Pure and Applied Mathematics Journal
2026, Vol. 15, No. 1, pp. 6–10
https://doi.org/10.11648/j.pamj.20261501.12

Research Article

Norm of General Derivation on Tensor Product of C*-Algebras

Benjamin Kimeu Daniel*, Kinyanjui Jeremiah Ndungu, Zachary Kaunda Kayiita

Department of Pure & Applied Sciences, Kirinyaga University, Kerugoya, Kenya

Abstract

The study of operator theory, specifically the determination of the norm of general derivations in C* algebras, has attracted significant attention in recent years. This problem has been approached through various methods across different spaces, yielding several results. The norm of a general derivation is crucial for understanding the structure and behavior of derivations in the context of C* algebras, with implications for functional analysis and operator theory. Despite previous efforts, a complete understanding remains elusive. The purpose of this paper is to investigate this problem using the finite rank operator in the tensor product of C*-algebras. By leveraging the tensor product structure, we explore how finite rank operators can provide insights into the norm of general derivations. The research utilizes a mathematical framework that examines the behavior of bounded linear operators within the tensor product of Hilbert spaces and C*-algebras. Through this approach, we aim to advance existing knowledge and offer new results that could potentially contribute to the field. The final conclusion of the study confirms that the use of finite rank operators in the tensor product of C*-algebras offers a valuable method for approximating and understanding the norm of general derivations, providing a more refined approach than previous techniques.

Keywords

General Derivation Operator, Finite Rank Operator, Tensor Product of C*-Algebra

1. Introduction

1.1. C^* Algebras and Their Tensor Product Structure in Hilbert Spaces

Definition 1.1.1. Tensor product. Daniel *et al.*, *[6] and* Muiruri *et al.*, *[10]*

Consider two complex Hilbert spaces, $Z = \{u_1, \ u_2 \ \ldots\}$ and $L = \{v_1, \ v_1 \ \ldots \}$ with inner products defined as $\langle u_1, u_2 \rangle$ and $\langle v_1, v_2 \rangle$ respectively. A tensor product of Z and L is a Hilbert space $Z \otimes L$ where $\otimes: Z \times L \to Z \otimes L, \otimes (u, v) \to u \otimes v$ is a bilinear mapping: The vectors $u \otimes v$ form a total subset of $Z \otimes L < u1 \otimes v1, u2 \otimes v2 >=< u_1, v_1 >< u_2, \ v_2 >, \forall \ u_1, u_2 \in Z, \ v1, \ v2 \in L$. This implies that $\|u \otimes v\| = \|u\| \|v\| \ \forall \ u \in Z, \ v \in L$.If $E \in B(Z)$, $F \in B(L)$, then $B(Z \otimes L)$ is a Hilbert space and for $E \otimes F \in B(Z \otimes L)$ we have $E \otimes F(u \otimes v) = Eu \otimes Fv \ \forall \ u \in Z, v \in L$.

Fundamental Properties of Operators in $B(Z \otimes L)$ hold:

1) i).$(E \otimes F)(G \otimes H) = EG \otimes FH, \forall \ E \in B(Z), \ G \in$

*Correspondence: Benjamin Kimeu Daniel (benjaminkimeu722@gmail.com)

Received: 22 January 2026; **Accepted:** 4 February 2026; **Published:** 26 February 2026

$B(Z)$ and $F \in B(L)$, $H \in B(L)$.(Associativity and commutativity).

2) ii).$\|E \otimes F\| = \|E\|\|F\|$ $\forall E \in B(Z)$ and $F \in B(L)$ (Distributivity under tensor product).

3) iii). Linearity of the Tensor Product Map $\otimes (u, v) \to u \otimes v$ and its Properties:

a) $(u_1 + u_2) \otimes v = (u_1 \otimes v) + (u_2 \otimes v)$

b) $(\psi u) \otimes v = \psi(u \otimes v)$.

c) $u \otimes (v_1 + v_2) = u \otimes v_1 + u \otimes v_2$

d) $u \otimes (\psi v) = \psi(u \otimes v)$.

The set of all vectors $\otimes (u, v)$, $u \epsilon Z$ and $v \epsilon L$ form a total subset of $Z \otimes L$.

Definition 1.1.2. Elementary operator in a tensor product. Daniel et al., [5] and King'ang'I et al., [7]

Let Z and L be complex Hilbert spaces, and $B(Z \otimes L)$ denote the set of all bounded linear operators on the tensor product space $Z \otimes L$. For fixed elements $E \otimes F$ and $G \otimes H$ in $B(Z \otimes L)$, where $E, G \in B(Z)$ and $F, H \in B(L)$, we define the elementary operator as follows:

$E_n (Z \otimes L) = \sum_n (Ei \otimes Fi)(U \otimes V)(Gi \otimes Hi)$

for every $U \otimes V \epsilon B(Z \otimes L)$, $Ei \otimes Fi$, $Gi \otimes Hi \in B(Z \otimes L)$.

Substituting $n = 1$, we obtain the basic elementary operator:

$$E(Z \otimes L) = (E \otimes F)(U \otimes V)(G \otimes H) \quad (1)$$

From equation (1), the basic elementary operator can be expressed as,

$E(Z \otimes L) = (E \otimes F)(U \otimes V)(G \otimes H) = (EUG) \otimes (FVH)$.

Definition 1.1.3: General Derivation. Nyamwala and Agure, [11] and Okelo and Ambongo, [12]

The general derivation operator is defined by

$\delta_{A,B} = AX - XB$ where A and B are coefficient operators in $B(X)$.

1.2. Norm of General Derivation

Stampfli [14] and Timoney [15] determined the $\|\delta\|_D$in a Banach algebra $B(L)$ by maximal numerical range and proved theorem 1.2.1:

Theorem 1.2.1. Stampfli [14] and Timoney [15],

Let A, B $\in$ B(H), the algebra of all bounded linear operators on a Hilbert space H. Then

$\|\delta_{A,B}\| = \sup\{\|AX - XB\|: X \in B(H), \quad \|X\| = 1\}$
$= \inf_{\lambda\epsilon C}\|A - \lambda I\| + \|B - \lambda I\|$

Let A be an irreducible C*-algebra on H. Let A $\in$ B(A). Then

$\|\delta_A\| = 2inf_{\lambda\epsilon C}\|A - \lambda I\| = 2d(A)$

Nyamwala and Agure [11] and Society [13] determined the converse of Stampfli [14], results, and established that the inner derivation derived from hyponormal operators has its range closed whenever the spectrum of the operator is defined.

Kittaneh [8] extended the work of Stampfli [14] in norm ideals and determined the range and kernel of a normal derivation restricted to unitary invariant norms.

Barraa and Pedersen [2] determined the conditions necessary for the product $\delta_{(C,D)}, \delta_{(A,B)}$ of 2 general derivations to satisfy the conditions of a derivation and proved theorem 1.2.2:

Theorem 1.2.2. Barraa and Pedersen [2].

Let A, B, C, and D be bounded operators on a Banach space $\mathcal{T}$.

1) If $A \notin \mathcal{T}$ and $B \notin \mathcal{T}$, then $\delta_{C,D}\delta_{A,B}$ is a generalized derivation if and only iff $C = aA + cI$ *and* $D = -aB + dI$ for some scalars a, c, and d.
2) If A $\in \mathcal{T}$ and B $\notin \mathcal{T}$, then $\delta_{C,D}\delta_{A,B}$ is a generalized derivation if and only if C $\in \mathcal{T}$.
3) If A $\notin \mathcal{T}$ and B $\in \mathcal{T}$, then $\delta_{C,D}\delta_{A,B}$ is a generalized derivation if and only if D $\in \mathcal{T}$.
4) If $A \in \mathcal{T}$ and $B \in \mathcal{T}$, then $\delta_{C,D}\delta_{A,B}$ is a generalized derivation.

Barraa [1] and Barraa and Pedersen [2] determined the norm of the sum of two bounded operators on a Hilbert space and provided sufficient conditions under which the norm equals the sum of their norms

Muhol *et al*. [9] determined the norm of a general derivation acting on a norm ideal and proved Theorem 1.2.3.

Theorem 1.2.3. Muhol *et al*., *[9]*

Let $A, B \in B(H)$ be S-universal operators and let $\mathcal{J}$ be a norm ideal in $B(H)$. Then

$\|\delta_{A,B}\| = \|\delta_{A,B,\mathcal{J}}\|$

Bonyo and Agure [4] determined the norms of derivations implemented by S-universal operators. Bonyo and Agure [4] focused on understanding the behavior of derivations within norm ideals, particularly the conditions under which these derivations can be analyzed using S-universal operators. Their work contributes to the broader field of operator theory by providing insights into the relationship between derivations and norm ideals, helping to establish criteria for when these derivations attain their norms.

Beatrice *et al*. [3] determined the norm of a general derivation on an infinite-dimensional complex Hilbert space H using finite rank one operators and proved Theorem 1.2.4.

Theorem 1.2.4. Beatrice *et al*. [3],

Let $A, B \in B(H)$ and $\delta_{A,B}: B(H) \to B(H)$. Then

$\|\delta_{A,B}\ B(H)\| = \|A\| + \|B\|$.

Further, Beatrice *et al*. [3] showed that this equality holds using Stampfli's maximal numerical range and proved Theorem 1.2.5.

Theorem 1.2.5. Beatrice *et al*., [3],

Let $d(A) = Inf\{\|A - \lambda I\|: \lambda \in C\}$, and $d(B) = inf\{\|B - \lambda I\|: \lambda \in C\}$.

Then $\|\delta_{A,B}\ B(H)\| = \|A\| + \|B\|$

2. Main Results

Norm of General Derivation $\boldsymbol{\delta}$ *on a Tensor Product of* C^*-*Algebras*

Let $H \otimes K$ be the tensor product of Hilbert spaces H and K, and $B(H \otimes K)$ be the set of bounded linear operators on $H \otimes K$. Then, for $X \otimes Y \in B(H \otimes K)$ with $\|X \otimes Y\| = 1$, we have:

$\|\delta_{A\otimes C,B\otimes D} \,|(B(H \otimes K)\| = \|A\|\|B\| + \|C\|\|D\|$

Proof

By definition,

$\|\delta_{A\otimes C,B\otimes D}\| = Sup\{\|\delta(A\otimes C, B\otimes D)(X \otimes Y)\|: X \otimes Y \in B(H \otimes K), \|X \otimes Y\| = 1$

$= \sup\{\|(A \otimes B)(X \otimes Y) - (X \otimes Y)(C \otimes D\|: X\otimes Y \in B(H\otimes K), \|X \otimes Y\| = 1$

Therefore,

$\|\delta_{A\otimes C,B\otimes D}\| = \sup\{\delta_{A\otimes B,C\otimes D}(X\otimes Y)\|: X\otimes Y \in B(H \otimes K), \|X\otimes Y\| = 1\}$

Taking an arbitrary $\epsilon > 0$, we have

$\|\delta_{A\otimes B,C\otimes D}\| - \epsilon < \|\delta_{A\otimes B,C\otimes D}(X\otimes Y)\|$

for *all* $X \in B(H), Y \in B(K)$, and $\|X \otimes Y\| = 1$.

$\|\delta_{A\otimes B,C\otimes D}\| - \epsilon < \|(A\otimes B)(X\otimes Y) - (X\otimes Y)(C\otimes D)$

Since

$\|(A \otimes B)(X \otimes Y) - (X \otimes Y)(C \otimes D)\| \leq \|A\|\|B\| + \|C\|\|D\|$

and letting $\epsilon \to 0$, then we have

$$\|\delta_{A\otimes B,C\otimes D}\| \leq \|A\|\|B\| + \|C\|\|D\| \tag{2}$$

Conversely, let there exist a unit vector $e \otimes f$ where $e \in H$ and $f \in K$, then we have

$\|\delta_{A\otimes B,C\otimes D}(X\otimes Y)(e\otimes f)\| \leq \|\delta_{A\otimes B,C\otimes D}(X\otimes Y)\|\|e\|\|f\| \leq \|\delta_{A\otimes B,C\otimes D}(X \otimes Y)\|\|e\|\|f\|$

given that X, Y, e, and f are unit operators and vectors, respectively, and $\|X \otimes Y\| = \|X\|\|Y\|$ then

$\|\delta_{A\otimes B,C\otimes D}\|\|X\otimes Y\|\|e\otimes f\| = \|\delta_{A\otimes B,C\otimes D}\|$

Thus,

$\|\delta_{A\otimes B,C\otimes D}\| \geq \|\delta_{A\otimes B,C\otimes D}(X\otimes Y)(e\otimes f)\|$

$\geq \|\{(A\otimes B)(X\otimes Y) - (X\otimes Y)(C\otimes D)\}(e\otimes f)\|$

By tensor product property, $(A\otimes B)(X\otimes Y) = AX\otimes BY$ we have

$$\|AX\otimes BY\| = \|AX\|\|BY\| \tag{3}$$

From equation (3), then

$= \|(AX\otimes BY)(e\otimes f) - (X\otimes Y)(C\otimes D)(e\otimes f)\|$

$= \|AXe\otimes BY f - CXe\otimes DY f\|$

since $\|e\otimes f\| = 1$

therefore,

$\delta_{A\otimes B,C\otimes D}\| \geq \|AXe \otimes BY f - CXe \otimes DY f\|$

Squaring both sides, we get

$$\|\delta_{(A\otimes B,C\otimes D)}\|^2 \geq \|AXe \otimes BY f - CXe \otimes DY f\|^2$$

$= \langle AXe \otimes BY f - CXe \otimes DY f, AXe \otimes BY f - CXe \otimes DY f\rangle$

$= \langle AXe \otimes BY f, AXe \otimes BY f\rangle - \langle AXe \otimes BY f, CXe \otimes DY f\rangle - \langle CXe \otimes DY f, AXe \otimes BY f\rangle + \langle CXe \otimes DY f, CXe \otimes DY f\rangle$

$= \|AXe \otimes BY f\|^2 - \langle AXe \otimes BY f, CXe \otimes DY f\rangle - \langle CXe \otimes DY f, AXe \otimes BY f\rangle + \|CXe \otimes DY f\|^2.$

By properties of the tensor product, $\langle x_1\otimes y_1, x_2 \otimes y_2\rangle = \langle x_1, x_2\rangle\langle y_1, y_2\rangle\ \forall\ x_1, x_2 \in H$ and $\forall\ y_1, y_2 \in K$ we have;

$$\|\delta_{(A\otimes B,C\otimes D)}\|^2 \geq \|AXe\|^2\|BY f\|^2 - \langle AXe \otimes BY f, CXe \otimes DY f\rangle - \langle CXe \otimes DY f, AXe \otimes BY f\rangle + \|CXe\|^2\|DY f\|^2. \tag{4}$$

If $\mu: H \to \mathbb{R}$ and $\nu: K \to \mathbb{R}$ are functionals. For unit vectors $y \in H$ and $z \in K$, define rank one operators:

$A = \mu \otimes y, y \in H, \|y\| = 1, by\ Ae = (\mu \otimes y)e = \mu(e)y$

$B = \nu \otimes z, z \in K, \|z\| = 1, by\ Bf = (\nu \otimes z)f = \nu(f)z$

Observe that:

$$\|A\| = sup\{\|(\mu \otimes y)e\|: e \in H\}$$

$$= sup\{\|\mu(e)y\|: e \in H\}$$

$$= \sup\{|\mu(e)|\|e\|: e \in H$$

$$= \sup\{|\mu(e)|: e \in H\}$$

$$= |\mu(e)|$$

Thus, $\|A\| = |\mu(e)|: e \in H.$

Similarly, using the same concept above, the norm of B is

$$\|B\| = |\nu(f)|: f \in K.$$

Moreover,

$$\|AXe\|^2 = \|(\mu y)xe\|^2$$

$$= \|\mu(e)yxe\|^2$$

$$= |\mu(e)|^2\|xe\|^2$$

$$= |\mu(e)|^2$$

since x, e are unit vectors $\|xe\|^2 = 1$ thus

$$|\mu(e)|^2 = \|A\|^2$$

Thus

$$\|AXe\|^2 = \|A\|^2 \quad (5)$$

Using the same concept from equation (5), we have Now,

$$\|BY\ f\|^2 = \|B\|^2 \quad (6)$$

$$\|CXe\|^2 = \|C\|^2 \quad (7)$$

$$\|Dyf\|^2 = \|D\|^2 \quad (8)$$

Now, from equation (4)

$$= -\langle Axe \otimes Byf, Cxe \otimes Dyf\ \rangle$$

$$= -\langle Axe \otimes Byf, Cxe \otimes Dyf\ \rangle$$

$$= -\langle Axe, Cxe\rangle\langle Byf, Dyf\ \rangle$$

hence

$$-\langle Axe, Cxe\rangle = \langle -(\mu \otimes y)eX, (\nu \otimes y)eX\rangle$$

$$= \langle -\mu(e)yx, \mu(e)yx\rangle$$

$$= -\mu(e)x \cdot \mu(e)x.\langle y, y\rangle \text{ since } \langle y, y\rangle = 1$$

$$= -\mu(e)x \cdot \mu(e)x$$

but $\mu(e)$ are positive real numbers, therefore

$$-\mu(e)x = |\mu(e)| = \mu(e)x$$

Therefore,
$-\mu(e) = |\mu(e)| = \| A \|$, and $\mu(e) = |\mu(e)| = \| C \|$.
Hence, $-\langle Axe, Cxe\rangle = \| A \|\| C \|$, Similarly, $\langle Byf, Dyf\ \rangle = \| B \|\| D \|$.
Thus,

$$-\langle Axe, Cxe\rangle\langle Byf, Dyf\ \rangle = \| A \|\| B \|\| C \|\| D \|. \quad (9)$$

$$-\langle Cxe \otimes Dyf, Axe \otimes Byf\ \rangle = -\ \langle Cxe, Axe\rangle\langle Dyf, Byf\ \rangle$$

$$= \| C \|\| A \|\| D \|\| B \|.$$

$$= \| A \|\| B \|\| C \|\| D \|. \quad (10)$$

Substituting equations (5), (6), (7), (8), (9), and (10) in (4), we have
$\left\|\delta_{(A\otimes B, C\otimes D)}\right\|^2 \geq \|A\|^2\|B\|^2 + 2\|A\|\|B\|\|C\|\|D\| + \|C\|^2\|D\|^2$
Taking the square roots of both sides, we get:

$$\left\|\delta_{(A\otimes B, C\otimes D)}|(B(H \otimes K)\right\| \geq \|A\|\|B\| + \|C\|\|D\| \quad (11)$$

Therefore, from (2) and (11)
$\|\delta_{A\otimes C, B\otimes D}\ |(B(H \otimes K)\| = \| A \|\| B \| + \| C \|\| D \|$

3. Conclusion

From the study, it can be concluded that using a finite rank operator, the Norm of general derivation in the tensor product of C*Algebras is

$\|\delta_{A\otimes C, B\otimes D}\ |(B(H \otimes K)\| = \| A \|\| B \| + \| C \|\| D \|.$

Abbreviations

C* algebras	C-star Algebras
H	Hilbert Space
$B(H)$	Bounded Linear Operators on a Hilbert Space H
$B(Z \otimes L)$	Bounded Linear Operators on the Tensor Product of Hilbert Spaces Z and L
$\delta_{A,B}$	General Derivation Between Two Operators A and B
$S-universal\ \ operators$	A Class of Operators Used in Norm Ideals

$\|.\|$	Norm Notation
I	Identity Operator
J	Norm Ideal in B(H)
$x \otimes y$	Tensor Product of Operators X and Y
$\in$	Arbitrary Small Positive Number (Epsilon)
λ	Scalar Value
R	Real Numbers

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Barraa, M. (2002). Convexoid and generalized derivations. Linear algebra and its applications, 350(1-3): 289–292. https://doi.org/10.1016/s0024-3795(02)00311-7

[2] Barraa, M. and Pedersen, S. (1999). On the product of two generalized derivations. Proceedings of the American Mathematical Society, pages 2679–2683. https://www.jstor.org/stable/119569

[3] Beatrice, O. A., Agure, J., and Nyamwala, F. (2019). On the norm of a generalized derivation. https://doi.org/10.12988/pms.2019.9810

[4] Bonyo, J. and Agure, J. (2011). Norms of derivations implemented by s-universal operators.

[5] Daniel, B. K., Wabomba, M. S., and Ndungu, K. J. (2023). An application of maximal numerical range on norm of an elementary operator of length two in tensor product. International Journal Of Mathematics And Computer Research, 11(11): 3837–3842.

[6] Daniel, B., Musundi, S., and Ndungu, K. (2022). An application of maximal numerical range on norm of basic elementary operator in tensor product. Journal of Progressive Research in Mathematics, 19(1): 73–81.

[7] King'ang'i, D. N., Agure, J. O., and Nyamwala, F. O. (2014). On the norm of elementary operator. http://dx.doi.org/10.4236/apm.2014.47041

[8] Kittaneh, F. (1995). Normal derivations in norm ideals. Proceedings of the American Mathematical Society, 123(6): 1779–1785. https://doi.org/10.2307/2160991

[9] Muholo, J., Bonyo, J., and Agure, J. (2019). Norm properties of generalized derivations on norm ideals.

[10] Muiruri, P., King'ang'i, D., & Musundi, S. (2019). On the Norm of Basic Elementary Operator in a Tensor Product.

[11] Nyamwala, F. and Agure, J. (2008). Norms of elementary operators in banach algebras. Int. Journal of Math. Analysis, 2(9): 411–424.

[12] Okelo, N., A. J. and Ambongo, D. (2010). norm of elementary operator and characterization of norm-attained operators. int. Journal of Math. Analysis, 4(11971204).

[13] Society, L. M. (1926). Journal of the London Mathematical Society, volume 1. London Mathematical Society.

[14] Stampfli, J. (1970). The norm of a derivation. Pacific journal of mathematics, 33(3), 737-747.

[15] Timoney, R. M. (2007). Some formulae for norms of elementary operators. Journal of Operator Theory, pages 121–145. https://www.jstor.org/stable/24715761