Research & Development
2026, Vol. 7, No. 1, pp. 54–62
https://doi.org/10.11648/j.rd.20260701.15

Research Article

# Construction of Empirical Bayes Estimators for Truncation Parameters of Two-sided Truncated Distributions Using Size Two Samples and Their Asymptotic Optimality

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## Abstract

Bayesian methods are effective statistical one that combine prior information with samples, but the inference results are different because subjectivity is involved in representing prior information as a prior distribution. The empirical Bayes approach is a statistical method that performs continuous Bayesian decision making when the prior distribution is unknown but given a large number of past data, which overcomes the above drawback and is now a common way because of a vast amount of data available. The inference for truncation parameters is important in evaluating the lower or upper bounds of the population and is required in many fields such as reliability, meteorology, medicine, etc. Many authors have considered the case of one-sided truncated distribution in the empirical Bayes framework, but in practice we are faced with the case of two-sided one. We consider the empirical Bayes estimation for truncation parameters of two-sided truncated distribution under the squared error loss. First, Bayesian estimators of the truncation parameters are derived to minimize the Bayes risk using sample of size two. And, based on the Bayesian estimators and kernel density estimate of the population density function, empirical Bayes estimators of the truncation parameters are constructed. Next, under some assumptions asymptotic optimality of empirical Bayes estimators is proved when the sample size goes to infinity, and the convergence rate is also evaluated. It is also proved that the probability that the lower and upper bounds are reversely estimated approaches zero. Finally, an example is presented to show the validity of the assumptions and the performance of the proposed empirical Bayes estimators is evaluated through simulation on the example. New proposal of using size two samples could be extended to the case of multi-dimensional truncated distributions defined on hyper-cubic domain.

## Keywords

Asymptotic Optimality, Empirical Bayes Estimator, Squared Error Loss, Truncation Parameter, Size Two Samples

## 1. Introduction

The empirical Bayes approach that have its origin in Robbins ([13, 14]) is applicable to statistical situations when one deals with an independent sequence of Bayes decision problems each having similar structure. In these problems the sta-

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**Received:** 13 November 2025; **Accepted:** 1 December 2025; **Published:** 26 March 2026

tistical similarity are included in the assumption that prior distribution is unknown (see [7]).

Many authors have concentrated on developing empirical Bayes frameworks and their asymptotic properties (see [8]) and systematic description and bibliographical details can be seen in [3]. An empirical Bayes predictive density has been proposed to estimate the density of a future outcome from a multivariate normal model and its asymptotic optimality obtained in infinite dimensional parameter spaces by Xu and Zhou [17]. A computational framework has been proposed for empirical Bayes estimation of parameter and the convergence of the algorithm discussed by Atchade [1] and Varian [16]. An empirical Bayes procedure for adaptively selecting an experiment wise significance threshold has been proposed by Chaoyu and Hoff [4] in order to control the sign error rate. Murray et al. [12] have described a collection of web-based statistical tools that enable investigators to incorporate historical control data into conduct of randomized clinical trials using empirical Bayes method. Liang [11] has used the squared and linear exponential (LINEX) error losses for empirical Bayes estimation for the reproduction parameter of Borel-Tanner distributions. The empirical Bayes estimation of guarantee lifetime in a two parameter exponential distribution has been studied by Huang and Huang [6].

In this paper we consider a probability density function (pdf) with truncation parameter $\theta=(\theta_1, \theta_2)$ as follows.

$$f(x\mid\theta)=u(x)\varphi(\theta)I(\theta_1\le x\le\theta_2)\ ,$$

$$(\theta_1, \theta_2)\in\Theta=[a_1, b_1]\times[a_2, b_2], \tag{1}$$

$$(-\infty< a_1< b_1\le a_2< b_2< \infty),$$

where $u(x)>0$, $x\in[a_1, b_2]$ is continuous function, $\varphi(\theta_1,\theta_2)=\left(\int_{\theta_1}^{\theta_2}u(x)dx\right)^{-1}$ and $I(A)$ is indicate function of $A$. We assume that truncation parameter $\theta$ has a prior distribution function $G(\theta)$ on $\Theta$ with $G(a_1, a_2) = 0$ and $G(b_1, b_2) = 1$ and its probability density function (pdf) $g(\theta)$.

The prior distribution function $G(\theta)$ (or $g(\theta)$) is usually unknown in practice. In such situation, the Bayes estimator is not available. The empirical Bayes method is used in the case when the prior distribution is unknown but the past data are given.

The empirical Bayes procedure for the truncation parameter distributions has been considered by several authors. The empirical Bayes estimations of one-side truncation parameters and its asymptotic optimality have been studied under LINEX error loss and squared error loss respectively by Huang and Liang [5] and Yimin and Balakrishnan [18]. Li and Gupta ([9, 10]) have considered empirical Bayes tests for truncation parameters. Yimin [19] has studied the empirical Bayes estimation for the truncation parameters of two-side truncated distribution under LINEX error loss and it is one parameter estimation, because lower and upper bounds are given as $\theta$, $m\theta$ (for a known constant $m$), respectively. Such assumption restrict to the range of application. He used size one samples as most papers on empirical Bayes estimation. In order to construct empirical Bayes estimators for independent lower and upper bounds, we use size two samples.

Let $Y_1$, $Y_2$ be sample of size two from the distribution with the pdf given by Eq. (1) and $X_1= Y_1\wedge Y_2$, $X_2 = Y_1\vee Y_2$. Then the pdf of $(X_1, X_2)$ is

$$f(x_1,x_2\mid\theta_1,\theta_2)=2\prod_{i=1}^{2}f(x_i\mid\theta_1,\theta_2)$$

$$=2u(x_1)u(x_2)\varphi^2(\theta_1,\theta_2),\ \ \theta_1\le x_1\le x_2\le\theta_2 \tag{2}$$

Since the pdf of sample and parameter $(X_1, X_2, \theta_1, \theta_2)$ are $f(x_1,x_2\mid\theta_1,\theta_2)g(\theta_1,\theta_2)$, the marginal pdf of $(X_1, X_2)$ is written as

$$f(x_1,x_2)=2u(x_1)u(x_2)V(x_1,x_2),\ \ a_1< x_1\le x_2< b_2\ , \tag{3}$$

where

$$V(x_1,x_2)=\int_{a_1}^{x_1\wedge b_1}\int_{x_2\vee a_2}^{b_2}\varphi^2(\theta)g(\theta)d\theta_2 d\theta_1\ . \tag{4}$$

As a loss function, we use the squared error loss as

$$L(\delta,\theta)=\sum_{j=1}^{2}w_j(\delta_j-\theta_j)^2\ ,$$

where $\delta = (\delta_1, \delta_2)$ is estimator of parameter $\theta$ and $w_j > 0$, $j$=1, 2 are weight constants. The risk of $\delta$ with respect to $G(\theta_1, \theta_2)$ can be written as

$$R(\delta,G)=EL(\delta,\theta)=\sum_{j=1}^{2}w_j\iint f(x_1,x_2)\left[\iint_{\Theta}(\delta_j-\theta_j)^2h(\theta_1,\theta_2\mid x_1,x_2)d\theta_1 d\theta_2\right]dx_1dx_2\ ,$$

where $h(\theta_1,\theta_2\mid x_1,x_2)$ is the conditional pdf of $(\theta_1, \theta_2)$ given $(X_1, X_2) = (x_1, x_2)$. Let $\delta_B = (\delta_{1B}, \delta_{2B})$ be the Bayesian estimator of $\theta$, i.e. $R(\delta_B,G)=\inf_{\delta}R(\delta,G)$, then we have

$$\delta_{jB}(x_1,x_2)=E(\theta_j\big|x_1,x_2)=\frac{\int_{\Theta}\theta_j f(x_1,x_2\big|\theta)g(\theta)d\theta}{\int_{\Theta} f(x_1,x_2\big|\theta)g(\theta)d\theta}=\frac{\int_{a_1}^{x_1\wedge b_1}\int_{a_2\vee x_2}^{b_2}\theta_j\varphi^2(\theta)g(\theta)d\theta_2 d\theta_1}{\int_{a_1}^{x_1\wedge b_1}\int_{a_2\vee x_2}^{b_2}\varphi^2(\theta)g(\theta)d\theta_2 d\theta_1} \tag{5}$$

and

$$a_j\le\delta_{jB}(x_1,x_2)\le b_j,\ j=1,2. \tag{6}$$

If only $\theta_1$ (or $\theta_2$) is given in Eq. (1), then

$$f(x\,|\,\theta_1)=\int_{a_2}^{b_2} f(x\,|\,\theta_1,\theta_2)g(\theta_1,\theta_2)d\theta_2=u(x)\int_{a_2\vee x}^{b_2}\varphi(\theta_1,\theta_2)g(\theta_1,\theta_2)d\theta_2,\ \theta_1\le x\le b_2.$$

$$(\ f(x\,|\,\theta_2)=\int_{a_1}^{b_1} f(x\,|\,\theta_1,\theta_2)g(\theta_1,\theta_2)d\theta_1=u(x)\int_{a_1}^{b_1\wedge x}\varphi(\theta_1,\theta_2)g(\theta_1,\theta_2)d\theta_1,\ a_1\le x\le\theta_2\ )$$

These parent pdfs differ from the start models of Yimin and Balakrishnan [18].

In this paper we propose the empirical Bayes estimators for the truncation parameters of two-side truncated distribution using size two samples and investigate the asymptotic properties of the estimators. We use the squared error loss function. To our best knowledge, earlier papers discussed the empirical Bayes estimation for one-sided truncated distributions. However, in practice we often encounter with the two-sided truncated distribution.

The organization of this paper is as follows. In section 2, we demonstrate the relation between the Bayes estimator and the marginal distribution of sample and on the base of it, construct the empirical Bayes estimators. In section 3, we study the asymptotic optimality of the proposed empirical Bayes estimators and show an example. Finally, we present simulation results on the performance of the empirical Bayes estimator in section 4.

## 2. Construction of Empirical Bayes Estimators

First we give the relation between the Bayes estimator $\delta_B$ and the marginal pdf $f(x_1, x_2)$ of sample $(X_1, X_2)$.

*Lemma 2.1.* The Bayes estimator of Eq. (5) is represented by

$$\delta_{jB}(x_1,x_2)=\frac{2u(x_1)u(x_2)}{f(x_1,x_2)}\cdot\psi_j(x_1,x_2),\ \ j=1,2, \tag{7}$$

where

$$\begin{aligned}\psi_1(x_1,x_2)&=(x_1\wedge b_1)\frac{f(x_1\wedge b_1,x_2)}{2u(x_1\wedge b_1)u(x_2)}-a_1\frac{f(a_1,x_2)}{2u(a_1)u(x_2)}+\Delta_1(x_1,x_2),\\ \psi_2(x_1,x_2)&=(x_2\vee a_2)\frac{f(x_1,x_2\vee a_2)}{2u(x_1)u(x_2\vee a_2)}-b_2\frac{f(x_1,b_2)}{2u(x_1)u(b_2)}+\Delta_2(x_1,x_2)\end{aligned} \tag{8}$$

$$\Delta_1(x_1,x_2)=-\frac{1}{2u(x_2)}\int_{a_1}^{x_1\wedge b_1}\frac{1}{u(\theta_1)}f(\theta_1,x_2)d\theta_1,\ \ \Delta_2(x_1,x_2)=\frac{1}{2u(x_1)}\int_{x_2\vee a_2}^{b_2}\frac{1}{u(\theta_2)}f(x_1,\theta_2)d\theta_2, \tag{9}$$

*Proof.* From (4)

$$\frac{\partial}{\partial\theta_1}V(\theta_1,x_2)=\int_{x_2\vee a_2}^{b_2}\varphi^2(\theta)g(\theta)d\theta_2,\ a_1<\theta_1<b_1,$$
$$\frac{\partial}{\partial\theta_2}V(x_1,\theta_2)=\int_{a_1}^{x_1\wedge b_1}\varphi^2(\theta)g(\theta)d\theta_1,\ a_2<\theta_2<b_2 \tag{10}$$

Therefore using Eq. (2), Eq. (3) and the integration method by part, $\delta_{1B}(x_1, x_2)$ of (5) is derived as

$$\delta_{1B}(x_1,x_2)=\frac{2u(x_1)u(x_2)}{f(x_1,x_2)}\int_{a_1}^{x_1\wedge b_1}\theta_1(\int_{x_2\wedge a_2}^{b_2}\varphi^2(\theta)g(\theta)d\theta_2)d\theta_1=$$

$$=\frac{2u(x_1)u(x_2)}{f(x_1,x_2)}\int_{a_1}^{x_1\wedge b_1}\theta_1\frac{\partial}{\partial\theta_1}V(\theta_1,x_2)d\theta_1=\frac{2u(x_1)u(x_2)}{f(x_1,x_2)}\psi_1(x_1,x_2)$$

Similarly, we have $i$=2 case of Eq. (7).

Next, we construct the empirical Bayes estimators of the truncation parameter $\theta$.

In the empirical Bayes method, it is assumed that there are past data $(X^{(i)}, \theta^{(i)})$, $i$=1, …, $n$ and present data $(X, \theta)$, where $\theta^{(i)}$, $i$=1, …, $n$ and $\theta$ are not observable with unknown prior distribution $G(\theta)$, and $X^{(i)}$, $i$=1, …, $n$ and $X$ are observable. Here and after, we assume that $(X^{(i)}, \theta^{(i)})$, $i$=1, …, $n$ and $(X, \theta)$ are independent and identically distributed (i.i.d.) random variables.

We use a kernel density estimation method to estimate $f(x_1, x_2)$, the marginal pdf of $X = (X_1, X_2)$. Let $K(x_1, x_2)$ is a kernel function satisfying following two assumptions.

Assumption $K$. (i) $K(x_1, x_2)$ is symmetric about (0, 0) and continuously differentiable.

(ii) $\iint_{R^2}K(x_1,x_2)dx_1dx_2=1,\ \lim_{\|(x_1,x_2)\|\to\infty}\|(x_1,x_2)\|^2K(x_1,x_2)=0,$

$\iint_{R^2}\|(x_1,x_2)\|^3K(x_1,x_2)dx_1,dx_2<\infty,\ \|K(x_1,x_2)\|_\infty<\infty.$

With the past data $(X_1^{(i)},X_2^{(i)})$, $i$=1, ⋯ $n$, we use a kernel estimator of $f(x_1, x_2)$ defined as

$$f_n(x_1,x_2)=\frac{1}{nh_{1n}^2}\sum_{i=1}^{n}K\left(\frac{X_1^{(i)}-x_1}{h_{1n}},\frac{X_2^{(i)}-x_2}{h_{1n}}\right),$$

where $h_{1n}\to0$ as $n\to\infty$. Furthermore, let estimators for the functions $\Delta_j$, $\psi_j$, $\tau_j$, $j$ =1, 2 be

$$\psi_{1n}(x_1,x_2)=(x_1\wedge b_1)\frac{f_n(x_1\wedge b_1,x_2)}{2u(x_1\wedge b_1)u(x_2)}-a_1\frac{f_n(a_1,x_2)}{2u(a_1)u(x_2)}+\Delta_{1n}(x_1,x_2),$$
$$\psi_{2n}(x_1,x_2)=(x_2\vee a_2)\frac{f_n(x_1,x_2\vee a_2)}{2u(x_1)u(x_2\vee a_2)}-b_2\frac{f_n(x_1,b_2)}{u(x_1)u(b_2)}+\Delta_{2n}(x_1,x_2), \tag{11}$$

$$\Delta_{1n}(x_1,x_2)=-\frac{1}{2u(x_2)}\cdot\frac{1}{nh_{2n}^v}\sum_{i=1}^{n}\frac{1}{u(X_1^{(i)})}I(a_1<X_1^{(i)}<x_1\wedge b_1,x_2<X_2^{(i)}<x_2+h_{2n}^v),$$
$$\Delta_{2n}(x_1,x_2)=\frac{1}{2u(x_1)}\cdot\frac{1}{nh_{2n}^v}\sum_{i=1}^{n}\frac{1}{u(X_2^{(i)})}I(x_1<X_1^{(i)}<x_1+h_{2n}^v,x_2\vee a_2<X_2^{(i)}<b_2), \tag{12}$$

where $v$>0 is a constant defined later and $h_{2n}\to0$ as $n\to\infty$. The estimators $\Delta_{jn}(x_1, x_2)$ ($j$ =1, 2) of the functions $\Delta_j(x_1, x_2)$ is asymptotically unbiased as $n\to\infty$. Then estimators for the Bayes estimators $\delta_{jB}(x_1, x_2)$, $j$ =1, 2 of Eq. (7) are defined as

$$\delta_{jn}(x_1,x_2)=\left[\frac{2u(x_1)u(x_2)}{f_n(x_1,x_2)}\cdot\psi_{jn}(x_1,x_2)\right]_{h_{3n}^{-1}}, \tag{13}$$

where

$$[z]_c=\begin{cases}z,\ if\ |z|\le -c\\ \mathrm{sgn}(z)c,\ \ if\ \ |z|>c\end{cases}$$

and $h_{3n}\to0$ as $n\to\infty$. With the present data $X = (X_1, X_2)$, we compute the empirical Bayes estimator of $\theta = (\theta_1, \theta_2)$ as

$$\delta_n = (\delta_{1n}(X_1, X_2), \delta_{2n}(X_1, X_2)) \tag{14}$$

## 3. Asymptotic Optimality

In this section, we investigate asymptotic properties of the empirical estimator $\delta_n$ of Eq. (14). Since $\delta_B$ is the Bayes estimator of the truncation parameter $\theta$, the inequality $R(\delta_n, G) \ge R(\delta_B, G)$ holds. By similar manner to Huang and Liang [5], we have

$$R(\delta_n, G) - R(\delta_B, G) = \sum_{j=1}^{2} w_j E(\delta_{jn} - \delta_{jB})^2 = \sum_{i=1}^{2} w_j E[E_n(\delta_{jn}(X_1, X_2) - \delta_{jB}(X_1, X_2))^2], \tag{15}$$

where $E_n$ denotes the conditional expectation with respect to $X^{(i)}$, $i$=1, …, $n$. As usual in empirical Bayes framework, we use the nonnegative difference $J_n = R(\delta_n, G)\text{-}R(\delta_B, G)$ as criteria to evaluate performance of the empirical Bayes estimator $\delta_n$.

*Definition 3.1.* [18]. A sequence $\{\delta_n\}$ of empirical Bayes estimators is said to be asymptotically optimal as least of the order $q_n$ with respect to the prior $G$, if $J_n \le O(q_n)$, as $n\to\infty$, where $q_n > 0$ and $q_n\to\infty$ as $n\to\infty$.

Suppose that $f(x_1, x_2)$ is twice differentiable and $\left\| f^{(l)} \right\|_\infty < \infty$, $l$=0, 1, 2, where $f^{(l)}$ denotes any partial derivative of order $l$. Let $h_{1n} = d_1 n^{-\frac{1}{6}}$, then from theorem 3 in Bosq [2].

$$E_n[|f_n(x_1, x_2) - f(x_1, x_2)|^2] \le O(n^{-\frac{2}{3}}) \cdot$$

Using Hölder inequality, for any $\gamma\in(0, 2]$.

$$E_n[|f_n(x_1, x_2) - f(x_1, x_2)|^\gamma] \le O(n^{-\frac{\gamma}{3}}) \cdot \tag{16}$$

We give some lemmas used in proof of the asymptotic optimality of the sequence $\{\delta_n\}$ of the empirical Bayes estimator $\delta_n$ of Eq. (14).

*Lemma 3.1.* If $h_{2n} = d_2 n^{-\frac{1}{6}}$ and $v$=2, we have

$$\mathrm{E}_n[|\Delta_{jn}(x_1, x_2) - \Delta_j(x_1, x_2)|^2] \le O(n^{-\frac{2}{3}}),\ j = 1, 2 \cdot \tag{17}$$

*Proof.* We prove $j$=1 case. $j$ =2 case is proved similarly. From Eq. (9) and Eq. (12).

$$|E_n\Delta_{1n}(x_1, x_2) - \Delta_1(x_1, x_2)| = \frac{1}{2u(x_2)} \left| \frac{1}{h_{2n}^v} \int_{a_1}^{x_1\wedge b_1} \int_{x_2}^{x_2+h_{2n}^v} \frac{1}{u(\theta_1)} (f(\theta_1, \theta_2) - f(\theta_1, x_2)) d\theta_2 d\theta_1 \right|$$

$$\le \frac{\left\| f^{(1)} \right\|_\infty}{2u(x_2) h_{2n}^v} \frac{h_{2n}^{2v}}{2} \int_{a_1}^{b_1} \frac{d\theta_1}{u(\theta_1)} \le C h_{2n}^v.$$

Let denote by $Var_n$ the variance with respect to $\{X^{(i)}, i=1, \dots, n\}$, then

$$Var_n(\Delta_{1n}(x_1, x_2) - \Delta_1(x_1, x_2)) = Var_n \Delta_{1n}(x_1, x_2) =$$

$$\le \frac{1}{4u^2(x_2)} \frac{1}{n h_{2n}^{2v}} E_n\left( \frac{1}{u^2(X_1^{(i)})} I(a_1 < X_1^{(i)} < x_1 \wedge b_1, x_2 < X_2^{(i)} < x_2 + h_{2n}^v) \right)$$

$$= \frac{1}{4u^2(x_2)} \frac{1}{n h_{2n}^{2v}} \int_{a_1}^{x_1\wedge b_1} \int_{x_2}^{x_2+h_{2n}^v} \frac{f(\theta_1, \theta_2)}{u^2(\theta_1)} d\theta_2 d\theta_1 \le \frac{\left\| f \right\|_\infty}{4u^2(x_2) n h_{2n}^{2v}} h_{2n}^v \int_{a_1}^{b_1} \frac{d\theta_1}{u^2(\theta_1)_1} = O(n^{-(1-\frac{v}{6})}).$$

Therefore

$$E_n|\Delta_{1n}(x_1, x_2) - \Delta_1(x_1, x_2)|^2 \le C(n^{-(1-\frac{v}{6})} + n^{-\frac{v}{3}}) = O(n^{-\frac{2}{3}}) \cdot$$

*Lemma 3.2.* Under the condition of lemma 3.1, if $\gamma\in$ (0, 2], then

$$E_n[|\psi_{jn}(x_1, x_2) - \psi_j(x_1, x_2)|^\gamma] \le O(n^{-\frac{\gamma}{3}}),\ j = 1, 2 \cdot \tag{18}$$

*Proof.* From Eq. (8), Eq. (12) and Eq. (17), we have

$$\mathrm{E}_n[|\psi_{jn}(x_1, x_2) - \psi_j(x_1, x_2)|^2] \le O(n^{-\frac{2}{3}}) \cdot$$

Therefore by Hölder inequality,

$$\mathrm{E}_n[|\psi_{jn}(x_1, x_2) - \psi_j(x_1, x_2)|^\gamma] \le O(n^{-\frac{\gamma}{3}}) \cdot$$

The following lemma is from [15].

*Lemma 3.3.* Let $y$, $z\neq0$ and $h>0$ be real numbers and $Y$, $Z$ be two real valued random variables. Then for any $\gamma\in(0, 2]$,

$$\begin{aligned}&E([y/z-Y/Z]_h)^\gamma\\&\le |z|^{-\gamma}[E|y-Y|^\gamma+(|y/z|+h)^\gamma E|z-Z|^\gamma]\end{aligned} \quad (19)$$

Next, we give the result on the asymptotic optimality of the sequence $\{\delta_n\}$ of the empirical Bayes estimators $\delta_n$.

*Theorem 3.1.* Suppose that $f(x_1, x_2)$ is twice differentiable and $\|f^{(l)}\|_\infty<\infty$, $l$=0, 1, 2 and the assumption $K$ holds. If $u(x)>0$, $a_1\le x\le b_2$ is continuous and for any $\gamma\in(0, 2]$,

$$\int_{a_1}^{b_2}\int_{x_1}^{b_2} f(x_1,x_2)^{1-\gamma}dx_2dx_1<\infty, \quad (20)$$

then with the choice of $h_{kn}=d_kn^{-\frac{1}{6}}, k=1,2,3$ and $v$=2, we have

$$J_n=R(\delta_n,G)-R(\delta_B,G)\le O(n^{-\frac{\gamma-1}{3}}).$$

*Proof.* We consider right hand of Eq. (15). From Eq. (6), we can see that $|\delta_{jB}|\le h_{3n}^{-1}$, for sufficiently great $n$. Using Eq. (16), Eq. (18) and Eq. (19), for $j$=1, 2.

$$E_n(\delta_{jn}(X_1,X_2)-\delta_{jB}(X_1,X_2))^2\le E_n\left(\left[\frac{2u(X_1)u(X_2)}{f_n(X_1,X_2)}\psi_{jn}(X_1,X_2)-\frac{2u(X_1)u(X_2)}{f(X_1,X_2)}\psi_j(X_1,X_2)\right]_{2h_{3n}^{-1}}\right)^2$$

$$\le(2h_{3n}^{-1})^{2-\gamma}E_n\left|\left[\frac{2u(X_1)u(X_2)}{f_n(X_1,X_2)}\psi_{jn}(X_1,X_2)-\frac{2u(X_1)u(X_2)}{f(X_1,X_2)}\psi_j(X_1,X_2)\right]_{2h_{3n}^{-1}}\right|^\gamma$$

$$\le(2h_{3n}^{\gamma-2})\frac{1}{|f(X_1,X_2)|^\gamma}\Big(E_n|f_n(X_1,X_2)-f(X_1,X_2)|^\gamma+2(h_{3n}^{-1}+2h_{3n}^{-1})^\gamma u(X_1)u(X_2)\times$$

$$\times E_n|\psi_{jn}(X_1,X_2)-\psi_j(X_1,X_2)|^\gamma\Big)\le\frac{C}{f(X_1,X_2)^\gamma}h_{3n}^{-2}\cdot n^{-\frac{\gamma}{3}}=\frac{C}{f(X_1,X_2)^\gamma}O(n^{-\frac{\gamma-1}{3}}) \quad (21)$$

Therefore

$$R(\delta_n,G)-R(\delta_B,G)\le\sum_{j=1}^{2}w_jE\left(\frac{1}{f(X_1,X_2)^\gamma}\right)O(n^{-\frac{\gamma-1}{3}})=O(n^{-\frac{\gamma-1}{3}}).$$

The estimator of a lower truncation parameter may be larger than the estimator of the upper one, because of randomness of samples. Therefore we evaluate the probability $p_{on}=P\{\delta_{1n}>\delta_{2n}\}$ of reversely estimating.

*Corollary 3.1.* Under the assumptions of theorem 3.1, we have

$$p_{on}=P\{\delta_{1n}(X_1,X_2)>\delta_{2n}(X_1,X_2)\}\le O(n^{-\frac{\gamma-1}{3}})$$

*Proof.* Using Eq. (6), Eq. (21) and Chebyshev inequality,

$$\begin{aligned}&P\{\delta_{1n}(X_1,X_2)-\delta_{2n}(X_1,X_2)>0\}\\&=P\{(\delta_{1n}-\delta_{1B})-(\delta_{2n}-\delta_{2B})>\delta_{1B}-\delta_{2B}\}\end{aligned}$$

$$\le P\{|\delta_{1n}-\delta_{1B}|>\delta_{1B}-\delta_{2B}\}+P\{|\delta_{2n}-\delta_{2B}|>\delta_{1B}-\delta_{2B}\}$$

$$\le P\{|\delta_{1n}-\delta_{1B}|>a_2-b_1\}+P\{|\delta_{2n}-\delta_{2B}|>a_2-b_1\}$$

$$\le(a_2-b_1)^{-2}\sum_{j=1}^{2}EE_n(\delta_{jn}-\delta_{jB})^2\le O(n^{-\frac{\gamma-1}{3}})$$

*Example 3.1.* We consider a two-sided truncated distribution with the pdf as

$$f(x|\theta_1,\theta_2)=e^{-x}\varphi(\theta_1,\theta_2),\ \theta_1\le x\le\theta_2, \quad (22)$$

where $(\theta_1, \theta_2)\in\Theta=[a_1, b_1]\times[a_2, b_2]$ and $0\le a_1<b_1\le a_2<b_2<\infty$. Then $\varphi(\theta_1,\theta_2)=(e^{-\theta_1}-e^{-\theta_2})^{-1}$, and $0<C_1\le\varphi^2(\theta_1,\theta_2)\le C_2$. Let $G(\theta_1, \theta_2)$ is a prior distribution with the pdf given by

$$g(\theta_1,\theta_2) = g_0(\theta_1-a_1)^{\lambda_1}(b_1-\theta_1)^{\lambda_2}(\theta_2-a_2)^{\lambda_3}(b_2-\theta_2)^{\lambda_4},\ (\theta_1,\theta_2)\in\Theta \tag{23}$$

where $\lambda_k > 1$, $k=1,\ldots,4$ are constants and $g_0>0$ is regular constant.

Let divide the sample space $D=\{(x_1,x_2)\,|\,a_1<x_1<x_2<b_2\}$ by

$$D_1=\{(x_1,x_2)\,|\,a_1<x_1<b_1,\ a_2\le x_2<b_2\}$$

$$D_2=\{(x_1,x_2)\,|\,x_1\ge b_1,\ a_2\le x_2<b_2,\ x_1<x_2\}$$

$$D_3=\{(x_1,x_2)\,|\,a_1<x_1<b_1,\ x_2<a_2,\ x_1<x_2\}$$

$$D_4=\{(x_1,x_2)\,|\,x_1\ge b_1,\ x_2<a_2,\ x_1<x_2\}$$

and denote by $D_i^0$ the interior of $D_i$. We can see that for the function $V(x_1, x_2)$ of Eq. (4), it holds

$$V(x_1,x_2)\le C\times\begin{cases}(x_1-a_1)^{\lambda_1+1}\cdot(b_2-x_2)^{\lambda_4+1}, & (x_1,x_2)\in D_1^0\\ (b_2-x_2)^{\lambda_4+1}, & (x_1,x_2)\in D_2^0\\ (x_1-a_1)^{\lambda_1+1}, & (x_1,x_2)\in D_3^0\\ 1, & (x_1,x_2)\in D_4^0\end{cases}$$

where $C>0$ is a constant related with $g_0$, $a_i$, $b_i$, $i=1, 2$ and $\lambda_k$, $k=1,\ldots,4$. From Eq. (3), we have

$$f(x_1,x_2)^{1-\gamma}\le C_1\times\begin{cases}(x_1-a_1)^{(\lambda_1+1)(1-\gamma)}(b_2-x_2)^{(\lambda_4+1)(1-\gamma)}, & (x_1,x_2)\in D_1^0\\ (b_2-x_2)^{(\lambda_4+1)(1-\gamma)}, & (x_1,x_2)\in D_2^0\\ (x_1-a_1)^{(\lambda_1+1)(1-\gamma)}, & (x_1,x_2)\in D_3^0\\ 1, & (x_1,x_2)\in D_4^0\end{cases},$$

If $(\lambda_k+1)(1-\gamma)>-1$ for $k=1, 4$, then Eq. (20) is satisfied. On the other hand, from Eq. (4) and Eq. (10), if $\lambda_k>1$, $k=1,\ldots,4$, then $\|V^{(l)}\|_\infty<\infty$, $l=0, 1, 2$, furthermore $\|f^{(l)}\|_\infty<\infty$, $l=0, 1, 2$. Therefore we can see that if

$$\begin{cases}(\lambda_k+1)(1-\gamma)>-1,\ k=1,4\\ \lambda_k\ge 1,\ k=1,\cdots,4\end{cases} \tag{24}$$

then the result in theorem 3.1 holds.

## 4. Simulation Study

Finally, we describe the Monte-Carlo Simulation on the performance of the empirical Bayes estimators.

In Eq. (22) and Eq. (23), we set $(a_1, b_1)=(0, 1/5)$, $(a_2, b_2)=(4/5, 1)$ and $\lambda_k = 2$, $k=1,\ldots,4$. In this setting, Eq. (24) is satisfied for $\gamma\in(0, 4/3)$. We take $d_1=5$, $d_2=1/5$, $d_3=0.35$ and $w_1= w_2=1$.

The simulation study is carried as follows. First, we iterate following steps (i) -(ii) 500 times. In the $k$th ($k=1,\ldots,500$) iteration:

(i) For $i = 1,\ldots,n+1$, we generate $\theta_1^{(i)}$, $\theta_2^{(i)}$ from the pdf of Eq. (23) and generate two samples $y_1^{(i)}, y_2^{(i)}$ from the pdf of Eq. (22). Then we set $x_1^{(i)}=y_1^{(i)}\wedge y_2^{(i)}$, $x_2^{(i)}=y_1^{(i)}\vee y_2^{(i)}$, $i=1,\ldots,n$ (past data) and $x_1=y_1^{(n+1)}\wedge y_2^{(n+1)}$, $x_2=y_1^{(n+1)}\vee y_2^{(n+1)}$ (present data).

(ii) The Bayes estimators $\delta_{jB}(x_1, x_2)$, $j=1, 2$, the empirical Bayes estimators $\delta_{jn}(x_1, x_2)$, $j=1, 2$ and

$$J_{nk}=\sum_{j=1}^{2}(\delta_{jn}(x_1,x_2)-\delta_{jB}(x_1,x_2))^2,$$
$$r_{nk}=I(\delta_{1n}(x_1,x_2)>\delta_{2n}(x_1,x_2))$$

are computed.

Next, we estimate $J_n$ and $p_{on}$ by sample means respectively as

$$\hat{J}_n=500^{-1}\sum_{k=1}^{500}J_{nk},\quad \hat{p}_{on}=500^{-1}\sum_{k=1}^{500}r_{nk}$$

The variances of above sample means are estimated respectively by

$$SE^2(\hat{J}_n)=(500\times 599)^{-1}\sum_{k=1}^{1000}(J_{nk}-\hat{J}_n)^2,\ SE^2(\hat{p}_{on})=500^{-1}\hat{p}_{on}(1-\hat{p}_{on})$$

The values of $\hat{J}_n$, $\hat{p}_{on}$, $SE(\hat{J}_n)$ and $SE(\hat{p}_{on})$ for $n \in M = \{10, 50, 100, 200, 300, 500, 700, 1,000, 1,500, 2,000\}$ are presented in Table 1. Finally, solving the minimization problem.

$$\min_{(C,\beta)} \sum_{n\in M} n(\hat{J}_{nk} - Cn^{-\beta})^2$$

gives β = 0.08452 and C = 0.07360. The values of $\hat{J}_n/(0.07360n^{-0.08452})$ are filled in 4th column of Table 1 and all they are close to 1. Numerical computations are carried by Matlab.

*Table 1. The performances of Estimators Via Estimates of $J_n$ and $p_{on}$ According to Different n.*

| n | $\hat{J}_n \times 10^2$ | $SE(\hat{J}_n)\times 10^2$ | $\hat{J}_n/(0.07360n^{-0.08452})$ | $\hat{p}_{on}\times 10^2$ | $SE(\hat{p}_{on})\times 10^2$ |
|---|---|---|---|---|---|
| 10 | 12.0427 | 23.029 | 1.7591 | 5.8 | 0.7392 |
| 20 | 7.5427 | 5.3255 | 1.1907 | 1.5 | 0.3844 |
| 30 | 6.7558 | 4.9921 | 1.116 | 1.4 | 0.3715 |
| 40 | 6.5257 | 4.9932 | 1.1133 | 1.6 | 0.3968 |
| 50 | 5.5564 | 4.44 | 0.972 | 0.6 | 0.2442 |
| 70 | 5.1461 | 4.692 | 0.9348 | 0.2 | 0.1413 |
| 100 | 4.9181 | 4.5479 | 0.9297 | 0.8 | 0.2817 |
| 150 | 4.3617 | 3.9638 | 0.8629 | 0.2 | 0.1413 |
| 200 | 4.3447 | 4.2461 | 0.8876 | 0.3 | 0.1729 |
| 300 | 4.4198 | 4.1612 | 0.9449 | 0.1 | 0.0999 |
| 600 | 4.365 | 3.8058 | 1.0086 | 0 | 0 |
| 1000 | 4.3067 | 3.8096 | 1.0537 | 0 | 0 |

Table 1 shows that $\hat{J}_n$ and variances of it tend to zero, as $n\to\infty$, respectively. We can also see that $\hat{p}_{on}$ and variances of it tend to zero, as $n\to\infty$, respectively. Therefore, $J_n$, the difference between empirical risk and theoretical risk, and $p_{on}$, the probability of reversely estimating, tend to zero, as $n\to\infty$, respectively. However, the speeds of the convergences are relatively slow as usual in empirical Bayes inferences.

## 5. Conclusion

The empirical Bayes estimation for truncation parameters of two-sided truncated distribution has been considered. We theoretically confirmed that by a sample of size one, the Bayesian estimators for two truncation parameters cannot be constructed. This is the motivation for using size two samples. Empirical Bayes estimators for the truncation parameters under squared error loss have been constructed and their asymptotic optimality have been proved. It has been also proved that the probability that the lower and upper bounds are reversely estimated approaches zero. An example was presented to verify applicability of the proposed estimation scheme to real problems. The simulation result confirms the asymptotic optimality of the proposed empirical Bayes estimators. New proposal of using size two samples could be extended to the case of multi-dimensional truncated distributions defined on hypercubic domain. The empirical Bayes inference using samples of size larger than one would be effective in many statistical problems.

## Abbreviations

| | |
|---|---|
| i.i.d. | Independent and Identically Distributed |
| LINEX | Linear Exponential |

## Acknowledgments

The authors would like to thank Alina Carlson, Rachel Nolan and the reviewers for their great efforts and valuable comments. Sung-Hyon Ri would like to express deeply gratitude to Prof. Erich Novak for providing selfless help for the publication of his first paper.

## Conflicts of Interest

The authors declare no conflicts of interest.

## References

[1] Atchade, Y. F., (2011). A computational framework for empirical Bayes inference. Stat Comput., 21: 463-473, https://doi.org/10.1007/s11222-010-9182-3

[2] Bosq, D., (1995). Optimal asymptotic quadratic error of density estimators for strong mixing or chaotic data. Statist. Probab. Lett., 22, 339-347.

[3] Carlin, B. P., Louis, T. A., (1996). Bayes and Empirical Bayes Methods for Data Analysis. Chapman & Hall, London.

[4] Chaoyu, Y., Hoff P. D., (2019). Adaptive sign error control. Journal of Statistical Planning and Inference, 201, 133-145.

[5] Huang, S. Y., Liang, T. C., (1997). Empirical Bayes estimation of the truncation parameter with linex loss. Statist. Sinica, 7, 755-769.

[6] Huang, W. T, Huang, H. H., (2006). Empirical Bayes estimation of guarantee lifetime in a two-parameter exponential distribution. Statist. Probab. Lett., 76, 1821-1829.

[7] Karunamuni, R. J., (1988). On empirical Bayes testing with sequential components. Ann. Statist., 16, 1270-1282.

[8] Karunamuni, R. J., Li J., Wu J., (2010). Robust empirical Bayes tests for continuous distributions. Journal of Statistical Planning and Inference, 140, 268-282.

[9] Li, J., Gupta, S. S., (2001). Monotone empirical Bayes tests with optimal rate of convergence for a truncation parameter. Statist. Decisions, 19, 223-237.

[10] Li, J., Gupta, S. S., (2003). Optimal rate of empirical Bayes tests for lower truncation parameters. Statist. Probab. Lett., 65, 177-185.

[11] Liang, T. C., (2009). Empirical Bayes estimation for Borel-Tanner distributions. Statist. Probab. Lett., 79, 2212-2219.

[12] Murray, T. A., Hobbs, B. P., Lystig, T. C., Carlin, B. P., (2014). Semiparametric Bayesian commensurate survival model for post-market medical device surveillance with non-exchangeable historical data. Biometrics, 70(1), 185-191.

[13] Robbins, H., (1956). An empirical Bayes approach to statistics. Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, vol. 1. University of California Press, Berkeley, CA, USA, pp. 157-163.

[14] Robbins, H., (1964). The empirical Bayes approach to statistical decision problems. Ann. Math. Statist., 35, 1-20.

[15] Singh, R. S., (1977). Applications of estimators of a density and its derivatives to certain statistical problems. J. Roy. Statist. Soc. Ser. B, 39, 357-363.

[16] Varian, H. R., (1975). A Bayesian approach to real estimate assessment, In: Feinberg, S. E., Zellner, A. (Eds.), Studies in Bayesian Econometrics and Statistics in honor of L. J. Savage. North-Holland, Amsterdam, 195-208.

[17] Xu, X., Zhou, D., (2011). Empirical Bayes predictive densities for high-dimensional normal models. Journal of Multivariate Analysis, 102, 1417-1428.

[18] Yimin, M., Balakrishnan, N., (2000). Empirical Bayes estimation for truncation parameters. J. Statist. Plann. Inference, 84, 111-120.

[19] Yimin, S., (2000). Empirical Bayes estimate for the perameter of two-side truncated distribution families. Appl. Math. J. Chinese Univ. Ser A, 15(4), 475-483.