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AXE-Filtrations of Semi-modules, Generalized Samuel Number $\overline{v}_{\varphi}(\theta)$, and Valuative Reduction

Kouadjo Pierre Brou*, Edjabrou Ulrich Blanchard Kablam, Grah Gelin Bogui

Applied Fundamental Sciences Department, Nangui Abrogoua University, Abidjan, Ivory Coast

Email address:
brou.kouadjopierre@yahoo.fr (Kouadjo Pierre Brou), bkablam11@gmail.com (Edjabrou Ulrich Blanchard Kablam),
gelinbogui10@gmail.com (Grah Gelin Bogui)

*Corresponding author

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Abstract: In this work, we develop an asymptotic theory for axe-filtrations within the broader framework of semi-modules, aiming to generalize the classical theory of filtrations from rings to these algebraic structures where addition is not necessarily invertible. This extension is crucial as semi-modules and semirings appear naturally in fields like geometry and computer science. Our first contribution is the introduction of the concept of an axe-filtration on a semi-module, which adapts the notion of a sequence of powers of an ideal in a ring. We then define a generalized Samuel number, denoted $\bar{v}_{\varphi}(\theta)$, designed to measure the relative asymptotic growth between two distinct axe-filtrations, φ and θ. The main result of this paper establishes a fundamental theorem: the existence of this Samuel number is guaranteed under the condition that one axe-filtration, φ, is a valuative reduction of the other, θ. This key finding extends the classical results of D. Rees by connecting the asymptotic behavior of these generalized filtrations to the well-established theory of discrete valuations. By doing so, we lay the groundwork for a robust asymptotic theory applicable to semi-modules. This work provides new analytical tools for studying non-symmetrizable algebraic structures and opens avenues for further research into the geometric and algebraic properties of systems described by semirings.

Keywords: Semiring, Semi-module, Filtration, Axe-filtration, Generalized Samuel Number, Valuative Reduction, Commutative Algebra

1. Introduction

The theory of filtrations, pioneered by D. Rees [14], is a fundamental tool in commutative algebra for analyzing rings and modules. This work extends these concepts to the broader framework of semi-modules over semirings algebraic structures with non-invertible addition that are important to fields like geometry and computer science [3, 5]. Building upon prior generalizations [9–12], we aim to adapt the techniques of asymptotic theory to these new structures, following recent lines of research [1, 2, 4, 6, 7]. Our first contribution is the introduction of the *axe-filtration on semi-module*, a structure for semi-modules that generalizes the sequence of powers of an ideal in a ring. This concept builds on foundational work in pseudo-valuations and generalized filtrations [10, 13]. Our second and main contribution is the definition of a *generalized Samuel number*, $\bar{v}_{\varphi}(\theta)$, which quantifies the relative growth rate between two axe-filtrations. We prove that this number, defined via a limit, always exists under the condition that one axe-filtration is a *valuative reduction* of the other. This key result extends classical results by Rees, linking the asymptotic behavior of filtrations to the theory of valuations and concepts like integral closure [8].

2. Preliminaries and Definitions

2.1. Semirings and Semi-modules

Definition 2.1. 1. A *monoid* is a set B equipped with an internal composition law $*$ that is associative, commutative, and has an identity element denoted by e_B.

2. Let $(B, *)$ be a monoid and A be a subset of B.
   We say that A equipped with the law $*$ of B is a *submonoid* of B if $(A, *)$ is a monoid such that $e_A = e_B$.
3. A *pre-semiring* is a set B equipped with an addition and a multiplication, for each of which it is a monoid.
4. A *semiring* is a set B equipped with an addition and a multiplication satisfying:
   (i) $(B, +, .)$ is a pre-semiring;
   (ii) Multiplication is distributive with respect to addition;
   (iii) 0 is an annihilator for B.
5. Let $(B, +, .)$ be a semiring and A be a subset of B. We say that A is a *subsemiring* of B if $(A, +, .)$ is a semiring such that $0_A = 0_B$ and $1_A = 1_B$.

Example 2.2. 1. Every commutative unitary ring is a semiring.

2. The sets $(\mathbb{N}, +, .), (\mathbb{R}_+, +, .), (\mathbb{Q}_+, +, .)$ are semirings. Thus, $\mathbb{N}$ is a subsemiring of $\mathbb{Q}_+$ and $\mathbb{Q}_+$ is a subsemiring of $\mathbb{R}_+$.
3. $\sqrt{2}\mathbb{N}$ is a submonoid of $(\mathbb{R}_+, +)$.

Definition 2.3. Let B be a semiring.

1. Let $(M, +)$ be a monoid.
   We say that M is a semi-B-module if the map $(b; x) \mapsto bx$ from $B \times M$ to M satisfies the following conditions:
   (a) $\forall b \in B$ and $\forall x_1, x_2 \in M, b(x_1+x_2) = bx_1+bx_2$;
   (b) $\forall b_1, b_2 \in B$ and $\forall x \in M, (b_1+b_2)x = b_1x+b_2x$;
   (c) $\forall b_1, b_2 \in B$ and $\forall x \in M, b_1(b_2x) = (b_1b_2)x$;
   (d) $\forall x \in M, 1_B \cdot x = x$;
   (e) $\forall b \in B, \forall x \in M, 0_B \cdot x = b \cdot 0_M = 0_M$.
2. Let $(M, +, .)$ be a semi-B-module and N be a subset of M.
   We say that N is a subsemi-module of M if $(N, +, .)$ is a semi-B-module with $0_N = 0_M$.

Example 2.4. 1. $\mathbb{Z}_-, \mathbb{Q}_-$ and $\mathbb{R}_-$ are semi-$\mathbb{N}$-modules.

2. $\sqrt{2}\mathbb{Z}_-$ is a subsemi-module of the semi-$\mathbb{N}$-module $\mathbb{R}_-$.

2.2. Quasi-filtrations and Axe-filtrations

Definition 2.5. Let B be a semiring.

1. We say that $g = (G_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ is a quasi-filtration of B if:
   (a) $G_0 = B$ is a subsemiring of B;
   (b) $G_\infty = (0)$;
   (c) $G_nG_m \subseteq G_{m+n}, \forall m, n \in \mathbb{N}$.
   (d) $G_{n+1} \subseteq G_n, \forall n \in \mathbb{N}^*$.
2. A quasi-filtration $g = (G_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ of the semiring B is called *AP (Approximable by Powers)* if there exists a sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that:
   (a) $G_{mk_n} \subseteq G_n^m, \quad \forall m, n \in \mathbb{N}$;
   (b) $\lim_{n\to+\infty} \frac{k_n}{n} = 1$.

Example 2.6. 1. All filtrations of a semiring B are quasi-filtrations of B.

2. Let $B = \mathbb{R}_+$ and $A = \mathbb{N}$.
   A is a subsemiring of the semiring B. If p is a prime number, then by taking $F = p\mathbb{N} + \sqrt{p}\mathbb{N}$, we have that F is a submonoid of $(B, +)$ such that $F^{n+1} \subseteq F^n$, $\forall n \in \mathbb{N}^*$.
   Thus $g = (G_n = F^n)_{n\in\mathbb{N}\cup\{+\infty\}}$, where $G_\infty = (0)$ and $F^0 = A$, is a quasi-filtration of the semiring B that is not a filtration of B.

3. Study of Axe-filtrations on a Semi-module

Definition 3.1. Let B be a semiring and M be a semi-B-module.

1. Let $\varphi = (M_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ be a sequence of submonoids of $(M, +)$. We say that φ is an axe-filtration of M if there exists a subsemiring A of B such that $M_{n+1} \subseteq AM_n, \forall n \in \mathbb{N}$ and $M_\infty = (0)$.
2. Let A be a subsemiring of B. $AF_A(M)$ is the set of axe-filtrations of the semi-B-module M relative to A, i.e., $\varphi = (M_n) \in AF_A(M) \implies M_{n+1} \subseteq AM_n, \forall n \in \mathbb{N}$.
   The set $AF_A(M)$ is ordered by the relation:
   $\varphi = (M_n) \leq \psi = (N_n)$, if $M_n \subseteq N_n$ for all $n \geq 0$.
3. Let $f = (F_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ be a quasi-filtration of the semiring B such that $F_0 = A$ and $\varphi = (M_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ be an axe-filtration of M.
   (a) We say that φ is axe-compatible with f if $F_mM_n \subseteq AM_{n+m}, \forall m, n \in \mathbb{N}$.
   (b) The axe-filtration φ of M is said to be weakly f-good if:
      i. φ is axe-compatible with f.
      ii. $\exists m \geq 1$ such that: $AM_n = \sum_{p=0}^{m} F_{n-p}M_p$, $\forall n \geq m$.
4. For any real number $\lambda > 0$, $\{\lambda\}$ is the smallest natural number greater than or equal to λ. If $\varphi = (M_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ is an axe-filtration of the semi-B-module M, then $\varphi^{(\lambda)} = (M_{\{n\lambda\}})_{n\in\mathbb{N}\cup\{+\infty\}}$ is an axe-filtration of M.
5. If $\varphi = (M_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ is an axe-filtration of M and $a \in \mathbb{N}$, we set
   $t_a\varphi = (M'_n)_{n\in\mathbb{N}\cup\{+\infty\}}$, where $M'_0 = M_0$ and $M'_n = M_{a+n}$ if $n \geq 1$.
   Then $t_a\varphi$ is an axe-filtration of M called the truncated axe-filtration of φ.

Example 3.2. 1. All filtrations of a semi-module M are axe-filtrations of M.

2. Let $B = \mathbb{R}_+$ and $M = \mathbb{R}_-$; $A = \mathbb{N} + \sqrt{2}\mathbb{N}$.
   $\varphi = (M_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ where $M_n = \sqrt{2}^n\mathbb{Z}_-$ and $M_\infty = (0)$. It is easily shown that φ is an axe-filtration of the semi-B-module M.
   It should be noted that φ is not a filtration of M.

4. Result

4.1. The Generalized Samuel Number

Proposition 4.1. Let B be a semiring, M be a semi-B-module, and $f = (F_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ be a quasi-filtration of B

such that $F_0 = A$. Then the sequence $g = (F_nM)_{n\in\mathbb{N}\cup\{+\infty\}}$ is an axe-filtration of M, called the axe-filtration of the semi-B-module M relative to the quasi-filtration f of B, denoted by f_M or fM.

Proof. (i) Since M is a semi-B-module and F_n is a submonoid of B for all $n \geq 0$, then F_nM is a submonoid of $(M, +)$ for all $n \geq 0$.

(ii) Since $F_0 = A$ is a subsemiring of B and $F_{n+1}M \subseteq AF_nM, \forall n \in \mathbb{N}$, then f_M is an axe-filtration of M.

Notation 4.2. Let B be a semiring, M be a semi-B-module, $\varphi = (M_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ be an axe-filtration of the semi-B-module M, and N be a submonoid of $(M, +)$.

Then there exists a subsemiring A of B such that $M_{n+1} \subseteq AM_n, \forall n \geq 1$. We set:

1. $v_\varphi(N) = \sup\{r \in \mathbb{N}; N \subseteq AM_r\}$, where $v_\varphi(N) = 0$ if $\{r \in \mathbb{N}; N \subseteq AM_r\}$ is empty.
2. If $\theta = (N_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ is an axe-filtration of M, then $\bar{v}_\varphi(\theta) = \lim_{n\to+\infty} \frac{1}{n} v_\varphi(N_n)$ if this limit exists.
3. If G is a submonoid of the semiring B, then we set:
   (a) $\overline{v}_{FM}(GM) = \overline{v}_{f_FM}(f_GM)$;
   (b) $\overline{v}_{FM}(gM) = \overline{v}_{f_FM}(g_M)$

   for any submonoid F of $(B, +)$ and for any quasi-filtration g of B.

Remark 4.3. Let $\varphi, \theta, \varphi^{'}$, and θ' be axe-filtrations of the semi-B-module M such that $\overline{v}_\varphi(\theta)$, $\overline{v}_\varphi(\theta^{'})$, and $\overline{v}_{\varphi'}(\theta)$ exist in $\mathbb{R}_+$. Then we have:

1. $\theta \leq \theta^{'} \Rightarrow \overline{v}_\varphi(\theta) \geq \overline{v}_\varphi(\theta^{'})$.
2. $\varphi \leq \varphi^{'} \Rightarrow \overline{v}_\varphi(\theta) \leq \overline{v}_{\varphi'}(\theta)$.

Remark 4.4. If $\varphi = (M_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ and $\theta = (F_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ are axe-filtrations of the semi-B-module M and $\lambda > 0$ is a real number, then the following assertions are equivalent:

1. $\overline{v}_\varphi(\theta)$ exists in $\overline{\mathbb{R}}_+$.
2. $\overline{v}_{\varphi^{(\lambda)}}(\theta)$ exists in $\overline{\mathbb{R}}_+$.
3. $\overline{v}_\varphi(\theta^{(\lambda)})$ exists in $\overline{\mathbb{R}}_+$.

If one of the equivalent assertions holds, then we have: $\overline{v}_\varphi(\theta) = \lambda\overline{v}_{\varphi^{(\lambda)}}(\theta) = \frac{1}{\lambda}\overline{v}_\varphi(\theta^{(\lambda)})$.

Proposition 4.5. Let $f = (F_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ be a quasi-filtration of the semiring B, G be a submonoid of B, and M be a semi-B-module. Then the sequence $(u_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ with general term $\dfrac{v_{fM}(G^nM)}{n}$ converges in $\overline{\mathbb{R}}_+$. We denote its limit by $\overline{v}_{fM}(GM)$.

Proof. 1) First, note that for all $n \in \mathbb{N}$, $v_{fM}(G^nM) \geq nv_{fM}(GM)$. Indeed, a) if $v_{fM}(GM) = +\infty$, then $n \neq 0$ implies $G^nM \subseteq GM \subseteq F_pM$ for all $p \in \mathbb{N}$, hence $v_{fM}(G^nM) = +\infty$ and the result holds. b) If $v_{fM}(GM) \in \mathbb{N}^*$, then we deduce that $G^nM \subseteq (F_{v_{fM}(GM)})^nM$, hence $G^nM \subseteq F_{nv_{fM}(GM)}M$, and consequently $v_{fM}(G^nM) \geq nv_{fM}(GM)$.

2) Now let's show that the sequence $(u_n)_{n\in\mathbb{N}^*}$ converges. Let $\overline{u}$ (resp. $\underline{u}$) be the upper limit (resp. lower limit) of the sequence u_n. We need to show that $\underline{u} = \overline{u}$. If $\underline{u} = +\infty$ or $\overline{u} = 0$, it is clear. So we assume $\underline{u}$ is finite and $\overline{u} > 0$. By definition, we have:

$(\star)$ $\forall\varepsilon > 0, \forall i \in \mathbb{N}, \exists j \geq i$; $u_j \leq \underline{u} + \varepsilon$. $(\star\star)$ $\forall\varepsilon, \forall i \in \mathbb{N}, \exists j \geq i$; $u_j \geq \overline{u} - \varepsilon$ (if $\overline{u} < +\infty$). $(\star\star\star)$ $\forall N \in \mathbb{N}, \forall i \in \mathbb{N}, \exists j \geq i; u_j \geq N$ (if $\overline{u} = +\infty$).

Fix $\varepsilon > 0$ (and if $\overline{u} = +\infty$ a natural number N) and choose an index i large enough so that $\frac{1}{i} < \frac{\varepsilon}{\overline{u}-\varepsilon}$ (resp. $\frac{\varepsilon}{N}$). By $(\star\star)$ (resp. $\star\star\star$), there exists $j \geq i$ such that $u_j \geq \overline{u} - \varepsilon$ (resp. N) and by $(\star)$, there exists $k \geq ij$ such that $u_k \leq \overline{u} + \varepsilon$. Divide k by j; $k = jl + q$, where $0 \leq q < j$, $q \in \mathbb{N}$. We then have $\overline{u} + \varepsilon \geq \frac{v_{fM}(G^{lj+q}M)}{jl+q} \geq \frac{v_{fM}(G^{lj}M)}{jl+q} \geq \frac{lv_{fM}(G^jM)}{jl+q} = \frac{v_{fM}(G^jM)}{j}\frac{lj}{k} = \frac{v_{fM}(G^jM)}{j}\frac{(k-q)}{k}$. but $q < j$ and $\frac{1}{k} \leq \frac{1}{ji}$ implies $\frac{q}{k} \leq \frac{1}{j}$. and we have:

$$1 - \frac{q}{k} > 1 - \frac{1}{i}$$

hence $\underline{u} + \varepsilon \geq (\overline{u} - \varepsilon)(1 - \frac{1}{i}) = \overline{u} - \varepsilon - \left(\frac{\overline{u}-\varepsilon}{i}\right) \geq \overline{u} - 2\varepsilon$ (resp. $N(1 - \frac{1}{i}) \geq N - \varepsilon$). Consequently, we have $\underline{u} + \varepsilon \geq \overline{u} - 2\varepsilon$ (resp. $\underline{u} + \varepsilon \geq N - \varepsilon$). Therefore, $\underline{u} = \overline{u}$.

Proposition 4.6. Let $f = (F_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ and $g = (G_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ be two quasi-filtrations of the semiring B and let M be a semi-B-module. Then:

1. The sequence $\left(\dfrac{\overline{v}_{fM}(G_nM)}{n}\right)_{n\in\mathbb{N}}$ converges in $\overline{\mathbb{R}}_+$. And we have $\overline{\lim}\,\dfrac{v_{fM}(G_nM)}{n} \leq \lim\limits_{n\to+\infty} \dfrac{\overline{v}_{fM}(G_nM)}{n}$.
2. if g is AP, then: $\overline{v}_{fM}(gM)$ exists in $\overline{\mathbb{R}}_+$, and we have $\overline{v}_{fM}(gM) = \lim_{n\to+\infty} \frac{\overline{v}_{fM}(G_nM)}{n}$.

Proof. i) If there exists $n_0 \in \mathbb{N}$ such that $\overline{v}_{fM}(G_{n_0}M) = +\infty$, then $\overline{v}_{fM}(G_nM) = +\infty$ for all $n \geq n_0$ and we have $\lim_{n\to+\infty} \frac{1}{n}\overline{v}_{fM}(G_nM) = +\infty$. Suppose now that $\overline{v}_{fM}(G_nM) < +\infty$ for all n. Then $v_{fM}(G_nM) < +\infty$. Let $(k, n) \in \mathbb{N}^2 - \{(0, 0)\}$. By Euclidean division, we have $n = q_nk + r_n$ with $0 \leq r_n < k$.

Thus $G_k^{q_n+1}M \subseteq G_{q_nk+k}M = G_{(q_n+1)k}M \subseteq G_nM$. For all $m \in \mathbb{N}^*$, we have $\frac{1}{m}v_{fM}(G_n^mM) \leq (q_n + 1)\frac{1}{(q_n+1)m}v_{fM}(G_k^{(q_n+1)m}M)$. As $m \to +\infty$, we get $\frac{\overline{v}_{fM}(G_nM)}{n} \leq \frac{q_n+1}{n}\overline{v}_{fM}(G_kM)$ and as $n \to +\infty$, we obtain $\overline{\lim}\,\frac{\overline{v}_{fM}(G_nM)}{n} \leq \frac{\overline{v}_{fM}(G_kM)}{k}$ for all $k \in \mathbb{N}^*$. Thus, the sequence $\left(\frac{\overline{v}_{fM}(G_nM)}{n}\right)_{n\in\mathbb{N}}$ converges in $\overline{\mathbb{R}}_+$. Since $v_{fM}(G_nM) \leq \overline{v}_{fM}(G_nM)$ for all n, we have i).

ii) For ii), it suffices to show that $\lim_{n\to+\infty} \frac{1}{n}\overline{v}_{fM}(G_nM) \leq \underline{\lim}\,\frac{1}{n}v_{fM}(G_nM)$ and then use i). Since g is AP, there exists a sequence of integers $(k_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ such that $G_{k_nm}M \subseteq G_n^mM$ for all n, m with $\frac{k_n}{n} \to 1$ as $n \to +\infty$. If we set $m = k_nq_m + r_n$ with $0 \leq r_m \leq k$. We have $G_{k_n}^{q_m+1}M \subseteq G_mM \subseteq G_{k_nq_m}M \subseteq G_n^{q_m}M$. Consequently, $\frac{q_m}{m}\frac{v_{fM}(G_n^{q_m}M)}{q_m} \leq \frac{v_{fM}(G_mM)}{m}$. As $m \to +\infty$, we get $\frac{n}{k_n}\frac{\overline{v}_{fM}(G_nM)}{n} \leq \underline{\lim}\,\frac{v_{fM}(G_mM)}{m}$. As $n \to +\infty$, we have ii).

Lemma 4.7. Let $f = (F_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ and $g = (G_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ be quasi-filtrations of the semiring B, and let M be a semi-B-module. Then we have $\overline{v}_{FM}(GM) > 1 \Rightarrow G^sM \subset F^sM$ for s large enough.

Proof. Let $l = \overline{v}_{FM}(GM) = \lim_{s\to+\infty} \frac{1}{s}v_{FM}(G^sM)$. We have $1 \leq l - \varepsilon < \frac{1}{s}v_{FM}(G^sM) \leq l + \varepsilon$ for all s

and for all ε. We deduce $\frac{1}{s}v_{FM}(G^sM) > 1$, which means $v_{FM}(G^sM) > s$. Hence $G^sM \subset F^sM$.

Proposition 4.8. Let $f = (F_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ and $g = (G_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ be two AP quasi-filtrations of the semiring B and let M be a semi-B-module. Then the sequence $(n\overline{v}_{F_nM}(gM))_{n\in\mathbb{N}\cup\{+\infty\}}$ converges in $\overline{\mathbb{R}}_+$ and we have:

$$\overline{v}_{fM}(gM) = \lim_{n\to+\infty} n\overline{v}_{F_nM}(gM).$$

Proof. We have $\overline{v}_{f_FM}(gM)$ for any submonoid F. Let's show that $m\overline{v}_{f_FM}(gM) \leq \overline{v}_{F_mM}(gM)$ for all $m \geq 1$. We can assume that $m\overline{v}_{f_FM}(gM) \neq 0$. Let p and q be integers greater than or equal to 1 such that $\frac{p}{q} < m\overline{v}_{F_mM}(gM)$. For n large enough, we have $\frac{m}{n}\overline{v}_{F_mM}(G_nM) > \frac{p}{q}$. Thus $G_n^{mqs}M \subseteq F_{mpns}M$ for s large enough (Lemma 4.7). It follows that $\overline{v}_{fM}(gM) \geq \frac{p}{q}$, hence $\overline{v}_{fM}(gM) > \overline{\lim}\, n\overline{v}_{F_nM}(gM)$. On the other hand, let p and q be integers greater than or equal to 1 such that $\frac{p}{q} < \overline{v}_{fM}(gM)$. Then $G_m^qM \subseteq F_{mp}M$ for m large enough. Let $(k_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ be a sequence of positive integers such that $F_{k_nm}M \subset F_n^mM$ for all m, n with $\lim\limits_{n\to+\infty}\dfrac{k_n}{n} = 1$. Set $m = k_nq_m + r_m$, $0 \leq r_m < k_n$. Then for m large enough, $G_m^qM \subseteq F_{mp}M \subseteq F_n^{q_mp}M$. So if $m \to +\infty$, we have $\overline{v}_{F_nM}(gM) \geq \frac{1}{k_n}\cdot\frac{p}{q}$, and if $n \to +\infty$, we obtain $\underline{\lim}\, n\overline{v}_{F_nM}(gM) \geq \frac{p}{q}$, which gives the result.

4.2. Valuative Reductions and Main Theorem

This notion of valuative reduction was introduced by D. Rees [14].

The purpose of this part is to link the Samuel number v_φ of two axis-filtrations on a semi-module by an integer and a comparison of axis-filtrations.

Definition 4.9. Let B be a semiring, $\varphi = (M_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ and $\theta = (U_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ be axe-filtrations of the semi-B-module M. For any $x \in M$, we set $v_\varphi(x) = \sup\{n \in \mathbb{N},\ x \in M_n\}$. We say that the axe-filtration φ is a valuative reduction of the filtration θ if there exists a constant $a \in \mathbb{N}$ such that for all $x \in M$ we have: $0 \leq v_\theta(x) - v_\varphi(x) \leq a$ with $v_\varphi(x) = +\infty$ if and only if $v_\theta(x) = +\infty$.

Lemma 4.10. Let B be a semiring, $\varphi = (M_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ be an axe-filtration of the semi-B-module M, and N be a submonoid of $(M, +)$. The following assertions are equivalent:

1. $v_\varphi(N) = +\infty$;
2. for all $a \in \mathbb{N}, v_{t_a\varphi}(N) = +\infty$;
3. there exists $a \in \mathbb{N}$ such that $v_{t_a\varphi}(N) = +\infty$.

Proof. a) Let's show that i) implies ii). If $v_\varphi(N) = +\infty$, then $N \subseteq M_p$ for all $p \in \mathbb{N}$. Thus, for all $a \in \mathbb{N}$, $N \subseteq M_{a+q}$ for all $q \in \mathbb{N}$, hence $v_{t_a\varphi}(N) = +\infty$.

b) ii) implies iii) is obvious.

c) Let's show that iii) implies i). Let $a \in \mathbb{N}$ be such that $v_{t_a\varphi}(N) = +\infty$. Then $N \subseteq M_{a+q}$ for all $q \in \mathbb{N}^*$, and we have $N \subseteq M_p$ for all $p \geq a$. We deduce that $N \subseteq M_p$ for all $p \in \mathbb{N}$, thus $v_\varphi(N) = +\infty$.

Theorem 4.11. Let B be a semiring, $\varphi = (M_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ and $\theta = (U_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ be axe-filtrations of the semi-B-module M. The following assertions are equivalent:

1. φ is a valuative reduction of θ;
2. there exists an integer $b \geq 0$ such that we have $t_b\theta \leq \varphi \leq \theta$.

Proof. a) Let's show that i) implies ii). Assume i) is true. Then there exists $a \in \mathbb{N}$ such that $0 \leq v_\theta(x) - v_\varphi(x) \leq a$ for all $x \in M$. We have $v_\varphi(x) \geq v_\theta(x) - a$. If $n \geq a$, then $x \in U_n$ implies $v_\theta(x) \geq n$, so $v_\varphi(x) \geq n - a$, hence $x \in M_{n-a}$. Since $v_\varphi(x) \geq v_\theta(x)$, then $M_n \subseteq U_n$ for all $n \in \mathbb{N}$. For all $n \geq a$, we have: $M_n \subseteq U_n \subseteq M_{n-a} \subseteq U_{n-a}$. Letting $n \geq a$, we have $U_{p+a} \subseteq M_p \subseteq U_p$ for all $p \in \mathbb{N}$, hence $t_a\theta \leq \varphi \leq \theta$, and we deduce that i) implies ii).

b) Now let's show that ii) implies i). Assume ii) is true. Then there exists $b \in \mathbb{N}$ such that $t_b\theta \leq \varphi \leq \theta$. Hence we have $v_{t_b\theta}(x) \leq v_\varphi(x) \leq v_\theta(x)$ for all $x \in M$. (1) If $v_\varphi(x) = +\infty$, then $v_\theta(x) = +\infty$ (from (1)). 1) If $v_\theta(x) = +\infty$, then $v_{t_b\theta}(x) = +\infty$ (Lemma 4.10). (1) then implies $v_\varphi(x) = +\infty$; consequently, we have: $v_\varphi(x) = +\infty$ if and only if $v_\theta(x) = +\infty$. We then deduce that $v_\varphi(x) \in \mathbb{N}$ if and only if $v_\theta(x) \in \mathbb{N}$. (2) If $v_\theta(x) \in \mathbb{N}$, let's show that $v_\theta(x) \leq v_{t_b\theta}(x) + 1 + b$. Let $v_{t_b\theta}(x) = r \in \mathbb{N}$. If $r \neq 0$, then $x \in U_{b+r}$ and $x \notin U_{b+r+1}$, we have $v_{t_b\theta}(x) + b \leq v_\theta(x) \leq v_{t_b\theta}(x) + 1 + b$. $(\star)$ 2) If $r = 0$, then $x \in U_0 = M$ and $x \notin U_{b+1}$, we have $0 = v_{t_b\theta}(x) \leq v_\theta(x) \leq v_{t_b\theta}(x) + 1 + b$. $(\star\star)$ $(\star)$ and $(\star\star)$ imply $v_\theta(x) < v_{t_b\theta}(x) + 1 + b$. (3) (1) and (3) imply $0 \leq v_\theta(x) - v_\varphi(x) \leq v_\theta(x) - v_{t_b\theta}(x) < 1 + b$, so we have $0 \leq v_\theta(x) - v_\varphi(x) \leq v_\theta(x) - v_{t_b\theta}(x) \leq b$. (4) (4) implies that φ is a valuative reduction of θ, and the result is proven.

Lemma 4.12. Let B be a semiring, $\varphi = (M_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ be an axe-filtration of the semi-B-module M, and N be a submonoid of $(M, +)$. The following assertions are equivalent:

1. $v_\varphi(N) = +\infty$;
2. for all $k \in \mathbb{N}^*, v_{\varphi^{(k)}}(N) = +\infty$;
3. there exists $k \in \mathbb{N}^*$ such that $v_{\varphi^{(k)}}(N) = +\infty$.

Proof. a) Let's show that i) implies ii). If $v_\varphi(N) = +\infty$, then $N \subseteq M_n$ for all $n \in \mathbb{N}$, hence $\forall k \in \mathbb{N}^*, N \subseteq M_{kn}$ for all $n \in \mathbb{N}$, so $v_{\varphi^{(k)}} = +\infty$.

b) ii) implies iii) is obvious.

c) Let's show that iii) implies i). If there exists $k \in \mathbb{N}^*$ such that $v_{\varphi^{(k)}}(N) = +\infty$, then $N \subseteq M_{kn}$ for all $n \in \mathbb{N}$. Consequently, $v_\varphi(N) \geq kn$ for all $n \in \mathbb{N}$, hence $v_\varphi(N) = +\infty$.

4.3. Generalized Samuel Numbers and Valuative Reductions

Lemma 4.13. Let B be a semiring, $\varphi = (M_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ be an axe-filtration of the semi-B-module M, and N be a subsemi-module of M. The following assertions are equivalent:

1. $w_\varphi(N) = +\infty$;
2. for all $a \in \mathbb{N}, w_{t_a\varphi}(N) = +\infty$;
3. there exists $a \in \mathbb{N}$ such that $w_{t_a\varphi}(N) = +\infty$.

Proof. a) Let's show that i) implies ii). If $w_\varphi(N) = +\infty$, then there is no $r \in \mathbb{N}$ such that $M_r \subseteq N$. Suppose $s = w_{t_a\varphi}(N) \neq +\infty$. Then we have $M_{s+a} \subset N$ with $s \in \mathbb{N}$, which implies $w_\varphi(N) \leq s + a$, a contradiction.

b) ii) implies iii) is obvious.

Proposition 4.14. Let B be a semiring, $\varphi = (M_n)_{n\in\mathbb{N}\cup\{+\infty\}}$, $\theta = (F_n)_{n\in\mathbb{N}\cup\{+\infty\}}$, and $\eta = (E_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ be axe-filtrations of the semi-B-module M. If φ is a valuative reduction of θ, then:

$$\overline{v}_\eta(\theta) \text{ exists if and only if } \overline{v}_\eta(\varphi) \text{ exists.}$$

We have: $\overline{v}_\eta(\theta)$= $\overline{v}_\eta(\varphi)$ if the defined terms exist.

Proof. Since the axe-filtration φ is a valuative reduction of the axe-filtration θ, there exists $a \in \mathbb{N}$ such that $F_{n+a} \subseteq M_n \subseteq F_n, \forall n$ (Theorem 4.11), and we have: $F_{n+2a} \subseteq M_{n+a} \subseteq F_{n+a} \subseteq M_n \subseteq F_n$.

a) We have: $\frac{n+2a}{n}\frac{v_\eta(F_{n+2a})}{n+2a} \geq \frac{n+a}{n}\frac{v_\eta(M_{n+a})}{n+a} \geq \frac{n+a}{n}\frac{v_\eta(F_{n+a})}{n+a} \geq \frac{v_\eta(M_n)}{n}$. As $n \to +\infty$, we get i), and if $\overline{v}_\eta(\theta)$ or $\overline{v}_\eta(\varphi)$ exists, we have $\overline{v}_\eta(\theta) = \overline{v}_\eta(\varphi)$.

Proposition 4.15. Let $\varphi = (M_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ and $\theta = (F_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ be axe-filtrations of the semi-B-module M. The following assertions are equivalent:

1. $\overline{v}_\varphi(\theta)$ exists in $\overline{\mathbb{R}}_+$.
2. For all $a \in \mathbb{N}$, $\overline{v}_{t_a\varphi}(\theta)$ exists in $\overline{\mathbb{R}}_+$..
3. There exists $a \in \mathbb{N}$ such that $\overline{v}_{t_a\varphi}(\theta)$ exists in $\overline{\mathbb{R}}_+$..

If one of the following assertions holds, then for all $a \in \mathbb{N}$ we have: $\overline{v}_\varphi(\theta) = \overline{v}_{t_a\varphi}(\theta)$.

Proof. a) Let's show that i) implies ii). Suppose there exists $n \in \mathbb{N}$ such that $v_\varphi(F_n) = +\infty$. Then $v_\varphi(F_k) = +\infty, \forall k \geq n$, hence for $a \in \mathbb{N}$, $\forall k \geq n$, we have $v_{t_a\varphi}(F_k) = +\infty$ (Lemma 4.10), and we deduce that i) implies ii). Suppose that $v_\varphi(F_n) \in \mathbb{N}, \forall n \in \mathbb{N}$; then for all $a \in \mathbb{N}, v_{t_a\varphi}(F_n) \in \mathbb{N}$ (Lemma 4.10). Let $v_{t_a\varphi}(F_n) = r \in \mathbb{N}$. 1) If $r \neq 0$, we have $F_n \subseteq M_{a+r}$ and $F_n \not\subseteq M_{a+r+1}$, so $a + v_{t_a\varphi}(F_n) \leq v_\varphi(F_n) < v_{t_a\varphi}(F_n) + 1 + a \quad (\star)$. 2) If $r = 0$, then $F_n \subseteq M_0$ and $F_n \not\subseteq M_{a+1}$. We have $0 = v_{t_a\varphi}(F_n) \leq v_\varphi(F_n) < v_{t_a\varphi}(F_n) + 1 + a \quad (\star\star)$. Inequalities $(\star)$ and $(\star\star)$ imply for all $n \in \mathbb{N}$, $\frac{v_\varphi(F_n)}{n} \leq \frac{v_{t_a\varphi}(F_n)+1+a}{n} \leq \frac{v_\varphi(F_n)+1+a}{n}$. As $n \to +\infty$, we deduce the existence of $\overline{v}_{t_a\varphi}(\theta)$ and we have $\overline{v}_\varphi(\theta) = \overline{v}_{t_a\varphi}(\theta)$.

b) ii) implies iii) is obvious.

c) Let's show that iii) implies i). Let $a \in \mathbb{N}$. If $v_{t_a\varphi}(F_n) = +\infty$, then $v_\varphi(F_n) = +\infty$ (Lemma 4.13), and as in a), we deduce that iii) implies i). Suppose that $v_{t_a\varphi}(F_n) \in \mathbb{N}, \forall n \in \mathbb{N}$; using relations $(\star)$ and $(\star\star)$ from a), we have: $\frac{v_{t_a\varphi}(F_n)}{n} \leq \frac{v_\varphi(F_n)}{n} \leq \frac{v_{t_a\varphi}(F_n)+1+a}{n}$. If $\overline{v}_{t_a\varphi}(\theta)$ exists in $\overline{\mathbb{R}}_+$ and as $n \to +\infty$, we deduce that $\overline{v}_\varphi(\theta)$ exists in $\overline{\mathbb{R}}_+$ and that $\overline{v}_{t_a\varphi}(\theta) = \overline{v}_\varphi(\theta)$.

Proposition 4.16. Let B be a semiring, and let φ, θ, and η be axe-filtrations of the semi-B-module M. If φ is a valuative reduction of θ, then the following assertions are equivalent:

1. $\overline{v}_\theta(\eta)$ exists in $\overline{\mathbb{R}}_+$.
2. $\overline{v}_\varphi(\eta)$ exists in $\overline{\mathbb{R}}_+$.

If one of the equivalent assertions holds, we have: $\overline{v}_\varphi(\eta) = \overline{v}_\theta(\eta)$.

Proof. a) Let's show that i) implies ii). Let $\eta = (E_n)_{n\in\mathbb{N}}$. Since φ is a valuative reduction of θ, there exists $a \in \mathbb{N}$ such that $t_a\theta \leq \varphi \leq \theta$, which implies $\frac{v_{t_a\theta}(E_n)}{n} \leq \frac{v_\varphi(E_n)}{n} \leq \frac{v_\theta(E_n)}{n}$. If $\overline{v}_\theta(\eta)$ exists and $n \to +\infty$, we have $\overline{v}_\theta(\eta) = \lim_{n\to+\infty}\frac{v_{t_a\theta}(E_n)}{n} = \lim_{n\to+\infty}\frac{v_\varphi(E_n)}{n}$ (Proposition 4.15). We deduce that $\lim_{n\to+\infty}\frac{v_\varphi(E_n)}{n}$ exists and that $\overline{v}_\varphi(\eta) = \overline{v}_\theta(\eta)$.

b) Let's show that ii) implies i). Since φ is a valuative reduction of θ, there exists $a \in \mathbb{N}$ such that $t_a\theta \leq \varphi \leq \theta$, which implies $t_a\varphi \leq t_a\theta \leq \varphi$. As in a), we can show that if $\overline{v}_\varphi(\eta)$ exists, then $\overline{v}_\theta(\eta)$ exists (Proposition 4.15), and we have $\overline{v}_\varphi(\eta) = \overline{v}_\theta(\eta)$.

Theorem 4.17. Let B be a semiring, φ and θ be axe-filtrations of the semi-B-module M, and $f = (F_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ and $g = (G_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ be quasi-filtrations of B. If φ is weakly f-good, θ is weakly g-good, and g is AP, then $\overline{v}_\varphi(\theta)$ exists in $\overline{\mathbb{R}}_+$ and we have $\overline{v}_\varphi(\theta) = \overline{v}_{fM}(gM) = \lim_{n\to+\infty}\frac{\overline{v}_{fM}(G_nM)}{n}$.

Proof. If $\varphi = (M_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ is weakly f-good, then fM is a valuative reduction of φ. Since φ is weakly f-good, we have $F_pM_q \subseteq M_{p+q}$ for all p, q, and there exists an integer $m \geq 1$ such that $M_n = \sum_{p=0}^m F_{n-p}M_p$ for all $n > m$, and $M_n \subseteq F_{n-m}M$ for all $n > m$. Therefore, $M_{n+m} \subseteq F_nM \subset M_n$ for all $n \in \mathbb{N}$. $t_m\varphi \leq fM \leq \varphi$, and consequently, fM is a valuative reduction of φ (Theorem 4.11). If g is AP, then $\overline{v}_{fM}(gM)$ exists in $\overline{\mathbb{R}}_+$ (Proposition 4.5) and we have $\overline{v}_{fM}(gM) = \overline{v}_\varphi(gM)$ (Proposition 4.16). Since θ is weakly g-good, gM is a valuative reduction of θ. Using Proposition 4.16, we obtain $\overline{v}_{fM}(gM) = \overline{v}_\varphi(gM) = \overline{v}_\varphi(\theta)$; hence the existence of $\overline{v}_\varphi(\theta)$ in $\overline{\mathbb{R}}_+$ and we have $\overline{v}_\varphi(\theta) = \overline{v}_{fM}(gM) = \lim_{n\to+\infty}\frac{\overline{v}_{fM}(G_nM)}{n}$ (Proposition 4.6).

4.4. Examples

Let $B = \mathbb{R}_+$, $A = \mathbb{N} + \sqrt{2}\mathbb{N}$ and $M = \mathbb{R}_-$. A is a subsemiring of B and M is a semi B-module.

(1) Let $M_n = \sqrt{2}^{\{\frac{n}{2}\}+1}\mathbb{Z}_-$, $N_n = \sqrt{2}^{n+1}\mathbb{Z}_-$ for all $n \in \mathbb{N}$ and $M_\infty = N_\infty = \{0\}$. Let $\varphi = (M_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ and $\theta = (N_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ be families of sub-monoids of M.

(i) We have: $M_{n+1} = \sqrt{2}^{\{\frac{n+1}{2}\}+1}\mathbb{Z}_-$ and $AM_n = \sqrt{2}^{\{\frac{n}{2}\}+1}\mathbb{Z}_- + \sqrt{2}^{\{\frac{n}{2}\}+2}\mathbb{Z}_-$ for any natural number n. We note that if n is even, then $M_{n+1} = \sqrt{2}^{\{\frac{n}{2}\}+2}\mathbb{Z}_-$ and if n is odd, then $M_{n+1} = \sqrt{2}^{\{\frac{n+1}{2}\}+2}\mathbb{Z}_-$.
In all cases, $M_{n+1} \subseteq AM_n$. Thus φ is an axis-filtration of M.

(ii) We can easily show that θ is also an axis-filtration of M.

(iii) $v_\varphi(N_n) = \sup\{r \in \mathbb{N} \mid N_n \subseteq AM_r\}$
then $v_\varphi(N_n) = 2n$;
hence $\bar{v}_\varphi(\theta) = \lim_{n\to+\infty}\frac{1}{n}v_\varphi(N_n) = 2$.

(2) Let $H_0 = H_1 = H_2 = 2\mathbb{Z}_- + \sqrt{2}\mathbb{Z}_-$ and $H_n = (0), \forall n \geq 3$.

Let $K_n = 2\mathbb{Z}_- + \sqrt{2}\mathbb{Z}_-$, for all $n \in \mathbb{N}$, and $H_\infty = K_\infty$.

The families $\eta = (H_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ and $\Delta = (K_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ are axis-filtrations of M. $v_\eta(K_n) = 2, \quad \forall n \in \mathbb{N}$; thus $\bar{v}_\eta(\Delta) = \lim_{n\to+\infty}\frac{1}{n}v_\eta(K_n) = 0$.

ORCID

0009-0000-5895-2264 (Kouadjo Pierre Brou)
0009-0009-1464-5049 (Edjabrou Ulrich Blanchard Kablam)
0009-0008-7475-2156 (Grah Gelin Bogui)

Abbreviations

AP Approximable by Powers

Author Contributions

Kouadjo Pierre Brou: Conceptualization, Funding acquisition, Project administration, Software, Supervision, Validation
Edjabrou Ulrich Blanchard Kablam: Data curation, Investigation, Resources, Writing - review & editing
Grah Gelin Bogui: Formal Analysis, Methodology, Visualization

Conflicts of Interest

The authors declare no conflicts of interest.

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