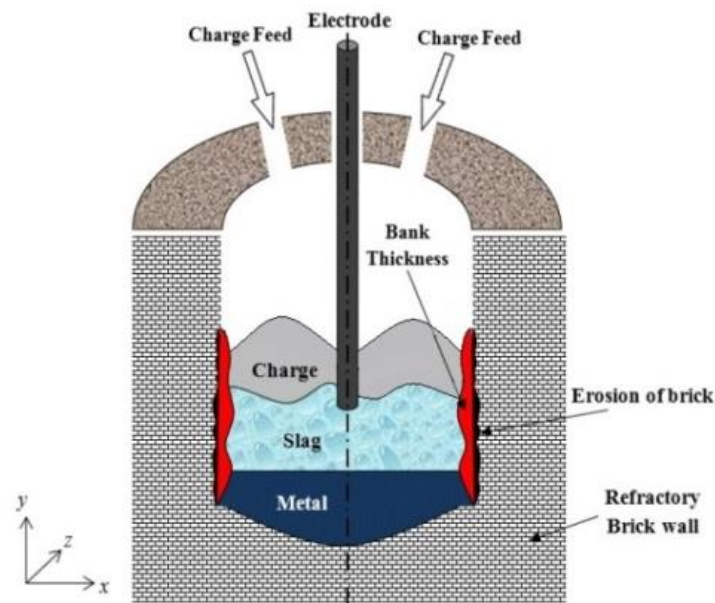


Figure 1. Section of a Melting Furnace.

The widely used method for inverse heat transfer problem is the Levenberg-Marquardt method. LMM is an optimization method that overcomes the shortcomings of the Newton method with narrow convergence-domain, which is often used in the numerical solution of inverse problem. Using LMM the time-varying heat flux of the furnace, i.e., the heat flux $q(t)$, in $(x = L_{Brick} + L_{PCM})$ (Figure 2), was predicted by solving the inverse problem. Once the heat flux is determined, the time-varying solidification layer thickness $E(t)$ covering the inner surface of the refractory brick wall can be calculated. Another problem that occurs in such furnaces is the erosion-corrosion of the inner surface of the refractory brick wall. This problem occurs when the bank is lost and the inner lining of the wall suddenly becomes exposed to the hostile molten material. Predicting erosion-corrosion wear is a crucial factor for determining the active life of the furnace. But, this prediction is difficult due to the prevailing physical and chemical conditions in the furnace.

For the study of these problems, the methods of inverse heat

transfer analysis of a nonlinear non-stationary one-dimensional phase change process, which can predict time-varying heat flux intensity, erosion-corrosion thickness and thermal properties inside the solidification layer of a high temperature melting furnace, were presented [9-12, 17]. It was then assumed that the density is independent of temperature and that the thermal contact resistance between the refractory brick wall and the phase change material is negligible. During the phase change, the densities for the solid and liquid phases are generally different. The mathematical model of the phase change process by enthalpy presented in the above-mentioned works is described in the absence of such density variations. In the study, a high-order modified Levenberg-Marquardt method (HMLMM) for solving nonlinear numerical equations was also proposed [4]. HMLMM is a method that saves more jacobian calculations and achieves faster convergence rate than the LMM.

In the present study, an inverse heat transfer procedure is proposed for predicting simultaneously the erosion-corrosion thickness $L_{Erosion}$ with the unknown time-varying heat flux

$q(t)$ by HMLMM. In this paper, we propose an inverse heat transfer numerical method using HMLMM combined with Broyden method (BM) to predict wall erosion and time-varying thickness of the protective bank covering the inner surface of refractory brick wall in a melting furnace when density, specific heat and thermal conductivity depend on the temperature and thermal contact resistance between refractory brick wall and phase change material is taken into account.

2. The Direct Problem

A direct problem was implemented for the whole melting furnace, i.e. the refractory brick wall and the PCM as phase change material. Here a phase-change material (PCM) consists of a solid layer, a mushy zone and a liquid layer. The inner surface of the refractory brick wall is covered by a protective bank whose time-varying thickness is $E(t)$. $E(t)$ shows the position of the solidification front of the PCM (Figure 2).

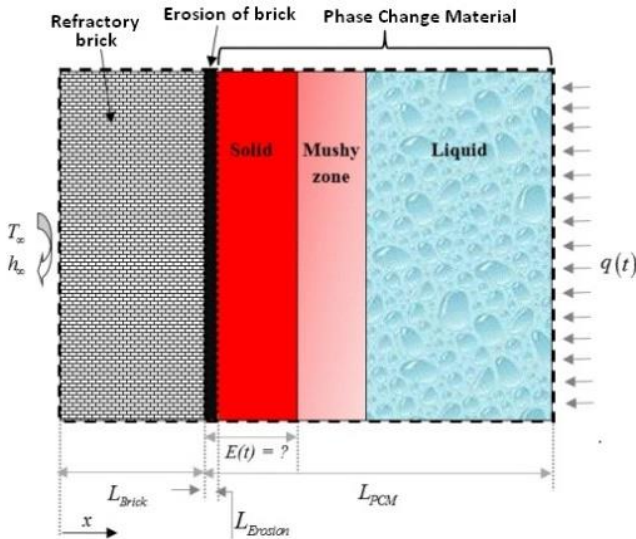


Figure 2. Schematic of the Direct Problem. $E(t)$ is the Unknown. It is Numerically Predicted by the Finite Difference Method.

The mathematical model of the melting furnace rests on the following assumptions:

- 1) The temperature gradients in the x direction are much larger than those in the other directions. Consequently, a one-dimensional analysis can be applied (Figure 2).
- 2) Thermal contact resistance between the refractory brick wall and the slag layer is considered.
- 3) The heat transfer inside the liquid phase of the PCM is conduction dominated.
- 4) The thermal properties in terms of density, specific heat and thermal conductivity are temperature dependent.
- 5) The phase change problem is non-isothermal. The melting process is depicted by three zones: a solid phase, a

mushy zone and a liquid phase.

Let $T_i = T_i(x, t)$ ($i = 1, 2$) be the temperatures of the PCM and the furnace body at the time t , respectively, and λ_i , C_{pi} , ρ_i ($i = 1, 2$) be the thermal conductivity, specific heat and density of the corresponding zone in the melting furnace, respectively. Then the mathematical model of the concentrate melting process in the PCM region $\Omega_1 = (L_{Brick}, L_{Brick} + L_{PCM})$ and the furnace body region $\Omega_2 = (0, L_{Brick})$ is described as a non-stationary one-dimensional phase change problem by enthalpy as follows:

$$\frac{\partial H(T_1)}{\partial t} = \frac{\partial}{\partial x} \left(\lambda_1(T_1) \frac{\partial T_1}{\partial x} \right), \text{ in } \Omega_1 \times I \quad (1)$$

$$\rho_2 C_{p2} \frac{\partial T_2}{\partial t} = \lambda_2 \frac{\partial^2 T_2}{\partial x^2}, \text{ in } \Omega_2 \times I \quad (2)$$

$$T_i = T_{i0}, \text{ in } \Omega_i, t = 0 \quad (i = 1, 2) \quad (3)$$

$$\lambda_1(T_1) \frac{\partial T_1}{\partial x} = q(t), x = L_{Brick} + L_{PCM}, t > 0 \quad (4)$$

$$\lambda_2 \frac{\partial T_2}{\partial x} = h_\infty (T_2 - T_\infty), x = 0, t > 0 \quad (5)$$

$$T_1 = T_2, \lambda_1(T_1) \frac{\partial T_1}{\partial x} = \lambda_2 \frac{\partial T_2}{\partial x}, x = L_{Brick}, t > 0 \quad (6)$$

Here $H(T_1)$ is defined as the enthalpy:

$$H(T_1) = \int_0^{T_1} C_{p1}(s) \rho_1(s) ds + L \rho_1(T_1) \eta(T_1),$$

$$\eta(T_1) = \begin{cases} 1, & T_1 > T^* \\ \frac{T_1 - T_*}{T^* - T_*}, & T_* \leq T_1 \leq T^* \\ 0, & T_1 < T_* \end{cases} \quad (7)$$

where $I = (0, t_{end})$ and L is the latent heat of PCM. The T_* , T^* are the solid and liquid critical temperatures, respectively. The values T_∞ , h_∞ are the air temperature and the heat transfer coefficient between the atmosphere and the furnace body, respectively. The $q(t)$ is the time-varying heat flux.

Equations (1)-(6) are mathematical model for the process of concentrate melting in a melting furnace described by enthalpy as a non-stationary one-dimensional phase-change coupling problem. The objective of direct problem is to determine the temperature field $T(x, t)$ and the time-varying thickness $E(t)$ of protective bank by the mathematical model presented above. The Kirchhoff transformation $u := \bar{F}(T_1) = \int_0^{T_1} \lambda_1(s) ds$ is applied to the mathematical model (1)-(6) for the concentrate melting process. Then, the numerical solution of the nonlinear system is obtained by constructing the nonlinear system by implicit approximation in the element with uniform mesh steps and constructing the numerical scheme by nonlinear successive relaxation. The flowchart of the numerical algorithm is shown in Figure 3.

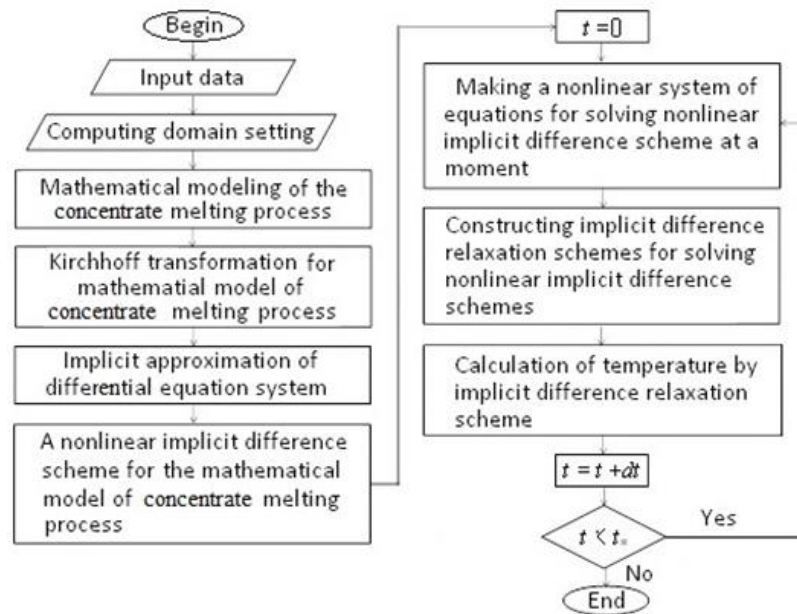


Figure 3. The Flowchart of the Numerical Algorithm.

In the direct problem all physical and geometrical properties are known. The thermophysical properties of the melting furnace (brick wall and PCM) are summarized in Table 1. The refractory wall is set to $L_{Brick} = 0.1m$ and the PCM layer (solid, mushy, liquid) to $L_{PCM} = 0.1m$. The ambient temperature is $T_{\infty} = 293K$, and the external average heat transfer coefficient is $h_{\infty} = 15 W/(m^2K)$. The time-varying heat flux $q(t)$ for $x = L_{Brick} + L_{PCM}$ is given by

$$q(t) = P_1 + P_2 \times \sin^2(2\pi t/t_{end}) \quad (8)$$

When the protective bank is lost, the inner surface of the refractory brick wall suddenly comes in direct contact with the melt. As a result, exposed brick walls can be corroded and damaged by erosion-corrosion. Indeed, the erosion of the refractory brick wall is a slow process. Therefore, the corroded part of the wall can be considered time-independent within the time interval $[0, 200\,000]$ simulated. The corroded part of the refractory wall $L_{Erosion}$ is set to:

$$L_{Erosion} = 0.01(m) = P_3 \quad (9)$$

The coefficients are given by

$$\begin{aligned} P_1 &= 5\,000(W/m^2), \\ P_2 &= 4\,000(W/m^2), \\ P_3 &= 0.01(m). \end{aligned} \quad (10)$$

Numerical simulations were executed with 1.4 relaxation coefficient on an Intel (R) Core (TM) i5 CPU @ 2.66 GHz. Numerical simulations were performed with 200 uniform meshes inside the PCM and the refractory brick wall. The time step was set equal to 100s. To validate the numerical results, the results of the numerical simulation analysis of the above mathematical model are compared with those obtained by COMSOL Multiphysics 6.2. The temperature profiles of the two results were compared for the 200 000s elapsed time at three fixed points with coordinates of 0.1, 0.15 and 0.19, respectively. As shown in Figure 4, it can be seen that our results are in good agreement with the numerical simulation results of COMSOL Multiphysics 6.2.

Table 1. Thermophysical Properties of the Refractory Brick Wall and of the PCM.

			PCM		
	Unit		Brick	Solid	Liquid
Density	ρ	kg/m ³	2 600	2 100	2 000
Specific heat	C_p	J/(kg K)	875	1 800	1 700
Thermal conductivity	λ	W/(m K)	16.8	1	10

		Unit	Brick	PCM	
				Solid	Liquid
Latent heat	L	J/kg	—	510 000	
Solidus temperature	T_*	K	—	1 213.15	
Liquidus temperature	T^*	K	—	1 233.15	

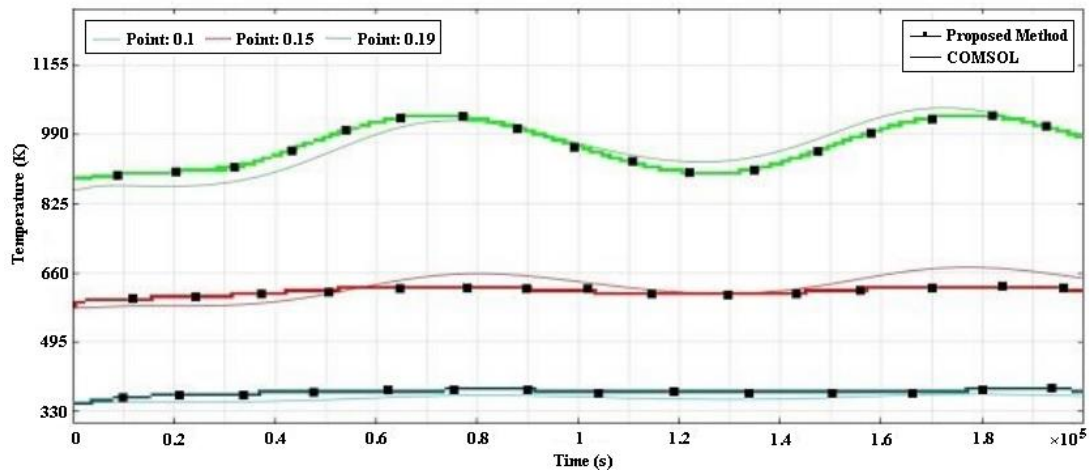


Figure 4. Temperature Profiles at Fixed Points by Our Method and COMSOL Multiphysics 6.2 for a Elapsed Time of 200 000s.

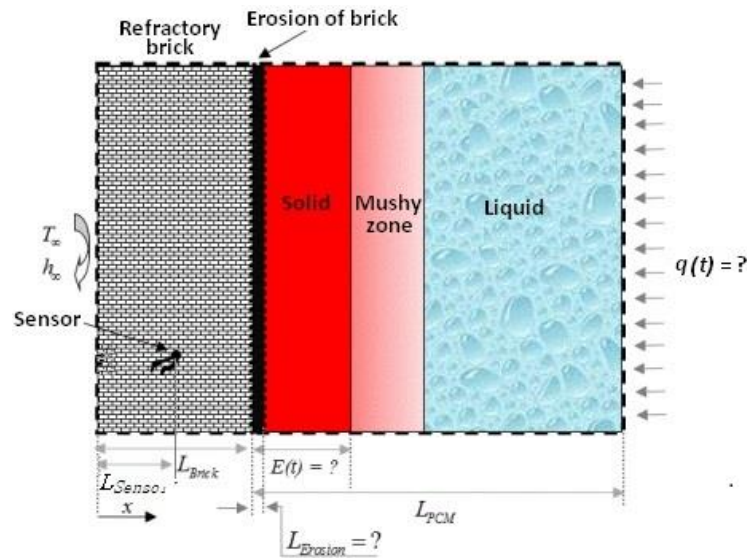


Figure 5. The Inverse Problem, $q(t)$ and $L_{Erosion}$ are Unknown.

3. The Inverse Problem

In the inverse problem, the time-varying heat flux $q(t)$ and the thickness of the corrosion wall $L_{Erosion}$ at $x = L_{Brick} + L_{PCM}$ are unknown. That is, P_1, P_2 and P_3 are unknown. The objective of the inverse problem is to determine

the unknown parameters using the temperature measurements obtained on the inner wall ($L_{Sensor} = 0.05m$) where the sensor is located (Figure 5). Once the time-varying heat flux $q(t)$ and the thickness of the corrosion wall $L_{Erosion}$ are determined, the time-varying thickness $E(t)$ of the protective bank may be estimated from the direct problem.

The solution of the inverse problem is found by minimizing

the norm $\psi(\vec{P})$:

$$\psi(\vec{P}) = \sum_{i=1}^l [Y(t_i) - \hat{T}(t_i, \vec{P})]^2 \quad (11)$$

where $\vec{P} = (P_1, P_2, P_3)^T$ is an unknown parameter vector,

$Y(t_i)$ is the measured temperature, and $\hat{T}(t_i, \vec{P})$ is the estimated temperature from the solution of the direct problem. l is the total number of measurements.

The minimizing of (11) is solved with HMLMM. The incremental value of the unknown parameter is given by

$$\Delta \vec{P}_{LMM}^k = [(J_k)^T J_k + \lambda_k I]^{-1} (J_k)^T (\vec{Y} - \vec{T}(\vec{P}^k)), \quad (12)$$

$$\Delta \vec{P}_{MLMM}^k = [(J_k)^T J_k + \lambda_k I]^{-1} (J_k)^T (\vec{Y} - \vec{T}(\vec{P}^k + \Delta \vec{P}_{LMM}^k)), \quad (13)$$

$$\Delta \vec{P}^k = [(J_k)^T J_k + \lambda_k I]^{-1} (J_k)^T (\vec{Y} - \vec{T}(\vec{P}^k + \Delta \vec{P}_{LMM}^k + \alpha_k \Delta \vec{P}_{MLMM}^k)), \quad (14)$$

where α_k is the step size, I, J_k are the 3×3 identity matrix and the Jacobian matrix, respectively. $\vec{Y}$ and $\vec{T}$ are l -dimensional vectors with $Y(t_i)$ ($i = \overline{1, l}$) and $\hat{T}(t_i, \vec{P})$ ($i = \overline{1, l}$) as their components, respectively. The superscript T represents the transpose of the matrix. J_k is defined as follows:

$$J_k = (J_{ij}^k)_{l \times 3} \quad (i = \overline{1, l}, j = \overline{1, 3}), \quad (15)$$

where $J_{ij}^k = \frac{\partial \hat{T}(t_i, \vec{P}^k)}{\partial \vec{P}_j^k}$. The Jacobian J_k is approximated by a finite difference approximation, i.e.

$$J_{ij}^k \approx \frac{\hat{T}(t_i, \vec{P}_1^k, \dots, \vec{P}_j^k + (\delta \vec{P}_j^k), \dots, \vec{P}_N^k)}{2(\delta \vec{P}_j^k)} - \frac{\hat{T}(t_i, \vec{P}_1^k, \dots, \vec{P}_j^k - (\delta \vec{P}_j^k), \dots, \vec{P}_N^k)}{2(\delta \vec{P}_j^k)} \quad (16)$$

The parameter perturbation $(\delta \vec{P}_j^k)$ is set to $\xi(1 + |\vec{P}_j^k|)$, where ξ is a small number. The subscripts i, j represent the number of time steps and parameters, respectively.

To reduce the computational cost, the Jacobian matrix is updated using BM. For the first iteration, for even iterations and for iterations where $\psi(\vec{P}^k + \Delta \vec{P}^k) > \psi(\vec{P}^k)$, the sensitivity coefficients of the Jacobian matrix are estimated with (16).

For every other iteration, the Jacobian matrix is updated by BM with the following relation:

$$J_k = J_{k-1} + \frac{[\vec{T}(\vec{P}^k) - \vec{T}(\vec{P}^{k-1}) - J_{k-1} \Delta \vec{P}^{k-1}](\Delta \vec{P}^{k-1})^T}{(\Delta \vec{P}^{k-1})^T \Delta \vec{P}^{k-1}}$$

The algorithm of HMLMM is as follows:

Step 1. Given $\vec{P}^1 \in R^3, \mu_1 > M > 0, 0 < p_0 < p_1 < p_2 < 1, 1 \leq \delta \leq 2, \hat{\alpha} > 1, k = 1$. In more detail $\mu_1 = 1, M = 10^{-8}, p_0 = 0.0001, p_1 = 0.25, p_2 = 0.75, \delta = 1, \hat{\alpha} = 1.85$.

Step 2. If $\|(J_k)^T (\vec{Y} - \vec{T}(\vec{P}^k))\| < \varepsilon, \psi(\vec{P}^{k+1}) < \varepsilon$, or $\|\vec{P}^{k+1} - \vec{P}^k\| < \varepsilon$, then stop, where ε is a small number. To obtain $\Delta \vec{P}_{LMM}^k$, solve $[(J_k)^T J_k + \lambda_k I] \Delta \vec{P}_{LMM}^k = (J_k)^T (\vec{Y} - \vec{T}(\vec{P}^k))$ with $\lambda_k = \mu_k \|\vec{Y} - \vec{T}(\vec{P}^k)\|^\delta$ and to obtain the $\Delta \vec{P}_{MLMM}^k$, solve $[(J_k)^T J_k + \lambda_k I] \Delta \vec{P}_{MLMM}^k = (J_k)^T (\vec{Y} - \vec{T}(\vec{P}^k + \Delta \vec{P}_{LMM}^k))$. Next to obtain $\Delta \vec{P}^k$, solve $[(J_k)^T J_k + \lambda_k I] \Delta \vec{P}^k = (J_k)^T (\vec{Y} - \vec{T}(\vec{P}^k + \Delta \vec{P}_{LMM}^k + \alpha_k \Delta \vec{P}_{MLMM}^k))$. And set $\vec{S}^k = \Delta \vec{P}^k + \Delta \vec{P}_{LMM}^k + \alpha_k \Delta \vec{P}_{MLMM}^k$, where α_k is the step size obtained by solving.

$$\max_{\alpha_k \in [1, \hat{\alpha}_k]} (\|\vec{Y} - \vec{T}(\vec{P}^k + \Delta \vec{P}_{LMM}^k)\|^2 - \|\vec{Y} - \vec{T}(\vec{P}^k + \Delta \vec{P}_{LMM}^k + \alpha_k \Delta \vec{P}_{MLMM}^k)\|^2).$$

Step 3. Compute $r_k = \text{Are} \Delta P^k / \text{Pre} \Delta P^k$, where

$$\text{Are} \Delta P^k = \|\vec{Y} - \vec{T}(\vec{P}^k)\|^2 - \|\vec{Y} - \vec{T}(\vec{P}^k + \Delta \vec{P}_{LMM}^k + \alpha_k \Delta \vec{P}_{MLMM}^k + \Delta \vec{P}^k)\|^2,$$

$$\begin{aligned} \text{Pre} \Delta P^k &= \|\vec{Y} - \vec{T}(\vec{P}^k)\|^2 - \|\vec{Y} - \vec{T}(\vec{P}^k + \Delta \vec{P}_{LMM}^k)\|^2 + \|\vec{Y} - \vec{T}(\vec{P}^k + \Delta \vec{P}_{LMM}^k + \Delta \vec{P}^k)\|^2 \\ &- \|\vec{Y} - \vec{T}(\vec{P}^k + \Delta \vec{P}_{LMM}^k + \alpha_k \Delta \vec{P}_{MLMM}^k)\|^2 + \|\vec{Y} - \vec{T}(\vec{P}^k + \Delta \vec{P}_{LMM}^k + \alpha_k \Delta \vec{P}_{MLMM}^k + \Delta \vec{P}^k)\|^2 \\ &- \|\vec{Y} - \vec{T}(\vec{P}^k + \Delta \vec{P}_{LMM}^k + \alpha_k \Delta \vec{P}_{MLMM}^k + \Delta \vec{P}^k)\|^2. \end{aligned}$$

Set

$$\vec{P}^{k+1} = \begin{cases} \vec{P}^k + \vec{S}^k: r_k \geq p_0, \\ \vec{P}^k: r_k < p_0. \end{cases}$$

Step 4. Choose μ_{k+1} as

$$\mu_{k+1} = \begin{cases} 4\mu_k, & \text{if } r_k < p_1, \\ \mu_k, & \text{if } r_k \in [p_1, p_2], \\ \max\left\{\frac{\mu_k}{4}, M\right\}, & \text{if } r_k > p_2. \end{cases}$$

Set $k = k + 1$ and go to Step 2.

The overall computational procedure for solving the inverse problem using HMLMM and BM is shown in Figure 6.

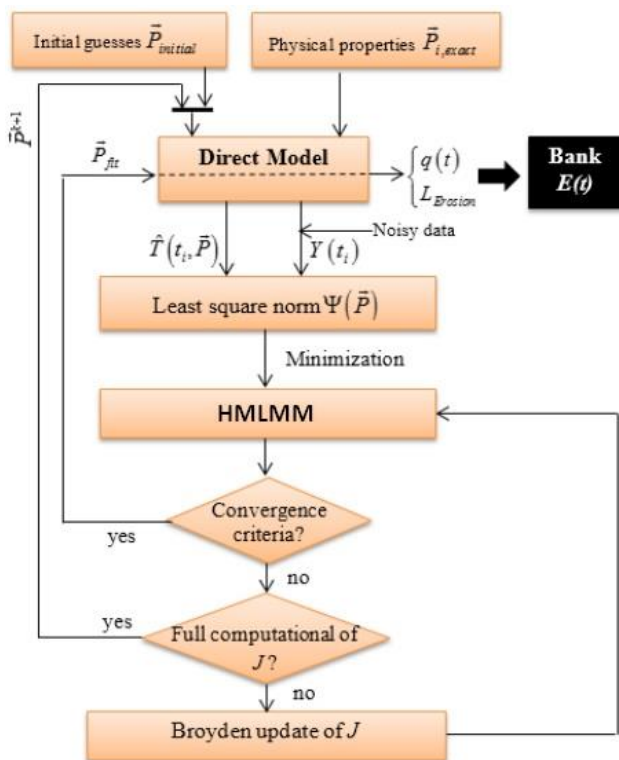


Figure 6. The Computational Flowchart of the Inverse Problem.

4. Results and Discussion

The computational procedure for solving the inverse problem shown in Figure 6 was used to simultaneously predict the unknown time-varying heat flux $q(t)$ and the erosion-corrosion thickness $L_{Erosion}$. Once the heat flux $q(t)$ and the corroded refractory wall thickness $L_{Erosion}$ are estimated, the bank thickness $E(t)$ is determined from the direct problem discussed above.

The temperature measurements Y_i were taken with a sensor ($L_{Sensor} = 0.05m$) placed inside the brick wall. The total temperature records in time $t_{end} = 200\ 000s$. Therefore, the

data-capture-frequency is $l = 2\ 000$. To simulate the measurement errors, random error noise $\vec{\omega}_i$ is added to the correct temperature $\vec{T}_{exact}$ obtained by the direct problem:

$$\vec{T}(t_i) = \vec{T}_{exact}(t_i) + \sigma \vec{\omega}_i,$$

where σ is the standard deviation of the measurement errors, which may take the value of $2\%T_{max}$ and $4\%T_{max}$. T_{max} is the maximum temperature measured by the sensor.

To demonstrate the accuracy of the numerical solution of the inverse problem for predicting the bank thickness, the test was performed in the case where the time-varying heat flux $q(t)$ at $x = L_{Brick} + L_{PCM}$ is $q(t) = P_1 + P_2 \times \sin^2(2\pi t/t_{end})$.

For the direct problem, the exact values are $P_1 = 5\ 000(W/m^2)$ and $P_2 = 4\ 000(W/m^2)$. For the sake of comparing the inverse solution (inverse problem) to the exact solution (direct problem), estimation errors are defined:

$$Error_{E(t)} = \frac{1}{l} \sum_{i=1}^l \frac{|E(t_i)_{inverse} - E(t_i)_{exact}|}{|E(t_i)_{exact}|} \times 100(\%),$$

$$Error_P = \frac{1}{l} \sum_{i=1}^l \left(\frac{1}{3} \sum_{j=1}^3 \frac{|P_j(t_i)_{inverse} - P_j(t_i)_{exact}|}{|P_j(t_i)_{exact}|} \right) \times 100(\%),$$

$$Error_T = \frac{1}{l} \sum_{i=1}^l \frac{|T_{inverse} - T_{exact}|}{|T_{exact}|} \times 100(\%).$$

The temperature distribution obtained from the inverse problem and the corresponding error $Error_T$ defined above are shown in Figure 7. It is shown that the error in the considered region is less than 0.08%. The numerical simulation results of the inverse problem by HMLMM are summarized in Table 2. Analyzing the data summarized in Table 2, it is shown that the relative error varies from 0.0 to 0.342%. The maximum difference occurs in the parameter prediction P_3 . P_3 is the smallest and most sensitive parameter. Figure 8 shows the time-varying profiles of the estimated temperature calculated by the inverse problem and the measured noise temperature (found by the one-dimensional direct problem with $2\%T_{max}$, $4\%T_{max}$ noise). The exact and the estimated (from the inverse model) time-varying bank thicknesses are depicted in Figure 9. The corrosion degree of the refractory brick wall is described by the negative layer thickness. This difference is increased when the noise is large, but the predicted value is shown to be kept stable.

Numerical simulation using conventional LMM is carried out and the results are summarized in Table 3. Table 4 shows the effectiveness of our method of inverse heat transfer problem using the HMLMM over the conventional LMM to predict wall erosion and time-varying thickness of the bank coating the inner surface of refractory brick wall of the melting furnace. Here, we analyze the results summarized in Tables (Table 2 and Table 3) and compare the solution error and CPU time for the efficiency evaluation of the HMLMM over the conventional LMM (Figure 10, Figure 11 and Table 4).

Consequently, when $\sigma = 2\%T_{max}$, the error $Error_p$, $Error_E$ and CPU time of our method are 0.56-, 0.899-, and 0.334-fold smaller than the error of the conventional LMM, and 0.03-, 0.875- and 0.293-fold lower than that of the conventional LMM when $\sigma = 4\%T_{max}$, respectively. Therefore,

we can see that our method is more effective than the conventional LMM for the numerical solution of the nonlinear inverse heat transfer problem in the melting furnace.

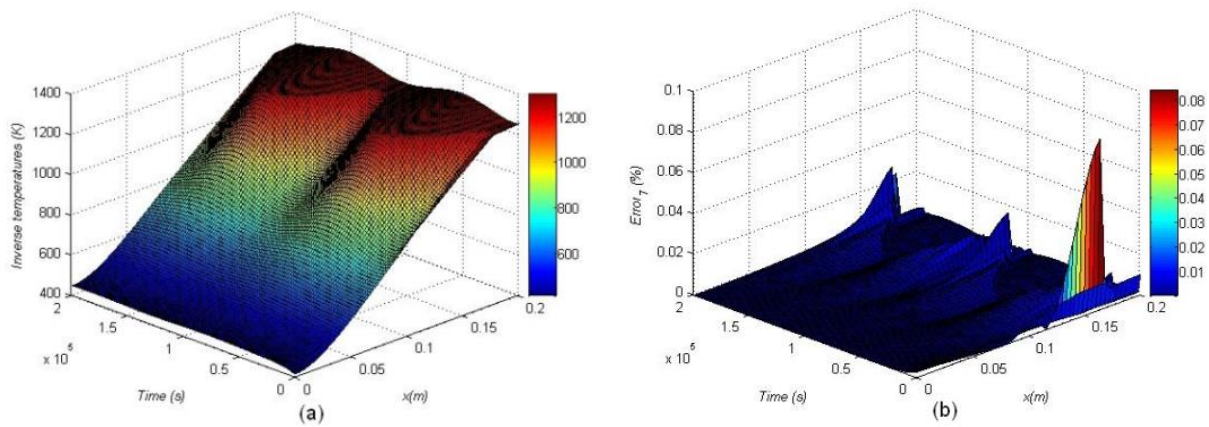
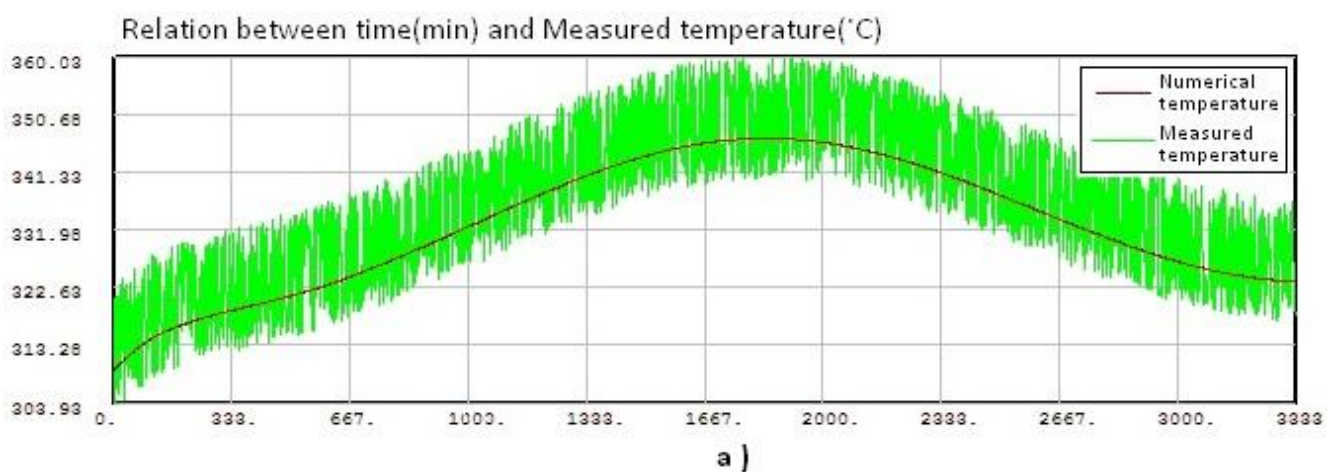


Figure 7. Inverse Predictions of the Temperature Distribution (a) and $Error_T$ (b).

Table 2. The Numerical Simulation Results of the Inverse Problem by HMLMM.

	$\sigma = 2\%T_{max}$				$\sigma = 4\%T_{max}$		
	P_{exact}	$P_{inverse}$	$Error_p$	$Error_E$	$P_{inverse}$	$Error_p$	$Error_E$
$P_1(W/m^2)$	5 000	5 000.0			5 000.0		
$P_2(W/m^2)$	4 000	4 000.0			4 000.0		
$P_3(mm)$	10	10.116	0.342	0.113	10.013 9	0.037 7	0.115 4
$E(mm)$	78	78.0			78.0		



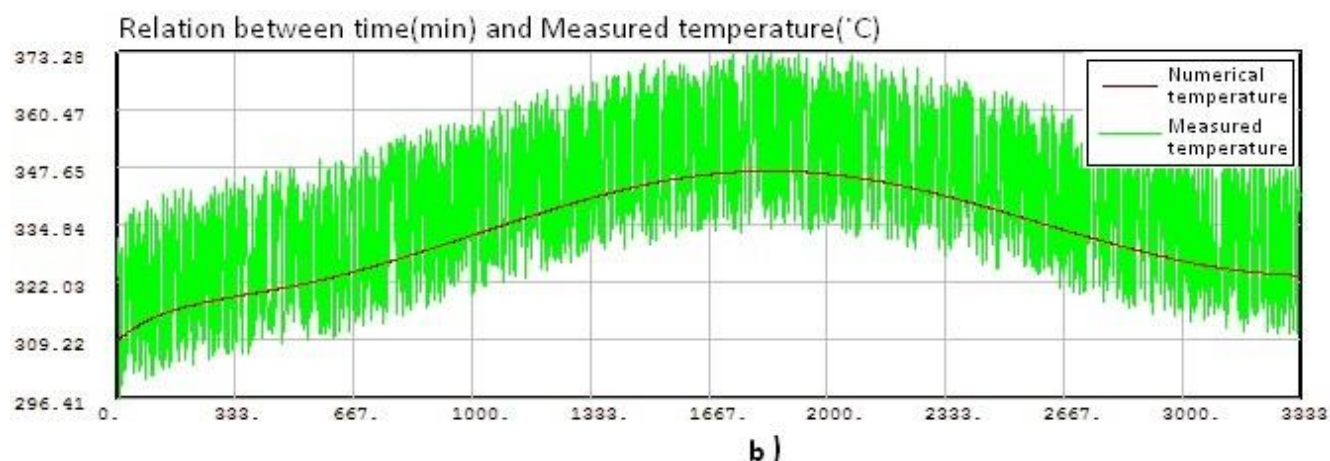


Figure 8. The Estimated Temperature Calculated by the Inverse Problem and the Measured Noise Temperature. a) $\sigma = 2\%T_{max}$, b) $\sigma = 4\%T_{max}$.

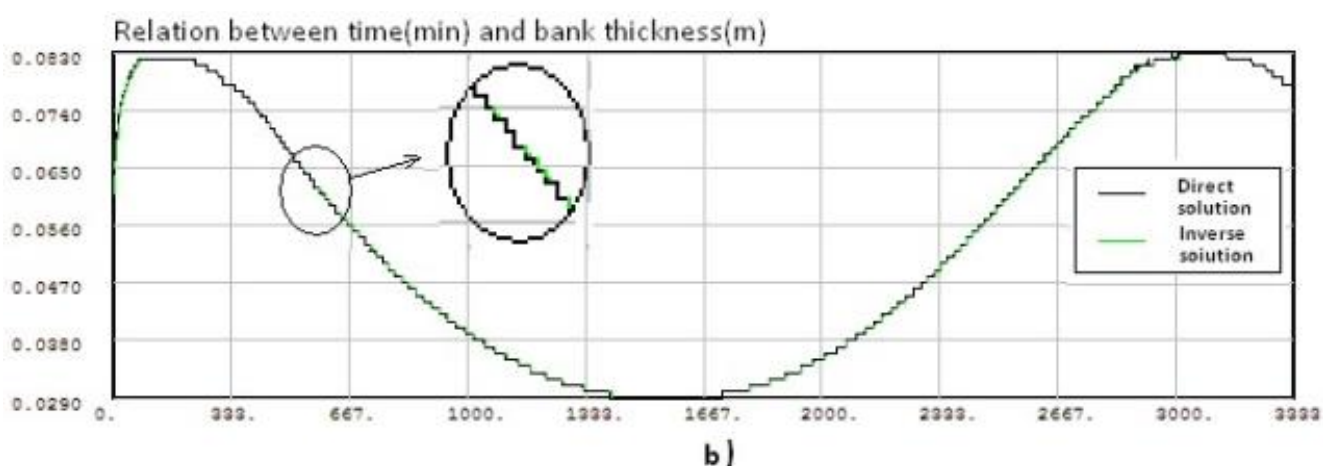
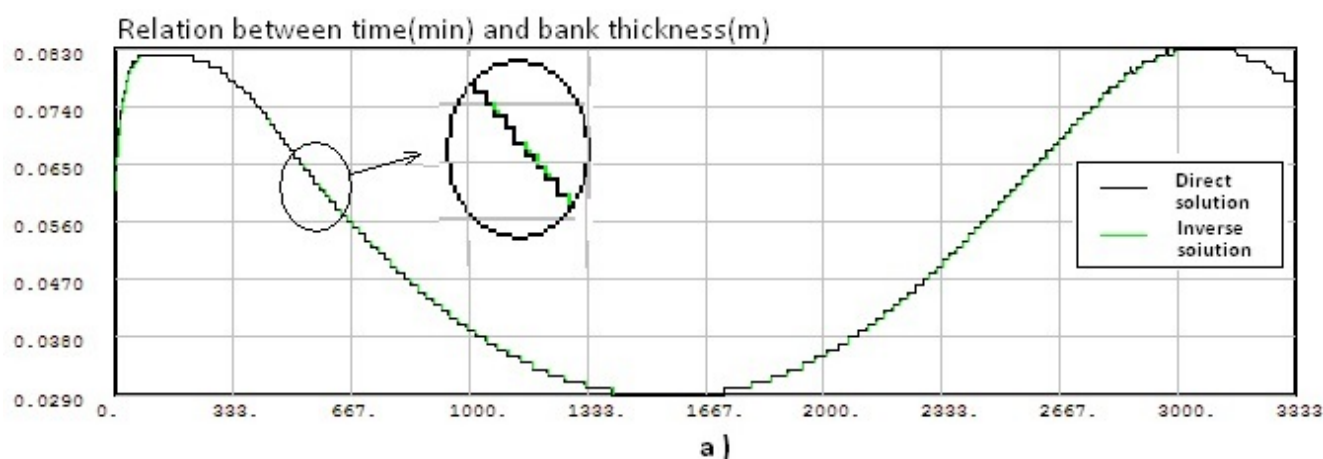


Figure 9. Predicted Time-Varying Bank Thickness. a) $\sigma = 2\%T_{max}$, b) $\sigma = 4\%T_{max}$.

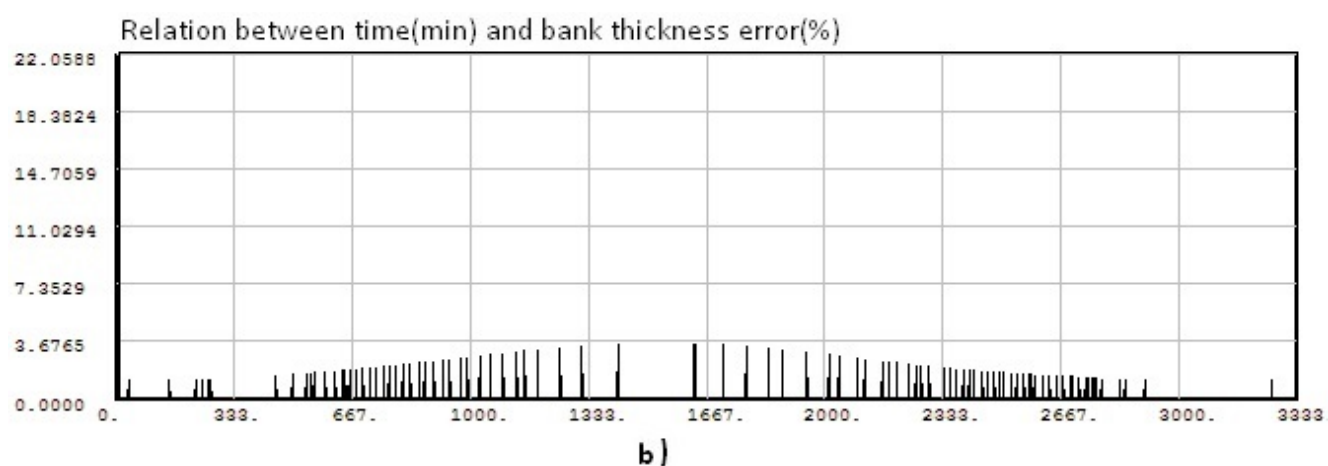
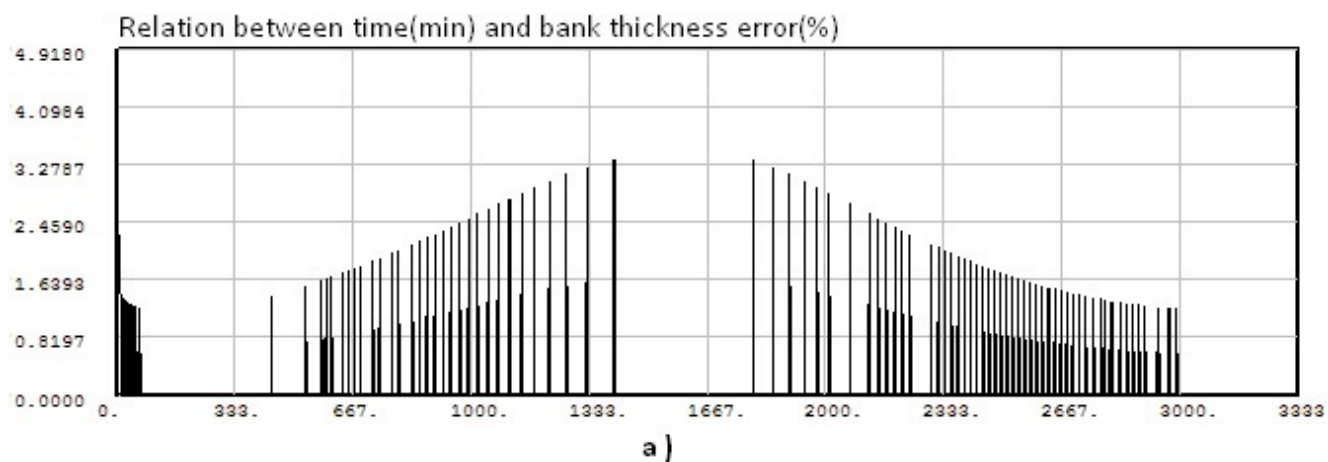
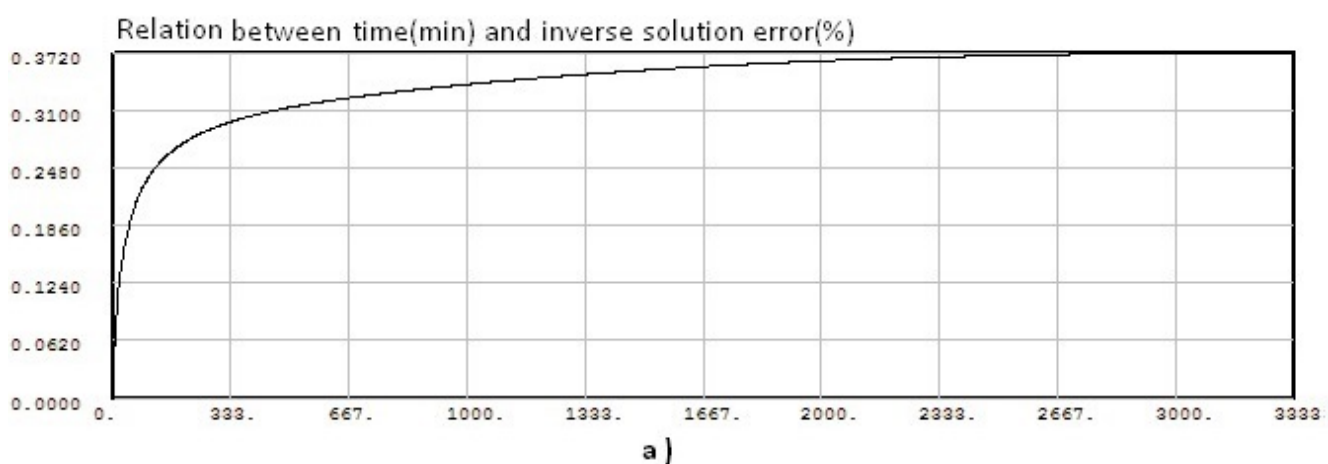


Figure 10. The Error of the Time-Varying Bank Thickness. a) HMLMM, b) LMM.



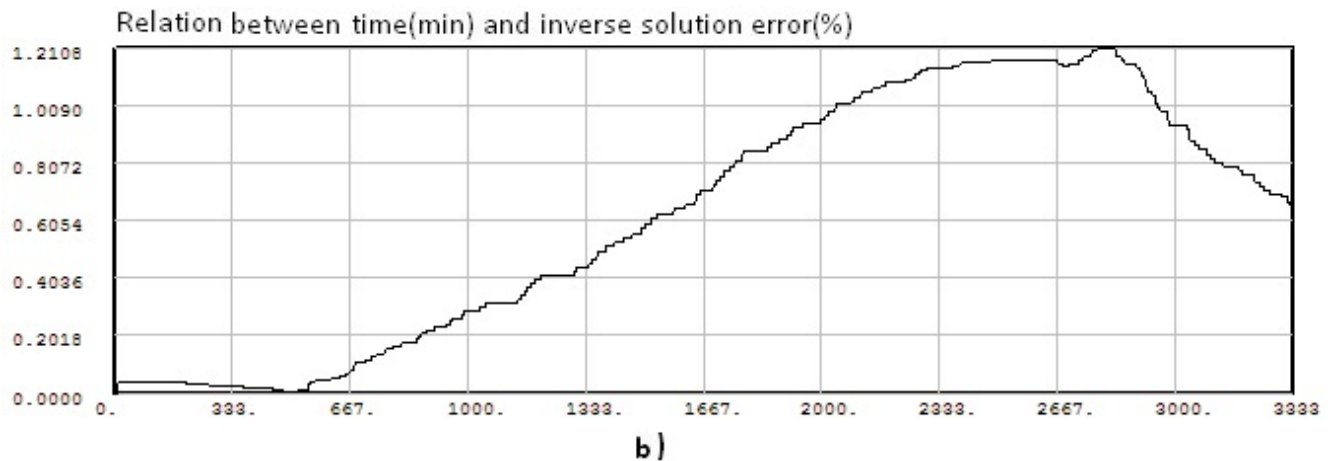


Figure 11. The Error of the Inverse Solution. a) HMLMM, b) LMM.

Table 3. The Numerical Simulation Results by LMM.

	$\sigma=2\%T_{max}$				$\sigma=4\%T_{max}$		
	P_{exact}	$P_{inverse}$	$Error_P$	$Error_E$	$P_{inverse}$	$Error_P$	$Error_E$
$P_1(W/m^2)$	5 000	5 000.967			5 002.34		
$P_2(W/m^2)$	4 000	4 000.77			4 001.87		
$P_3(mm)$	10	10.194	0.617 3	0.125 9	10.47	1.283 4	0.131 9
$E(mm)$	78	78.0			78.0		

Table 4. The Efficiency Evaluation of Our Method Using HMLMM Over the Conventional LMM.

	$\sigma = 2\%T_{max}$			$\sigma = 4\%T_{max}$		
	HMLMM	LMM	HMLMM/LMM	HMLMM	LMM	HMLMM/LMM
$Error_P(\%)$	0.342	0.617 3	0.56	0.037 7	1.283 4	0.03
$Error_E(\%)$	0.113 2	0.125 9	0.899	0.115 4	0.131 9	0.875
CPU time(s)	480	1 436	0.334	418	1 425	0.293

5. Conclusions

A numerical method of the nonlinear inverse heat transfer problem in a melting furnace was presented to predict wall erosion and time-varying thickness of the bank covering the inner surface of refractory brick wall. The direct problem is posed and solved as a non-stationary one-dimensional phase-change problem by enthalpy, assuming that the material properties of density, specific heat and thermal conductivity are temperature dependent and that the thermal contact resistance between the refractory brick wall and the slag layer is taken

into account. The inverse approach rests on the high-order modified Levenberg-Marquardt method (HMLMM) combined with the Broyden method (BM). And compared to LMM in the inverse approach, we have obtained the result that the proposed method has high accuracy of solution and stability of solution against noise and shortens the computational time. It was shown that our method of inverse heat transfer analysis using HMLMM for the selected object is more efficient than the conventional method of inverse heat transfer analysis using LMM. Future work on this method will be devoted to present more practical results of the inverse problem in the melting furnace.

Abbreviations

HMLMM	High-Order Modified Levenberg-Marquardt Method
LMM	Levenberg-Marquardt Method
BM	Broyden Method
PCM	Phase-Change Material

Author Contributions

Gwang-Ryong Yun: Conceptualization, Methodology, Writing – original draft

Guan-Gyu Ji: Software

Song-Chol Choe: Formal Analysis, Validation

Chol Ri: Writing – review & editing

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Data Availability Statement

The data is available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare no conflicts of interest.

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