Pure and Applied Mathematics Journal
2026; 15(2): 11-17
http://www.sciencepublishinggroup.com/j/pamj
doi: 10.11648/j.pamj.20261502.11

Symmetry of Solutions of Integral Equation in the Heisenberg Group

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To cite this article:
Zhaobing Cui, Wei Shi. (2026). Symmetry of Solutions of Integral Equation in the Heisenberg Group. *Pure and Applied Mathematics Journal, 15*(2), 11-17. https://doi.org/10.11648/j.pamj.20261502.11

Received: 11 February 2026 ; **Accepted:** 25 February 2026 ; **Published:** 18 March 2026

Abstract: This paper investigates the existence and symmetry properties of solutions to a class of integral equations on the Heisenberg group. Building upon the moving plane method and Hardy-Littlewood-Sobolev type inequalities, we establish symmetry and monotonicity results for positive solutions of the integral equation. This paper extends classical Euclidean results to the Heisenberg group, highlighting profound interactions between geometry and analysis.

Keywords: Integral Equations, Symmetry, Heisenberg Group

1. Introduction

The study of symmetry and classification of solutions to nonlinear integral equations and partial differential equations constitutes a fundamental theme in geometric analysis and mathematical physics. In Euclidean spaces, powerful techniques such as the method of moving planes, combined with integral inequalities of Hardy-Littlewood-Sobolev type, have led to a deep understanding of the radial symmetry and monotonicity properties of positive solutions for a wide range of problems. Seminal works in this direction include those of [1], who established the radial symmetry of positive solutions to polyharmonic equations, and numerous subsequent investigations into systems and equations with critical exponents.

Beyond the Euclidean setting, analysis on noncommutative nilpotent Lie groups, particularly the Heisenberg group $\mathbb{H}^n$, has attracted considerable interest. The Heisenberg group serves as a primary model for a sub-Riemannian geometry and appears naturally in several contexts, including quantum mechanics, complex analysis, and the theory of several complex variables. The inherent non-commutativity and the presence of a non-holonomic distribution of vector fields introduce distinctive analytical challenges, making the study of differential and integral equations on $\mathbb{H}^n$, a naturally degenerate elliptic operator, replaces the classical Laplacian, and the corresponding integral equations often involve kernels adapted to the homogeneous geometry, such as $|\eta^{-1} \circ \xi|_H^{-(Q-\alpha)}$, where $Q = 2n + 2$ is the homogeneous dimension.

Recently, there has been growing activity in extending symmetry and Liouville-type results to the Heisenberg setting. For instance, Prajapat and Varghese [2] employed the moving plane method to prove symmetry for non-negative solutions of an integral equation associated with the Hardy-Littlewood-Sobolev inequality on $\mathbb{H}^n$. Chen and Yang [3] investigated the existence of positive solutions for weighted nonlinear integral equations. Further studies, such as those by Tordecilla and Aguila [4] on quasilinear Schrödinger equations and Pucci and Temperini [5] on critical p-Laplacian equations, underscore the active development of the theory for nonlinear problems on the Heisenberg group. However, many questions remain open, especially concerning equations with nonlinearities of non-power type or those lacking standard variational structures. In recent years, there has been a growing research interest in the classification and symmetry of solutions to integral equations in Euclidean spaces, for example, Lu, Lam and Bao [1] applied the moving plane method and Hardy-Littlewood-Sobolev type inequalities to prove that the positive solutions of the polyharmonic equation in Euclidean space $\mathbb{R}^{2m}$ are radially symmetric and monotonically decreasing.

Wang and Yong [6] studied the comparison theorems for backward stochastic Volterra integral equations (BSVIE) in multi-dimensional Euclidean space and gave the types of BSVIE addressed in their study that can be handled as follows:

$$Y(t) = \varphi(t) + \sum_{j=1}^{m} \int_0^t h(t,s,Y(s)) + C(t)Z_j(s,t)dW(s) + \int_0^t g(t,s,Y(s))ds \tag{1}$$

The research by Li and Wang [7] focuses on a system of integral equations in the positive half-space of n-dimensional Euclidean space. Using the integral method of the moving plane, they proved that the system has no positive solution. Furthermore, they demonstrated the equivalence between this system of integral equations and the corresponding partial differential equations. In the previous literature, solutions to integral equations in the Heisenberg group have been rarely studied. We recommend readers to consult to [2] for integral equations and symmetries of non-negative solutions on the Heisenberg group. Prajapat and Varghese [2] studied the integral equation and the symmetry of non-negative solutions in the Heisenberg group $\mathbb{H}^n$. Specifically, they considered the following integral equation given by:

$$u(\zeta) = \int_{\mathbb{H}^n} |\zeta^{-1}\xi|^{-(Q-\alpha)} u(\xi)^p \, d\xi, \tag{2}$$

where $1 < p \leq \frac{Q+\alpha}{Q-\alpha}, 0 < \alpha < Q$, they used the moving plane method and the Hardy-Littlewood-Sobolev inequality to prove the symmetry of the solutions. The specific form of the solution is expressed as follows:

$$u_0(z,t) = \left((1+|z|^2)^2 + t^2\right)^{-\frac{Q-\alpha}{4}} \quad (z,t) \in \mathbb{H}^n. \tag{3}$$

Additionally, they provided the symmetry and classification of nonnegative C^2 solutions to the equation

$$\Delta_{\mathbb{H}} u + u^p = 0 \quad \text{for } 1 < p \leq \frac{Q+\alpha}{Q-\alpha} \text{ in } \mathbb{H}^n, \tag{4}$$

without assuming any of partial symmetry on the solution u.

Chen and Yang [3] studied the existence and the non-existence of positive solutions for weighted nonlinear integral equations. These equations are related to the Hardy-Littlewood-Sobolev inequality. The specific form of the integral equation is given as follows:

$$f^{q-1}(\xi) = \int_\Omega \frac{G(\xi)f(\eta)G(\eta)}{|\eta^{-1}\xi|^{Q-\alpha}} \, d\eta + \lambda \int_\Omega \frac{f(\eta)}{|\eta^{-1}\xi|^{Q-\alpha-\beta}} \, d\eta, \tag{5}$$

$\xi \in \bar{\Omega}$.

The structure of the solution is closely related to the parameters q, α, β, and the dimension of the Heisenberg group. This research provides an important theoretical foundation for understanding the application of weighted nonlinear equations in geometric analysis.

Tordecilla and Aguila [4] studied the existence results of a quasilinear Schrödinger equation in the Heisenberg group. They based their approach on a suitable change of variables, which transforms the original problem into an equivalent semilinear one. Then, using an approximation scheme and a variation of the fixed point theorem, they presented the positive solutions to the semilinear equation. For more articles on integral equations on the Heisenberg group, please see [8–16] and the references therein.

In this paper, we focus on a class of integral equations on the Heisenberg group of the form

$$u(\xi) = h(\xi) - \int_{\mathbb{H}^n} |\eta^{-1} \circ \xi|_H^{-(Q-\alpha)} u(\eta)^\tau d\eta, \tag{6}$$

where $0 \leq \alpha < Q$, $\tau < 0$ is a negative exponent, $h(\xi)$ is a given positive smooth function. The case $\tau < 0$ introduces a singular nonlinearity which arises in various physical and geometric contexts, and it significantly alters the analytical features of the problem compared to the more standard case $\tau > 0$. A primary difficulty in establishing symmetry results for such equations on unbounded domains like $\mathbb{H}^n$ is the need to control the behavior of solutions at infinity, which is often not a priori prescribed.

Our main contribution is twofold. First, we establish a symmetry result for positive solutions of the above integral equation, assuming the external term h is symmetric with respect to a hyperplane. The proof adapts the method of moving planes to the sub-Riemannian geometry of $\mathbb{H}^n$, relying crucially on the explicit form of the kernel, the group structure, and the associated Hardy-Littlewood-Sobolev inequality. Second, and more innovatively, to circumvent the lack of decay assumptions on u at infinity, we introduce and systematically exploit a Kelvin-type transformation tailored to the Heisenberg group. This transformation maps a potential singularity at infinity to a manageable singularity at a finite point, allowing the moving plane argument to proceed. This technique provides a unified approach to handle asymptotic behavior in such non-Euclidean settings and may be of independent interest.

The rest of this paper is organized as follows. In Section 2, we recall essential preliminaries concerning the Heisenberg group, its metric structure, the sub-Laplacian, and the relevant function space embeddings. Section 3 is devoted to stating and proving our main symmetry theorem. The proof is structured into several logical steps: setting up the moving plane, deriving the key integral identity, applying the Kelvin transformation to manage behavior at infinity, and completing the moving plane argument via a continuity method and measure estimates. Finally, we conclude with some remarks on possible extensions and open problems.

2. Preliminaries

This section collects the necessary background material concerning the Heisenberg group, its associated metric and analytic structures, as well as the specific transformations and inequalities that are fundamental to our analysis.

The Heisenberg group $\mathbb{H}^n$ is a nilpotent Lie group of step 2, identified with $\mathbb{R}^{2n+1}$ equipped with the non-commutative group law

$$\xi \circ \eta := \left(x + \tilde{x}, y + \tilde{y}, t + \tilde{t} + 2\sum_{i=1}^{n} (x_i \tilde{y}_i - y_i \tilde{x}_i) \right),$$

for $\xi = (x, y, t), \eta = (\tilde{x}, \tilde{y}, \tilde{t}) \in \mathbb{R}^{2n} \times \mathbb{R}$, where $x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^n$. The inverse element is $\xi^{-1} = -\xi$.

The family of dilations $\{\delta_\lambda\}_{\lambda>0}$, defined by

$$\delta_\lambda(\xi) := (\lambda x, \lambda y, \lambda^2 t),$$

is compatible with the group structure and introduces the homogeneous dimension

$$Q := 2n + 2.$$

A natural homogeneous gauge is given by

$$\rho(\xi) := |\xi|_H = \left(\left(\sum_{i=1}^{n} (x_i^2 + y_i^2) \right)^2 + t^2 \right)^{1/4},$$

which induces the left-invariant Korányi distance

$$d_H(\xi, \eta) := \rho(\eta^{-1} \circ \xi).$$

The ball centered at ξ with radius $R > 0$ is denoted by $B_R(\xi)$. Its Haar measure, which coincides with the Lebesgue measure on $\mathbb{R}^{2n+1}$, satisfies $|B_R(\xi)| = |B_1(0)|R^Q$.

The Lie algebra of $\mathbb{H}^n$ is spanned by the left-invariant vector fields

$$\begin{aligned} X_i &:= \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \\ Y_i &:= \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad (i = 1, \dots, n) \\ T &:= \frac{\partial}{\partial t}. \end{aligned}$$

These satisfy the canonical commutation relations $[X_i, Y_j] = -4\delta_{ij} T$, while all other commutators vanish.

The horizontal gradient of a function u is

$$\nabla_{\mathbb{H}} u := (X_1 u, \dots, X_n u, Y_1 u, \dots, Y_n u),$$

and the sub-Laplacian is the second-order operator

$$\Delta_{\mathbb{H}} u := \sum_{i=1}^{n} (X_i^2 + Y_i^2) u.$$

For $1 \le p < \infty$, the Lebesgue space $L^p(\mathbb{H}^n)$ is defined with respect to the Haar measure. The local space $L^p_{\text{loc}}(\mathbb{H}^n)$ consists of functions integrable to the p-th power on any compact subset.

A cornerstone of our analysis is the *Hardy-Littlewood-Sobolev (HLS) inequality* on $\mathbb{H}^n$ (see, e.g., [17]). For $0 < \alpha < Q$, $1 < p < q < \infty$ satisfying

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q},$$

there exists a constant $C = C(n, \alpha, p) > 0$ such that for all $f \in L^p(\mathbb{H}^n)$,

$$\left\| \int_{\mathbb{H}^n} \frac{f(\eta)}{|\eta^{-1} \circ \cdot|_H^{Q-\alpha}} \, d\eta \right\|_{L^q(\mathbb{H}^n)} \le C \|f\|_{L^p(\mathbb{H}^n)}. \tag{7}$$

This inequality governs the mapping properties of the convolution with the Riesz-type kernel $|\cdot|_H^{\alpha-Q}$.

3. Main Results

This section presents the core theorems of this paper along with their proofs. We begin by stating the precise form of the integral equation under study and the fundamental assumptions. Subsequently, two key preparatory lemmas are established, followed by a step-by-step proof of the main theorem.

We now present our main symmetry result for positive solutions of the integral equation (6) with singular nonlinearity $\tau < 0$.

Theorem 3.1. Let $u \in L^{\tau-1}_{\text{loc}}(\mathbb{H}^n)$ be a positive solution of the integral equation

$$u(\xi) = h(\xi) - \int_{\mathbb{H}^n} \frac{1}{|\eta^{-1} \circ \xi|_H^{Q-\alpha}} \, u(\eta)^\tau \, d\eta, \quad \xi \in \mathbb{H}^n,$$

where $0 < \alpha < Q$ and $\tau < 0$, $h : \mathbb{H}^n \to (0, \infty)$ is a given smooth external term, Assume that h is cylindrical symmetry with respect to the plane $t = 0$, Then u is also cylindrical symmetry with respect to the plane $t = 0$.

The proof of Theorem 3.1 relies on two key lemmas. The first establishes the fundamental integral identity obtained after reflection. For any positive solution u of equation (6), satisfies

$$u_\lambda(\xi)-u(\xi) = -\int_{\Sigma_\lambda}\left(\frac{1}{|\eta^{-1}\circ\xi|_H^{Q-\alpha}}-\frac{1}{|\eta^{-1}\circ\xi^\lambda|_H^{Q-\alpha}}\right) \times (u_\lambda(\eta)^\tau-u(\eta)^\tau)\,d\eta \tag{8}$$

Let $\Sigma_\lambda^c=\{\xi=(x_1,...,x_n,t)|t<\lambda\}$, then it is easy to see that

$$\begin{aligned} h-u(\xi) &= \int_{\Sigma_\lambda}\frac{1}{|\eta^{-1}\circ\xi|_H^{Q-\alpha}}u(\eta)^\tau d\eta + \int_{\Sigma_\lambda^c}\frac{1}{|\eta_\lambda^{-1}\circ\xi|_H^{Q-\alpha}}u(\eta)^\tau d\eta \\ &= \int_{\Sigma_\lambda}\frac{1}{|\eta^{-1}\circ\xi|_H^{Q-\alpha}}u(\eta)^\tau d\eta \\ &\quad + \int_{\Sigma_\lambda}\frac{1}{|\eta_\lambda^{-1}\circ\xi|_H^{Q-\alpha}}u(\eta^\lambda)^\tau d\eta \\ &= \int_{\Sigma_\lambda}\frac{1}{|\eta^{-1}\circ\xi|_H^{Q-\alpha}}u(\eta)^\tau d\eta + \int_{\Sigma_\lambda}\frac{1}{|\eta^{-1}\circ\xi^\lambda|_H^{Q-\alpha}}u_\lambda(\eta)^\tau d\eta, \end{aligned}$$

here we have used the fact that $|(\eta^\lambda)^{-1}\circ\xi|_H=|\eta^{-1}\circ\xi^\lambda|_H$.

Substituting ξ by ξ^λ, we get

$$h-u(\xi^\lambda)=\int_{\Sigma_\lambda}\frac{1}{|\eta^{-1}\circ\xi^\lambda|_H^{Q-\alpha}}u(\eta)^\tau d\eta + \int_{\Sigma_\lambda}\frac{1}{|\eta^{-1}\circ\xi|_H^{Q-\alpha}}u_\lambda(\eta)^\tau d\eta. \tag{9}$$

Thus,

$$u(\xi^\lambda)-u(\xi) = -\int_{\Sigma_\lambda}\left(\frac{1}{|\eta^{-1}\circ\xi|_H^{Q-\alpha}}-\frac{1}{|\eta^{-1}\circ\xi^\lambda|_H^{Q-\alpha}}\right) \times(u_\lambda(\eta)^\tau-u(\eta)^\tau)d\eta. \tag{10}$$

For any fixed $\lambda>0$, the function $u_\lambda, u\in L^{\tau-1}(\mathbb{H}^n)$, there existence $C>0$, such that

$$\|u_\lambda-u\|_{L^q(\Sigma_\lambda^-)}\le C(\int_{\Sigma_\lambda^-}u_\lambda^{(\tau-1)\beta})^{\frac{1}{\beta}}\|u_\lambda-u\|_{L^q(\Sigma_\lambda^-)}.$$

Define $\Sigma_\lambda^-=\{\xi|\xi\in\Sigma_\lambda, u(\xi)\ge u_\lambda(\xi)\}$ and $\Sigma_\lambda^+=\{\xi|\xi\in\Sigma_\lambda, u(\xi)<u_\lambda(\xi)\}$, then $\Sigma_\lambda=\Sigma_\lambda^+\cup\Sigma_\lambda^-$. We want to show that for sufficiently position value of λ, Σ_λ^- must be empty. Note that for $\eta\in\Sigma_\lambda^-$, we have $u(\eta)^\tau\le u_\lambda(\eta)^\tau$, whenever $\xi,\eta\in\Sigma_\lambda$, we have that $|\eta^{-1}\circ\xi|_H\le|\eta^{-1}\circ\xi^\lambda|_H$ and $|\eta^{-1}\circ\xi|_H^{\alpha-Q}\ge|\eta^{-1}\circ\xi^\lambda|_H^{\alpha-Q}$, then by Lemma 3, we have for $\xi\in\Sigma_\lambda^-$,

$$\begin{aligned} &u(\xi)-u_\lambda(\xi) \\ &= -\int_{\Sigma_\lambda}\left(|\eta^{-1}\circ\xi|_H^{\alpha-Q}-|\eta^{-1}\circ\xi^\lambda|_H^{\alpha-Q}\right) \times(u(\eta)^\tau-u_\lambda(\eta)^\tau)\,d\eta \\ &\le \int_{\Sigma_\lambda^-}|\eta^{-1}\circ\xi|_H^{\alpha-Q}(u_\lambda(\eta)^\tau-u(\eta)^\tau)d\eta \\ &\le -\tau\int_{\Sigma_\lambda^-}|\eta^{-1}\circ\xi|_H^{\alpha-Q} \times[u_\lambda(\xi)^{\tau-1}(u_\lambda^{\tau-1}(\eta)(u(\eta)-u_\lambda(\eta)))]d\eta, \end{aligned}$$

it follow from the Hölder inequality that

$$\begin{aligned} &\|u_\lambda-u\|_{L^q(\Sigma_\lambda^-)} \\ &\le C\left\|\int_{\Sigma_{\lambda-}}|\eta^{-1}\circ\xi|_H^{\alpha-Q}[u_\lambda^{\tau-1}(u_\lambda-u)](\eta)d\eta\right\|_{L^q(\Sigma_\lambda^-)} \\ &\le C\left\|\int_{\Sigma_\lambda^-}u_\lambda^{\tau-1}(u_\lambda-u)(\eta)d\eta\right\|_{L^p(\Sigma_\lambda^-)} \\ &\le C\left(\int_{\Sigma_\lambda^-}u_\lambda^{(\tau-1)\beta}\right)^{\frac{1}{\beta}}\left(\int_{\Sigma_\lambda^-}(u_\lambda-u)^q\right)^{\frac{1}{q}} \\ &\le C\left(\int_{\Sigma_\lambda^-}u_\lambda^{(\tau-1)\beta}\right)^{\frac{1}{\beta}}\|u_\lambda-u\|_{L^q(\Sigma_\lambda^-)}, \end{aligned}$$

where $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{Q}$, $\frac{1}{\beta}=1-\frac{1}{q}$.

We begin by stating and proving several key lemmas that will be used to handle the CR inversion transformation and the measure estimates of reflection sets.

Lemma 3.1 (CR inversion preserves the equation form). Let $u\in L_{\text{loc}}^{\tau-1}(\mathbb{H}^n)$ be a positive solution of equation (6). Define the CR inversion as

$$v(\xi)=\frac{1}{\rho(\xi)^{Q-\alpha}}u(\widetilde{\xi}), \qquad \xi=(x,y,t)\in\mathbb{H}^n\setminus\{0\},$$

where

$$\widetilde{\xi}=(\widetilde{x},\widetilde{y},\widetilde{t}), \quad \widetilde{x}=\frac{xt+yr^2}{\rho^4}, \quad \widetilde{y}=\frac{yt-xr^2}{\rho^4}, \quad \widetilde{t}=-\frac{t}{\rho^4},$$

with $r^2=\sum_{i=1}^n(x_i^2+y_i^2)$ and $\rho(\xi)=|\xi|_H$. Then v satisfies the following integral equation

$$v(\xi) - \tilde{h}(\xi) = -\int_{\mathbb{H}^n\setminus\{0\}} \frac{1}{|\eta^{-1}\circ\xi|_H^{Q-\alpha}} \frac{1}{\rho(\eta)^{(Q-\alpha)-\tau(Q-\alpha)}} v(\eta)^\tau \, d\eta, \tag{11}$$

where $\tilde{h}(\xi) = \dfrac{1}{\rho(\xi)^{Q-\alpha}} h(\widetilde{\xi})$.

Proof: Using the conformal property of the CR inversion on the Heisenberg group, we have the distance relation

$$|\widetilde{\eta}^{-1}\circ\widetilde{\xi}|_H = \frac{|\eta^{-1}\circ\xi|_H}{\rho(\xi)\,\rho(\eta)}.$$

Meanwhile, the volume element transforms as $d\widetilde{\eta} = \rho(\eta)^{-2Q}\, d\eta$. Substituting $u(\xi) = \rho(\xi)^{Q-\alpha} v(\xi)$ into equation (6) and changing variable $\eta \mapsto \widetilde{\eta}$, a direct computation yields

$$\begin{aligned}\rho(\xi)^{Q-\alpha}v(\xi) &= h(\xi) - \int_{\mathbb{H}^n} \frac{1}{|\eta^{-1}\circ\xi|_H^{Q-\alpha}}\,\rho(\eta)^{(Q-\alpha)\tau} v(\eta)^\tau \, d\eta \\ &= h(\xi) - \int_{\mathbb{H}^n\setminus\{0\}} \frac{\rho(\xi)^{Q-\alpha}\rho(\eta)^{Q-\alpha}}{|\widetilde{\eta}^{-1}\circ\widetilde{\xi}|_H^{Q-\alpha}} \\ &\quad \times \rho(\eta)^{(Q-\alpha)\tau} v(\eta)^\tau\, \rho(\eta)^{-2Q} d\widetilde{\eta}.\end{aligned}$$

Noting the homogeneous dimension $Q = 2n+2$ and the exponent relation $(Q-\alpha)\tau - 2Q + (Q-\alpha) < 0$, we obtain (11) with $\tilde{h}(\xi) = \rho(\xi)^{\alpha-2} h(\widetilde{\xi})$. Since h is assumed smooth, $\tilde{h}$ is also smooth away from the origin.

Lemma 3.2 (Integrability near the singularity). Assume u is a positive solution of (6) with $u \in L^{\tau-1}_{\text{loc}}(\mathbb{H}^n)$. Then the function v defined in Lemma 3.1 satisfies for any compact set $K \subset \mathbb{H}^n \setminus \{0\}$,

$$\int_K v(\eta)^{\tau+1}\, d\eta < \infty.$$

In particular, if u is bounded (or decays) at infinity, then v is continuous at the origin.

Proof: Since the CR inversion maps a neighborhood of the origin to a neighborhood of infinity and u is locally integrable, for any compact set K away from the origin, there exists a constant $C_K > 0$ such that

$$\begin{aligned}\textstyle\int_K v(\eta)^{\tau+1}\, d\eta &= \textstyle\int_K \rho(\eta)^{-(Q-\alpha)(\tau+1)} u(\widetilde{\eta})^{\tau+1}\, d\eta \\ &\le C_K \textstyle\int_{\widetilde{K}} u(\zeta)^{\tau+1}\, d\zeta < \infty,\end{aligned} \tag{12}$$

where $\widetilde{K}$ is the image of K under the inversion and is compact. The second inequality uses the control of the Jacobian of the inversion. If u decays at infinity, then as $\xi \to 0$, $\widetilde{\xi} \to \infty$, and hence $v(\xi) \sim \rho(\xi)^{2-Q} u(\infty)$, so v is continuous at the origin.

Lemma 3.3 (Measure estimate of the reflection set). Let u be a positive solution of (6) with $u^{\tau-1} \in L^\beta(\mathbb{H}^n)$ for some $\beta > 1$. For a given λ, define

$$\Sigma_\lambda^- = \{\xi \in \Sigma_\lambda : u(\xi) < u_\lambda(\xi)\},$$

and let $(\Sigma_\lambda^-)^*$ denote the reflection of Σ_λ^- about the plane $t = \lambda$. Then there exists a constant $C > 0$ and a sufficiently large $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$,

$$\int_{(\Sigma_\lambda^-)^*} u(\eta)^{(\tau-1)\beta}\, d\eta \le \frac{1}{2C^\beta}.$$

Proof: Since $u^{\tau-1} \in L^\beta(\mathbb{H}^n)$, by the absolute continuity of the integral, for any $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{\mathbb{H}^n\setminus B_R(0)} u(\eta)^{(\tau-1)\beta}\, d\eta < \varepsilon,$$

where $B_R(0)$ is the Heisenberg ball centered at the origin with radius R. When λ is sufficiently large, the reflected set $(\Sigma_\lambda^-)^*$ is contained in $\mathbb{H}^n \setminus B_R(0)$, and therefore

$$\int_{(\Sigma_\lambda^-)^*} u(\eta)^{(\tau-1)\beta}\, d\eta < \varepsilon.$$

Choosing $\varepsilon = 1/(2C^\beta)$ yields the desired estimate.

We now prove the main theorem. The proof is divided into four steps, combining the method of moving planes, the CR inversion transformation, and the lemmas above.

Proof of Theorem 3.1: Step 1: CR inversion and problem transformation. Let u be a positive solution of (6). We perform the CR inversion and define v as in Lemma 3.1. By this lemma, v satisfies equation (11), and the singularity at infinity is transferred to the origin. According to Lemma 3.2, v is locally integrable on $\mathbb{H}^n \setminus \{0\}$, so we can apply the method of moving planes to v on $\mathbb{H}^n \setminus \{0\}$. Now we start the moving plane on v. Fix a direction and consider reflection about the plane $\pi_\lambda : t = \lambda$. Define

$$\Sigma_\lambda = \{\xi = (x, y, t) \in \mathbb{H}^n : t > \lambda\},$$

the reflected point $\xi^\lambda = (x, y, 2\lambda - t)$, and $v_\lambda(\xi) = v(\xi^\lambda)$. We start with λ sufficiently negative. As $\lambda \to -\infty$, the reflected point ξ^λ is closer to the origin, and v takes larger values near the origin (due to the singularity). By directly estimating the integral equation (11), one can prove that there exists $N > 0$ such that for all $\lambda \le -N$, we have $v \ge v_\lambda$ on $\Sigma_\lambda \setminus \{0\}$.

Step 2: Continuous moving of the plane. Let

$$\lambda_0 = \sup\{\lambda \in \mathbb{R} : v \geq v_\alpha \text{ on } \Sigma_\alpha \setminus \{0\} \text{ for all } \alpha \leq \lambda\}.$$

We claim that $\lambda_0 = 0$. Suppose $\lambda_0 < 0$. We will show that the plane can be moved further to the right, i.e., there exists $\epsilon > 0$ such that for $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$, we still have $v \geq v_\lambda$ on $\Sigma_\lambda \setminus \{0\}$.

At λ_0, by continuity we have $v \geq v_{\lambda_0}$. If there exists $\xi_0 \in \Sigma_{\lambda_0} \setminus \{0\}$ such that $v(\xi_0) = v_{\lambda_0}(\xi_0)$, then by the strong maximum principle for integral equations (see [2], Lemma 3.1), we obtain $v \equiv v_{\lambda_0}$ on $\Sigma_{\lambda_0} \setminus \{0\}$, so symmetry is achieved earlier. Otherwise, we have $v > v_{\lambda_0}$ strictly in the interior of $\Sigma_{\lambda_0} \setminus \{0\}$.

Now, suppose there exists a sequence $\lambda_k \searrow \lambda_0$ and points $\xi_k \in \Sigma_{\lambda_k} \setminus \{0\}$ such that $v(\xi_k) < v_{\lambda_k}(\xi_k)$. Define the set

$$\Sigma_\lambda^- = \{\xi \in \Sigma_\lambda \setminus \{0\} : v(\xi) < v_\lambda(\xi)\}.$$

Applying Lemma 3 to v (the proof works equally for v), we obtain the integral identity

$$\begin{aligned} &v_\lambda(\xi) - v(\xi) \\ &\quad = -\int_{\Sigma_\lambda} \left(\frac{1}{|\eta^{-1} \circ \xi|_H^{Q-\alpha}} \right. \\ &\qquad\qquad \left. - \frac{1}{|\eta^{-1} \circ \xi^\lambda|_H^{Q-\alpha}} \right) \big(v_\lambda(\eta)^\tau - v(\eta)^\tau\big)\, d\eta. \end{aligned} \tag{13}$$

On Σ_λ^-, using $\tau < 0$ and the monotonicity of the kernel, we derive

$$\begin{aligned} &v(\xi) - v_\lambda(\xi) \\ &\leq -\tau \int_{\Sigma_\lambda^-} \frac{1}{|\eta^{-1} \circ \xi|_H^{Q-\alpha}}\, v_\lambda(\eta)^{\tau-1} \big(v_\lambda(\eta) - v(\eta)\big)\, d\eta. \end{aligned}$$

Now, applying the Hardy–Littlewood–Sobolev inequality (see [17], Theorem 1.1) together with Hölder inequality, we get

$$\begin{aligned} &\|v_\lambda - v\|_{L^q(\Sigma_\lambda^-)} \\ &\leq C \left(\int_{(\Sigma_\lambda^-)^*} v(\eta)^{(\tau-1)\beta}\, d\eta \right)^{1/\beta} \|v_\lambda - v\|_{L^q(\Sigma_\lambda^-)}, \end{aligned}$$

where $1/q = 1/p - \alpha/Q$ and $1/\beta = 1 - 1/q$. By Lemma 3.3, when λ is sufficiently close to λ_0, we have

$$C \left(\int_{(\Sigma_\lambda^-)^*} v(\eta)^{(\tau-1)\beta}\, d\eta \right)^{1/\beta} \leq \frac{1}{2}.$$

Hence, $\|v_\lambda - v\|_{L^q(\Sigma_\lambda^-)} = 0$, so $\Sigma_\lambda^- = \emptyset$, a contradiction. Therefore, the plane can be moved continuously until $\lambda_0 = 0$.

Step 3: Derivation of symmetry. When the plane reaches $\lambda = 0$, we have $v \geq v_0$ on $\Sigma_0 \setminus \{0\}$. By the definition of reflection, we symmetrically obtain $v \leq v_0$ on $\Sigma_0 \setminus \{0\}$, so $v = v_0$ on $\Sigma_0 \setminus \{0\}$. This means v is symmetric about the plane $t = 0$.

If symmetry is achieved earlier at some $\lambda_0 < 0$, then v is symmetric about the plane $t = \lambda_0$. In this case, v has no singularity at the origin, and by Lemma 3.2, u decays nicely at infinity. Then we can directly apply the method of moving planes to u to obtain symmetry about some plane.

Step 4: From symmetry of v to symmetry of u. Since v is symmetric about the plane $t = 0$ and u is related to v via the CR inversion, using the geometric properties of the inversion, we conclude that u is symmetric about a plane corresponding to the symmetry of h. Specifically, if h is symmetric about a plane π, then through the inversion, u is also symmetric about π.

4. Conclusions

This paper studies the symmetry of solutions to a class of integral equations with negative exponent nonlinearities on the Heisenberg group. By introducing a CR inversion transformation, we convert the singularity at infinity to a finite point, thereby successfully implementing the method of moving planes in a non-Euclidean structure. This approach is general and can be applied to other geometric spaces with non-commutative structures, such as Carnot groups or more general stratified Lie groups.

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Funding

This research was supported by the National Natural Science Foundation of China (Grant No. 12561093) and Xinjiang Natural Science Foundation (Grant No. 2024D01B38).

Conflicts of Interest

The authors declare no conflicts of interest.

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