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Modular Resolution by Polyseries

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Abstract: We study the modular resolution method using a new tool called a Polyserie introduced by Wildberger N.J. & Rubine D. to prove an equivalence theorem of the existence and the unicity of the solution of the equation of the form $H_t^2(x) - H_t(x) + t \equiv 0 \pmod{t^n}$. These equations admits solutions for a fixed t but when t changes in a range of several values, then the equation becomes parametric and hence the solution should be parametric too. Prof. Wildberger N.J has studied such equations using Catalan numbers sequence using the recurrence formula between their terms $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$, which allow us to easily prove the main result of the present article. We study Wildberger's Polynumber Sequences, Binomial Chu-Vandermonde Identity, Truncated Polyseries and finally Modular Resolution as Application.

Keywords: Truncated Polyseries, Binomial Chu-Vandermonde Identity, Polynumber Sequences, Exponential Euler Polyseries, Newton Polyserie Catalan Numbers, Modular Resolution

1. Discretization

Modular resolutions have long played a central role in understanding the structure of modules over rings with positive characteristic. In this paper, we introduce a novel technique based on polyseries expansions to construct and analyze such resolutions.

Remark 1. Most of the present work is indebted to the wise efforts of Prof. Wildberger N.J. Therefore The author declare no conflicts of interest unless the permission of the diffusion. For more details, see [14–19]

Definition 1 (Informally). A sequence is a set of related events, movements, or items that follow each other in a particular order. See [19]

Example 1.

1. Integer sequences:

$$s := a_0, a_1, a_2, \cdots$$

2. Forward difference operator of a sequence:

$$\Delta(s) := a_1 - a_0, a_2 - a_1, \cdots$$

3. Summation operator of a sequence:

$$\sum(s) := 0, a_0, a_0 + a_1, a_0 + a_1 + a_2$$

Thomas Harriot (1560-1621) was the first to introduce the difference tables

$s = [1\rangle$	$\Delta(s)$	$\Delta^2(s)$	$\Delta^3(s)$	$\Delta^4(s)$	$\Delta^5(s)$
0	1	14	36	24	0
1	15	50	60	24	0
16	65	110	84	24	⋮
81	175	194	108	⋮	⋮
256	369	302	⋮	⋮	⋮
625	671	⋮	⋮	⋮	⋮
1296	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮

1.1. Harriot's Triangular Summation Tables

Example 2. The columns of the ongoing Harriot's polyseries summation table are:

$s=[1\rangle$	$\sum(s)$	$\sum^2(s)$	$\sum^3(s)$	$\sum^4(s)$	$\sum^5(s)$	
1	0	0	0	0	0	$\cdots$
1	1	0	0	0	0	$\cdots$
1	2	1	0	0	0	$\cdots$
1	3	3	1	0	0	$\cdots$
1	4	6	4	1	0	$\cdots$
1	5	10	10	5	1	$\cdots$
1	6	15	20	15	6	$\cdots$
$\vdots$	7	21	35	35	21	$\cdots$
$\vdots$	8	28	56	70	56	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The initial ongoing Harriot's polyserie

$$h_0 \equiv [1\rangle := 1,1,1,\cdots$$

The first summation of it is

$$H_t \equiv [n\rangle := 0,1,1,\cdots$$

its second one is

$$H'_t \equiv \left[\binom{n}{2}\right\rangle = \left[\frac{n\,(n-1)}{2!}\right\rangle := 0,0,1,3,6,10,$$
$$\cdots \quad \text{(Triangular Numbers)}$$

The third is:

$$H''_t \equiv \left[\binom{n}{3}\right\rangle = \left[\frac{n(n-1)(n-2)}{3!}\right\rangle$$
$$:= 0,0,0,1,4,10,20,35,\cdots \quad \text{(Pyramidal Numbers)}$$

The general term of the ongoing Harriot's polyserie is:

$$h_{nk} = \left[\binom{n}{k}\right\rangle = \left[\frac{n\,(n-1)\cdots(n-k+1)}{k!}\right\rangle = \left[\frac{n^{\underline{k}}}{k!}\right\rangle \quad (1)$$

1. $n^{\underline{k}} = n\,(n-1)\cdots(n-k+1)$ is the n to the k falling (Knuth. D. notation)
2. Harriot's Triangular/Binomial sequences $h_0, H_t, H'_t\cdots$, by using the binomial formula (2)

$$h_{nk} \equiv h_k\,(n) = \binom{n}{k}, \quad k,n = 0,1,2,\cdots \quad (2)$$

We get the following table:

$$H = [h_{nk}\rangle = \begin{array}{c|cccccc} & k & & & & & \\ n & & & & & & \\ \hline 1 & 0 & 0 & 0 & \cdots & \cdots \\ 1 & 1 & 0 & 0 & \cdots & \cdots \\ 1 & 2 & 1 & 0 & \cdots & \cdots \\ 1 & 3 & 3 & 1 & \cdots & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \quad (3)$$

Theorem 2 (Harriot's Difference Theorem).

$$\Delta\,(h_k) = h_{k-1} \qquad (\text{For } k = 1,2,3,\cdots) \quad (4)$$

Proof By definition:

$$h_k\,(n) = \binom{n}{k}.$$

So we get:

$$\Delta\,(h_k)\,(n) = h_k\,(n+1) - h_k\,(n) = \binom{n+1}{k} - \binom{n}{k} = \binom{n}{k-1} = h_{k-1}\,(n)$$

Corollary 1. For any sequence $S := a_0, a_1, a_2\cdots$ we can generate difference or summation table.

Example 3. From the sequence:

$$s := 1,1,3,13,37,81,212$$

we generate the following summation table:

0	0	6	2	0	1	0	0	0	
0	0	6	8	2	1	1	0	0	
0	0	6	14	10	3	2	1	0	
0	0	0	20	24	13	5	3	1	
0	0	0	0	44	37	18	8	4	
0	0	0	0	0	81	55	26	12	
0	0	0	0	0	0	131	81	38	
0	0	$\underset{\Leftarrow}{\Delta}$	0	0	0	0	212	219	$\underset{\Rightarrow}{\sum}$

(5)

1.2. Chains

Definition 2. For a fixed counting number k and an integer n, the k-chain on n is

$$c \equiv n, n+1, \cdots n+k-1 \tag{6}$$

Example 4.
1. The 3-chain on 17 is 17,18,19
2. The 1-chain on -51 is -51
3. The 8-chain on -2 is -2,-1,0,1,2,3,4,5

Notation 1.

$$c\left(5,11\right) := 5,6,7,8,9,10,11 = c$$

1.3. Integral Sequences

Definition 3. An integral $k-$sequence is an explicit assignment of integers to the elements of a k-chain

Example 5. From this table:

Table 1. *Values of S(n) for integer shifts.*

n	-1	0	+1	+2	+3	+4
$s\left(n\right)$	+1	+2	-1	+3	0	+2

We deduce the sequence:latex

$s = 1_{-1}, 2_0, -1_1, 3_2, 0_3, 2_4$

$s_{-1}^{4} = 1, 2, -1, 3, 0, 2$ the limits are: -1 and 4, s_{-1}^{4} the size is 6

Remark 3.
1. The default beginning limit is 0.
2. The term sequence means k-sequence for some counting number k.
3. We only consider finite sequences.

1.4. Clips

Definition 4. A clip is a representation of a (consecutive) portion of a sequence

Example 6. The sequence $s = 1_{-1}, 2_0, -1_1, 3_2, 0_3, 2_4$ has clips:
1. $\ell = 1_{-1}, 2_0, -1_1, 3_2, \cdots$ (left clip)
2. $j = \cdots, 3_2, 0_3, 2_4$ (right clip)
3. $k = \cdots, -1, 3_2, \cdots$ (double clip)
4. $s = 1_{-1}, 2_0, -1_1, 3_2, 0_3, 2_4$ (total clip)

Remark 4.
1. A clip must have at least one element.
2. The default clip kind is the left one starting at 0.
3. A given sequence will have many clips: partial representation of the sequence.

Example 7. From the sequence:

$$s = 10, 9, 8, 7, 6, 5, 4$$

we get:

clip	**assignement data**
$10, 9, 8, \cdots$	right data (accepted)
$\cdots, 7, 6, \cdots$	wrong data (non accepted)
$\cdots, 7_3, 6_4, \cdots$	right data
$\cdots, 4_6$	right
$\cdots$	wrong

1.5. Polynumber Sequences

Example 8. If

$$p = \begin{vmatrix} +2 \\ -1 \\ +3 \end{vmatrix} = 2 - \alpha + 3\alpha^2$$

is a polynumber then we may define sequences using p

$$\begin{aligned} \left[p\left(n\right)\right\rangle_1^4 &= p\left(1\right), p\left(2\right), p\left(3\right), p\left(4\right) \\ &= 4, 12, 26, 46 \end{aligned}$$

Summary 1. We say $\left[p\left(n\right)\right\rangle_1^4 = \left[2 - n + 3n^2\right\rangle_1^4$ is a polynumber (polynomial) sequence.

1.6. Ongoing Polyseries

We want to consider the finite sequences that keep going on, in a limited way (finite computable way).

Example 9. The ongoing polyserie $\left[2 - n + 3n^2\right\rangle_1^+$ is an ongoing sequence for the polynumber $2 - n + 3n^2$ starting from the integer 1

$$\left[2 - n + 3n^2\right\rangle_1^+ := 4, 12, 26, 46, \cdots$$

It shows a clip of the ongoing sequence.

Algorithm 1 (Square root). We search a solution for p of the equation:

$$p^2 = 1 + \alpha$$

such that:

$$\alpha = \left[\begin{array}{cccc} 0 & 1 & 0 & \cdots \end{array}\right\rangle.$$

We have:

$$\begin{array}{rcl} p^2 = 1 + \alpha & \Leftrightarrow & \left[\begin{array}{cccccc} p_0 & p_1 & p_2 & p_3 & \cdots \end{array}\right\rangle \left[\begin{array}{cccccc} p_0 & p_1 & p_2 & p_3 & \cdots \end{array}\right\rangle = 1 + \alpha \\ & \Leftrightarrow & \left[\begin{array}{ccccc} p_0^2 & 2p_0p_1 & p_1^2 + 2p_0p_2 & \cdots \end{array}\right\rangle = 1 + \alpha \end{array}$$

Algorithm 2 (Square root). We deduce that:

$$\begin{cases} p_0^2 = 1 \\ 2p_0p_1 = 1 \\ p_1^2 + 2p_0p_2 = 0 \\ p_0p_3 + p_1p_2 + p_2p_1 + p_3p_0 = 0 \end{cases}$$

Then the Solution of p is given by its coordinates:

$$\left[p_0 = -1, p_1 = -\frac{1}{2}, p_2 = \frac{1}{8}, p_3 = -\frac{1}{16}\right], \text{or } \left[p_0 = 1, p_1 = \frac{1}{2}, p_2 = -\frac{1}{8}, p_3 = \frac{1}{16}\right]$$

1.7. Binomial Chu-Vandermonde Identity

Theorem 5. For any natural number n and polynumbers α, β and t

$$(\alpha + \beta)^{n:t} = \sum_{k=0}^{n} \binom{n}{k} \alpha^{(n-k):t} \beta^{k:t} \tag{7}$$

Example 10 (Newton Polyserie). If $t = -1$, we get the Newton Polyseries identity

$$\sum_{k=0}^{n} \binom{n}{k} r^{(n-k):-1} s^{k:-1} = (r+s)^{n:-1}$$

We have

$$(1+\alpha\rangle^r = \sum_{n=0}^{+} \frac{r^{n:-1}}{n!}\alpha^n \quad \text{and} \quad (1+\alpha\rangle^s = \sum_{n=0}^{+} \frac{s^{n:-1}}{n!}\alpha^n$$

We deduce that:

$$\begin{aligned} (1+\alpha\rangle^r \times (1+\alpha\rangle^s &= \left(\sum_{n=0}^{+} \frac{r^{n:-1}}{n!}\alpha^n\right)\left(\sum_{n=0}^{+} \frac{s^{n:-1}}{n!}\alpha^n\right) \\ &= \sum_{n=0}^{+} \frac{1}{n!}\left(\sum_{k=0}^{n} n! \frac{r^{k:-1}}{k!}\frac{s^{n-k:-1}}{(n-k)!}\right)\alpha^n \\ &= \sum_{n=0}^{+} \frac{1}{n!}\left(\sum_{k=0}^{n} \binom{n}{k} \frac{r^{k:-1}}{k!}\frac{s^{n-k:-1}}{(n-k)!}\right)\alpha^n \\ &= \sum_{n=0}^{+} \frac{1}{n!}\left((r+s)^{n:-1}\right)\alpha^n = (1+\alpha\rangle^{r+s} \end{aligned}$$

Example 11 (Exponential Euler Polyseries). If $t = 0$ we get the Binomial identity:

$$(\alpha + \beta)^{n:0} = \sum_{k=0}^{n} \binom{n}{k} \alpha^{(n-k):0} \beta^{k:0} \tag{8}$$

We introduce then the exponential Polyseries:

$$(\exp\rangle^r = \sum_{n=0}^{+} \frac{r^n}{n!}\alpha^n = 1 + r\alpha + \frac{r^2}{2}\alpha^2 + \cdots$$

and

$$(\exp\rangle^s = \sum_{n=0}^{+} \frac{s^n}{n!}\alpha^n = 1 + r\alpha + \frac{r^2}{2}\alpha^2 + \cdots$$

such that:

$$\begin{aligned}(\exp\rangle^r \times (\exp\rangle^s &= \left(\sum_{n=0}^{+} \frac{r^{n:0}}{n!}\alpha^n\right)\left(\sum_{n=0}^{+} \frac{s^{n:0}}{n!}\alpha^n\right)\\ &= \left(\sum_{n=0}^{+}\left(\sum_{k=0}^{n} \frac{r^{k:0}}{k!}\frac{s^{n-k:0}}{(n-k)!}\right)\alpha^n\right)\\ &= \left(\sum_{n=0}^{+}\frac{1}{n!}\left(\sum_{k=0}^{n} n!\frac{r^{k:0}}{k!}\frac{s^{n-k:0}}{(n-k)!}\right)\alpha^n\right)\\ &= \left(\sum_{n=0}^{+}\frac{1}{n!}(r+s)^{n:0}\alpha^n\right) = (\exp\rangle^{r+s}\end{aligned}$$

Example 12 (Newton Reciprocal Polyserie). If $t = 1$, then:

$$\left(\sum_{k=0}^{+}\frac{r^{k:1}}{k!}\alpha^k\right)\times\left(\sum_{\ell=0}^{+}\frac{s^{\ell:1}}{\ell!}\alpha^\ell\right) = \left(\sum_{\ell=0}^{+}\frac{(r+s)^{n:1}}{(r+s)!}\alpha^n\right). \quad (9)$$

Case 1. We get the Newton Reciprocal Polyserie:

$$\alpha^{n:1} = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$$

Hence

$$\begin{aligned}(-\alpha)^{n:1} &= (-\alpha)(-\alpha+1)(-\alpha+2)\cdots(-\alpha+n-1)\\ &= (-1)^n\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)\\ &= (-1)^n\alpha^{n:-1}\end{aligned}$$

We have:

$$(-\alpha)^{n:1} = (-1)^n\alpha^{n:-1}$$

Then

$$\sum_{k=0}^{n}\binom{n}{k}r^{(n-k):1}s^{k:1} = (r+s)^{n:1}$$

We deduce

$$(1-\alpha\rangle^{-r} = \sum_{n=0}^{+}\frac{r^{n:1}}{n!}\alpha^n$$

and

$$(1-\alpha\rangle^{-s} = \sum_{n=0}^{+}\frac{s^{n:1}}{n!}\alpha^n$$

Finally we get:

$$(1-\alpha\rangle^{-r}\times(1-\alpha\rangle^{-s} = (1-\alpha\rangle^{-(r+s)} \quad (10)$$

1.8. Truncated Polyseries

An algebra (with identity) is a vector space $\mathbf{A}$ over $\mathbb{Q}$ with a multiplication $a \times b$ satisfying for all $a, b, c \in \mathbf{A}$ and $\lambda \in \mathbb{Q}$ the following properties:

1. *Associativity*

$$(a\times b)\times c = a\times(b\times c)$$

2. *Distributivity*

$$\begin{aligned}a\times(b+c) &= a\times b + a\times c,\\ (a+b)\times c &= a\times c + b\times c,\\ (\lambda a)\times b = a\times(\lambda b) &= \lambda(a\times b)\end{aligned}$$

3. *Identity*

$$1\times a = a\times 1 = a$$

Definition 5. A truncated Polyserie is defined finitely as a data-structure (ongoing up to a certain finite order k)

$$\alpha_k^1 \equiv \alpha_k \equiv \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} \Rightarrow \alpha_k^2 \equiv \begin{bmatrix}0\\0\\1\\\vdots\\0\end{bmatrix} \cdots \alpha_k^k \equiv \begin{bmatrix}0\\0\\0\\\vdots\\1\end{bmatrix}, \alpha_k^0 \equiv \begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix} \quad (11)$$

$$a = \begin{bmatrix}a_0\\a_1\\\vdots\\a_k\end{bmatrix} = a_0\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix} + a_1\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} + \cdots a_k\begin{bmatrix}0\\0\\0\\\vdots\\1\end{bmatrix} = a_0\alpha_k^0 + a_1\alpha_k^1 + a_2\alpha_k^2\cdots + a_k\alpha_k^k \quad (12)$$

Algorithm 3. If

$$a = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_k \end{bmatrix}$$

Then

$$a + b = \begin{bmatrix} a_0 + b_0 \\ a_1 + b_1 \\ \vdots \\ a_k + b_k \end{bmatrix} \quad ; \quad \lambda a = \begin{bmatrix} \lambda a_0 \\ \lambda a_1 \\ \vdots \\ \lambda a_k \end{bmatrix} \tag{13}$$

$$a \times b = \begin{bmatrix} a_0 b_0 \\ a_1 b_0 + a_1 b_0 \\ a_0 b_2 + a_1 b_1 + a_2 b_0 \\ \vdots \\ a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0 \end{bmatrix} \tag{14}$$

Theorem 6. The identity (15) holds for any truncated polyseries $a = \sum_{i=0}^{k} a_i \alpha_k^i$, and $b = \sum_{i=0}^{k} b_i \alpha_k^i$ expressed in term of the $k-$basis $\left[\alpha_k^0 \ \alpha_k^1 \ \alpha_k^2 \ \cdots \ \alpha_k^k\right]$

$$\left(\sum_{i=0}^{k} a_i \alpha_k^i\right)\left(\sum_{i=0}^{k} b_i \alpha_k^i\right) = \sum_{i=0}^{k}\left(\sum_{j=0}^{i} a_i b_{j-i}\right)\alpha_k^i \tag{15}$$

Proof

From (12) we have:

$$\begin{array}{rcl} a & = & a_0\alpha_k^0 + a_1\alpha_k^1 + a_2\alpha_k^2 \cdots + a_k\alpha_k^k = \sum_{i=0}^{k} a_i\alpha_k^i \\ & & \text{and} \\ b & = & b_0\alpha_k^0 + b_1\alpha_k^1 + b_2\alpha_k^2 \cdots + b_k\alpha_k^k = \sum_{i=0}^{k} b_i\alpha_k^i \end{array}$$

Using (14) we get:

$$\begin{array}{rcl} a \times b = \left(\sum_{i=0}^{k} a_i\alpha_k^i\right)\left(\sum_{i=0}^{k} b_i\alpha_k^i\right) = & & (a_0b_0)\,\alpha_k^0 + \\ & & +\,(a_1b_0 + a_1b_0)\,\alpha_k^1 + \\ & & +\,(a_0b_2 + a_1b_1 + a_2b_0)\,\alpha_k^2 + \\ & & \vdots \\ & & +\,(a_0b_k + a_1b_{k-1} + \cdots + a_{k-1}b_1 + a_kb_0)\,\alpha_k^k \\ & = & \sum_{i=0}^{k}\left(\sum_{j=0}^{i} a_i b_{j-i}\right)\alpha_k^i \end{array}$$

2. Modular Resolution

We introduce the following notations:

1. $\mathbb{A} = [a_1, a_2, \ldots\rangle$: Wildberger's Polyserie.
2. $\mathbb{N}_{\text{comp}} = [0, 1, \ldots\rangle$: Polyserie of computable natural numbers.
3. $\mathbb{P} = [p_1, p_2, \ldots\rangle$: Polyserie of primes.
4. $\mathbb{A}_k = [a_1, a_2, \ldots a_k\rangle = \mathbf{T}_k(\mathbb{A})$: Truncated polyserie of order k.
5. $\mathbb{F}_p \simeq \mathbb{Z}/p\mathbb{Z}$: Finite field of characteristic p, $(p \in \mathbb{P})$.

with modular equivalence:

$$\forall a, b \in \mathbb{N}_{\text{comp}} : [a \equiv b \pmod p] \Leftrightarrow [a \pmod p = b \pmod p] \tag{16}$$

1. Subsets:

$$\mathbb{F}_p^+ = \left\{1, \ldots, \frac{p-1}{2}\right\} = \left[1, \frac{p-1}{2}\right] \tag{17}$$

$$\mathbb{F}_p^- = \left\{\frac{p+1}{2}, \ldots, p-1\right\} = \left[\frac{p+1}{2}, p-1\right] \tag{18}$$

$$\mathbb{F}_p^{\times} = \mathbb{F}_p^{+} \cup \mathbb{F}_p^{-} = \mathbb{F}_p - \{0_p\} \tag{19}$$

2. Absolute value:

$$\forall a \in \mathbb{F}_p : \|a\|_p = \begin{cases} a & \text{if } a \in \left[1, \dfrac{p-1}{2}\right] \\ p-a & \text{if } a \in \left[\dfrac{p+1}{2}, p-1\right] \\ 0_p & \text{if } a = 0_p \end{cases} \tag{20}$$

3. Catalan Numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n = 0, 1, 2, \ldots \tag{21}$$

$$\begin{array}{l} C_0 = 1, \quad C_1 = 1, \quad C_2 = 2, \quad C_3 = 5, \quad C_4 = 14, \\ C_5 = 42, \quad C_6 = 132, \quad C_7 = 429, \quad C_8 = 1430, C_9 = 4862 \end{array} \tag{22}$$

$$C_0 = 1, \quad C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad (n \geq 0) \tag{23}$$

$$C_{n+1} = \frac{2(2n+1)}{n+2} C_n \tag{24}$$

$$G(t) = \frac{1-\sqrt{1-4t}}{2t} = \sum_{n=0}^{\infty} C_n t^n = 1 + tG(t)^2, \quad t \in \left(0, \frac{1}{4}\right) \tag{25}$$

Catalan numbers are often used in geometry and combinatorics

1. *Dyck paths:* Number of lattice paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$ that never dip below the x axis.
2. *Binary trees:* Number of rooted full binary trees with $n+1$ leaves.
3. *Triangulation:* Number of ways to triangulate a convex $(n+2)$ gon.
4. *Non-a-crossing partitions:* Number of ways to connect n points on a circle by non-a-crossing chords.

Theorem 7. Set $n \in \mathbb{N}_{\text{comp}}$, $n > 2$, $H : \mathbb{F}_p^{+} \longrightarrow \mathbb{F}_p^{+}$, $p > [t]^{n+1}$.

Then the following equivalence holds

$$\left(H_t^2(x) - H_t(x) + t \equiv 0 \pmod{t^n}\right) \Leftrightarrow \left(H_t(x) \equiv \sum_{k=1}^{n-1} C_{k-1}\, t^k \pmod{t^n}\right) \tag{26}$$

Proof

1. Assume that $H_t(x) \equiv \sum_{k=1}^{p} C_{k-1}\, t^k \pmod{t^n}$, then:

$$\left(\sum_{k=1}^{n-1} C_{k-1}\, t^k + \sum_{k \geq n} a_k t^k\right)^2 = (\alpha_n + \beta_n)^2 = \alpha_n^2 + \beta_n(2\alpha_n + \beta_n) \equiv \alpha_n^2 \pmod{t^n}$$

Hence, we deduce that:

$$\left(\sum_{k=1}^{n-1} C_{k-1}\, t^k + \sum_{k \geq n} a_k t^k\right)^2 \equiv \left(\sum_{k=1}^{n-1} C_{k-1}\, t^k\right)^2 \pmod{t^n}.$$

Such that:

$$H_t^2(x) \equiv \left(\sum_{k=1}^{n-1} C_{k-1}\, t^k\right)^2 \pmod{t^n} \tag{27}$$

On the other hand we have:

$$\begin{aligned}\left(\sum_{k=1}^{n-1} C_{k-1}\, t^k\right)^2 &= t^2 \left(\sum_{i=0}^{n-1} C_i\, t^i\right)\left(\sum_{j=0}^{n-1} C_j\, t^j\right) = t^2 \sum_{k=0}^{2n-1}\left(\sum_{\ell=0}^{k} C_\ell\, C_{k-\ell}\right) t^k \\ &= \sum_{k=0}^{2n-1} C_{k+1} t^{k+2} = -t + \sum_{k=0}^{2n-1} C_k t^{k+1} = -t + \sum_{m=1}^{2n} C_{m-1} t^m \\ &= -t + \sum_{k=1}^{n-1} C_{k-1}\, t^k + \sum_{k=n}^{2n} C_{k-1}\, t^k \equiv -t + H_t(x) \pmod{t^n}\end{aligned}$$

From (27) we get $H_t^2(x) \equiv -t + H_t(x) \pmod{t^n}$.

2. Set $C(H_t)(x) = H_t^2(x) - H_t(x) + t,\ t \in \mathbb{F}_p^+$.
For any two functions $H_t : \mathbb{F}_p^+ \longrightarrow \mathbb{F}_p^+$, $H_t' : \mathbb{F}_p^+ \longrightarrow \mathbb{F}_p^+$ and for all $x \in \mathbb{F}_p^+$:

$$\begin{aligned}\left|\left(C(H_t) - C(H_t')\right)(x)\right|_p &= \left|H_t^2(x) - H_t(x) - H_t'^2(x) + H_t'(x)\right|_p \\ &= \left|\left(H_t - H_t'\right)\left(H_t + H_t' - 1\right)(x)\right|_p.\end{aligned}$$

Since $H_t(x) \in \mathbb{F}_p^+$ and $H_t'(x) \in \mathbb{F}_p^+$, then:

$$\begin{aligned}H_t(x), H_t'(x) \in \left[1, \tfrac{p-1}{2}\right] &\Rightarrow H_t(x) + H_t'(x) \in [2, p-1] \\ &\Rightarrow \left(H_t + H_t' - 1\right)(x) \in [1, p-2] \\ &\Rightarrow \left(H_t + H_t' - 1\right)(x) \in \mathbb{F}_p^\times \\ &\Rightarrow \left(H_t + H_t' - 1\right)(x) \neq 0_p\end{aligned}$$

We have:

$$\begin{aligned}\left|\left(C(H_t) - C(H_t')\right)(x)\right|_p = 0_p &\Leftrightarrow \left|\left(H_t - H_t'\right)\left(H_t + H_t' - 1\right)(x)\right|_p = 0_p \\ &\Leftrightarrow \left|\left(H_t - H_t'\right)(x)\right|_p = 0_p\end{aligned}$$

Since $p > [t]^{n+1}$, if $C(H_t)(x) = C(H_t')(x) \equiv 0 \pmod{t^n}$, then:

$$H_t(x) \equiv H_t'(x) \equiv \sum_{k=1}^{n-1} C_{k-1}\, t^k \pmod{t^n} \tag{28}$$

Example 13. The solutions of the modular equation

$$t + ax + x^2 \equiv 0 \pmod{t^n} \tag{29}$$

are of the form:

$$x \equiv \sum_{k=1}^{n-1} C_{k-1} \frac{1}{a^{2k-1}} t^k \pmod{t^n}. \tag{30}$$

Since $\mathbb{F}_p^\times = \mathbb{F}_p^+ \cup \mathbb{F}_p^-$, we can easily deduce the second solution form in $\mathbb{F}_p^-$ using its modular conjugate in $\mathbb{F}_p^+$.

$$\begin{aligned}x^2 - ax + t \equiv 0 \pmod{t^n} &\Leftrightarrow \left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right) + \frac{t}{a^2} \equiv 0 \pmod{t^n} \\ &\Leftrightarrow H^2 - H + u \equiv \pmod{t^n}\end{aligned}$$

From Theorem (7) we get:

$$\frac{x}{a} = H = \sum_{k=1}^{n-1} C_{k-1}\, u^k + a_n u^n = \sum_{k=1}^{n-1} C_{k-1} \left(\frac{t}{a^2}\right)^k + a_n \left(\frac{t}{a^2}\right)^n$$

Hence

$$x = a \sum_{k=1}^{n-1} C_{k-1} \frac{t^k}{a^{2k}} + b_n t^n = \sum_{k=1}^{n-1} C_{k-1} \frac{t^k}{a^{2k-1}} + \frac{b_n}{a} t^n$$

We deduce that

$$x \equiv \sum_{k=1}^{n-1} C_{k-1} \frac{1}{a^{2k-1}} t^k \pmod{t^n}.$$

If $a = -1$, and $n = 10$, then the two modular solutions of the quadratic equation

$$t + x + x^2 \equiv 0 \pmod{t^{10}}$$

are:

$$x_1 \equiv -t - t^2 - 2t^3 - 5t^4 - 14t^5 - 42t^6 - 132t^7 - 429t^8 - 1430t^9 \pmod{t^{10}}$$

and

$$x_2 \equiv -1 + t + t^2 + 2t^3 + 5t^4 + 14t^5 + 42t^6 + 132t^7 + 429t^8 + 1430t^9 \pmod{t^{10}}$$

up to degree 10, we get $H(x_1) = H(x_2) = H$, such that:

$$H = t^{10}\left(2044900\,t^8 + 1226940\,t^7 + 561561\,t^6 + 233376\,t^5 + 93500\,t^4 + 37400\,t^3 + 15470\,t^2 + 7072\,t + 4862\right)$$

For more complete examples see [20]

3. Conclusion

The equivalence (26) in the theorem (7), shows how strong is the algebraic notion of truncation of Polyseries in Finite fields introduced by Wildberger N.J & Rubine D. in their recent article [20]

Future Work:

1. Extend the method to higher degree polynomials and explore generalizations.
2. Investigate computational complexity and algorithmic implementation.
3. Apply the method to cryptography of protocols or error correcting codes.
4. Explore connections with p-adic analysis and modular forms.
5. The author declare no conflicts of interest.

Author Contributions

Mahdi-Tahar Brahimi is the sole author. The author read and approved the final manuscript.

Conflicts of Interest

The author declares no conflicts of interest.

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