Pure and Applied Mathematics Journal
2025; 14(6): 161-165
http://www.sciencepublishinggroup.com/j/pamj
doi: 10.11648/j.pamj.20251406.11

SciencePG
Science Publishing Group

A Study of Quasi-filtrations on a Semiring and Extension of the Generalized Samuel Number $\overline{w}_f(g)$ to Quasi-filtrations on a Semiring

Kouadjo Pierre Brou*, Edjabrou Ulrich Blanchard Kablam, Grah Gelin Bogui

Applied Fundamental Sciences Department, Nangui Abrogoua University, Abidjan, Ivory Coast

Email address:
brou.kouadjopierre@yahoo.fr (Kouadjo Pierre Brou), bkablam11@gmail.com (Edjabrou Ulrich Blanchard Kablam), gelinbogui10@gmail.com (Grah Gelin Bogui)
*Corresponding author

To cite this article:
Kouadjo Pierre Brou, Edjabrou Ulrich Blanchard Kablam, Grah Gelin Bogui. (2025). A Study of Quasi-filtrations on a Semiring and Extension of the Generalized Samuel Number $\overline{w}_f(g)$ to Quasi-filtrations on a Semiring. *Pure and Applied Mathematics Journal, 14*(6), 161-165. https://doi.org/10.11648/j.pamj.20251406.11

Received: 24 August 2025 ; **Accepted:** 15 September 2025 ; **Published:** 8 November 2025

Abstract: In this work, we develop a theory of asymptotic growth for quasi-filtrations in the framework of commutative semirings. A quasi-filtration $g = (G_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ is a family of submonoids of a semiring B that satisfies the conditions of a quasi-graduation and is also decreasing for indices $n \geq 1$. This structure generalizes the notion of a filtration by using submonoids instead of the more restrictive semi-ideals. In this context, we extend the Samuel number $w_I(J)$ of a pair of semi-ideals (I, J) to a quasi-filtration f and a submonoid J, denoted $w_f(J)$, and also to a pair of quasi-filtrations (f, g), which we will denote $w_f(g)$. The central question is the existence of generalized Samuel number, denoted $\overline{w}_f(g)$, which is defined as the limit of the radio $w_f(G_n)/n$ as n tends to infinity. Our main result provides a positive answer to this question under certain conditions. We demonstrate that if the quasi-filtration f is regular, then $w_f(J)$ exists for any submonoid J, and the generalized Samuel number $\overline{w}_f(g)$ is well-defined for any quasi-filtration g that is Approximable by Powers of submonoids (AP). Finally, we study the algebraic properties of this number. We prove that this number is non-negative and positively homogeneous under certain conditions. These results constitute an essential step towards the development of analytical tools for non-symmetrizable algebraic structures.

Keywords: Commutative Ring, Semi-ring, Submonoid, Filtration, Quasi-filtration, Generalized Samuel Number, Regular Quasi-filtration, AP Quasi-filtration

1. Introduction

In commutative algebra, the theory of ideal filtrations is a powerful tool for studying the asymptotic structure of rings. It allows for the definition of numerical invariants, such as the *Samuel number* introduced in [12–14]. This number captures deep geometric and algebraic information, particularly concerning the intersection multiplicity of algebraic varieties [8–10]. Although these tools were successfully developed for Noetherian rings, much research has aimed to extend their scope. The present work is part of this generalization effort. A first step, initiated by P. Ayegnon and D. Sangare [12], consisted of extending the Samuel number to the broader framework of filtrations. More recently, a significant line of research, led by E. D. Akeke, S. Ouattara, and P. Ayegnon [3, 4, 7], has sought to adapt these concepts to *semirings. In the same vein, Y. M. Diagana and S. Ouattara [5] extended the Samuel number* $\overline{v}_f(g)$ *to decreasing quasi-graduations and E. D. Akeke, P. K. Ayegnon and H. Dichi extended this number to bifiltrations in [2].* These algebraic structures, where addition is not necessarily invertible, arise naturally in various fields. Fundamental examples include the sets of natural numbers $(\mathbb{N}, +, .)$ or positive real numbers $(\mathbb{R}_+, +, .)$, as well as any unital commutative ring. Our goal is to develop an asymptotic theory for these structures. The first contribution of this paper is to introduce the notion of a *quasi-filtration* on a semiring. This is a decreasing family $(G_n)_{n\in\mathbb{N}}$ of submonoids that generalizes the usual filtration by powers of an ideal. For example, on the semiring $B = \mathbb{R}_+$ with the sub-semiring $A = \mathbb{N}$, the family defined by $G_0 = A$ and $G_n = (0)$ for $n \geq 1$ is a trivial quasi-filtration. A richer case is given by $G_n = (p + \sqrt{p})^n\mathbb{N}$ for a prime number p. We particularly study the so-called "Approximable by Powers" (AP) quasi-filtrations, which are central to our main results. The second contribution is the extension of the Samuel number concept to a semimodule, as in [6]. We define a generalized number, denoted $\overline{w}_f(g)$, which measures the relative growth rate of a quasi-filtration g with respect to a reference filtration f. The central question is the existence of this number, defined as a limit. We prove that if f is an AP quasi-filtration, this limit always exists and can be computed as an infimum, which facilitates its analysis. *This number provides a new invariant to compare growth rates in non-Noetherian contexts, opening avenues for analyzing the complexity of systems described by semirings in algebraic geometry.* The paper is organized as follows. The first section is devoted to preliminaries, where we recall the basic definitions of semirings and submonoids. The second section formally introduces quasi-filtrations, their operations, and their properties. Finally, the third and final section is dedicated to the definition and in-depth study of the generalized Samuel number $\overline{w}_f(g)$, where we present our main theorems on its existence and properties.

2. Preliminaries

2.1. Definitions and Examples

Definition 2.1. (1) A *monoid* is a set B equipped with an internal composition law $*$ that is associative, commutative, and has an identity element, denoted e_B.

(2) Let $(B, *)$ be a monoid and A be a subset of B. We say that A equipped with the law $*$ from B is a *submonoid* of B if $(A, *)$ is a monoid such that $e_A = e_B$.

(3) A *pre-semiring* is a set B equipped with an addition and a multiplication, for each of which it is a monoid.

(4) A *semiring* is a set B equipped with an addition and a multiplication satisfying: (i) $(B, +, .)$ is a pre-semiring; (ii) Multiplication is distributive over addition; (iii) 0 is an annihilator for B.

(5) Let $(B, +, .)$ be a semiring and A be a subset of B. We say that A is a *sub-semiring* of B if $(A, +, .)$ is a semiring such that $0_A = 0_B$ and $1_A = 1_B$.

Example 2.1. (1) Any unital commutative ring is a semiring.

(2) The sets $(\mathbb{N}, +, .)$, $(\mathbb{R}_+, +, .)$, $(\mathbb{Q}_+, +, .)$ are semirings. Thus, $\mathbb{N}$ is a sub-semiring of $\mathbb{Q}_+$, and $\mathbb{Q}_+$ is a sub-semiring of $\mathbb{R}_+$.

(3) The set $\mathbb{N}[X]$ of polynomials in one variable with coefficients in $\mathbb{N}$, equipped with addition and multiplication, is a semiring.

2.2. Operations on Submonoids of a Semiring

2.2.1. Sum of Submonoids

Definition 2.2. Let H and G be two submonoids of a semiring B. We call the sum of the submonoids H and G the set of sums $h + g$, where $h \in H$ and $g \in G$. It is denoted: $H + G = \{h + g | h \in H \text{ and } g \in G\}$.

Proposition 2.1. If H and G are two submonoids of a semiring B, then $H + G$ is a submonoid of B. More generally, if $(G_s)_{s\in S}$ is a family of submonoids of B, the set of finite sums $\sum_{s\in S} g_s$ where $g_s \in G_s$ for all $s \in S$ is a submonoid of B, denoted $\sum_{s\in S} G_s$.

2.2.2. Product of Two Submonoids

Definition 2.3. Let H and G be two submonoids of a semiring B. The set of finite sums $\sum_{s\in S} h_s g_s$ where $h_s \in H$ and $g_s \in G$, is a submonoid of B denoted HG and called the product of the submonoids H and G.

Remark 2.1. The product of an arbitrary family of submonoids is defined analogously. In particular, we define the powers G^n $(n > 0)$. By convention, G^0 is a sub-semiring of B.

3. Quasi-filtrations of a Semiring

3.1. Definitions and Examples

Definition 3.1. Let B be a semiring and $g = (G_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be a family of submonoids of B. (1) We say that g is a *quasi-graduation* of B if: (i) $G_0 = A$ is a sub-semiring of B; (ii) $G_\infty = \{0\}$; (iii) $G_pG_q \subseteq G_{p+q}, \forall p, q \in \mathbb{N}$.

(2) We will say that g is a *quasi-filtration* of B, if: (i) g is a quasi-graduation of B; (ii) $G_{n+1} \subseteq G_n, \forall n \in \mathbb{N}^*$.

Example 3.1. Let $B = \mathbb{R}_+$ and $A = \mathbb{N}$. (1) The family $g = (G_n)_{n\in\mathbb{N}}$ defined by: $G_0 = A$ and $G_n = \{0\}, \forall n \in \mathbb{N}^*$, is a quasi-filtration of B. It is denoted $g = g_{(0)}$.

(2) The family $h = (H_n)_{n\in\mathbb{N}}$ defined by: $H_n = A, \forall n \in \mathbb{N}$, is a quasi-filtration of B. It is denoted $h = h_A$. More generally, $g_{(0)}$ and g_B are the trivial quasi-filtrations of the semiring B.

(3) If p is a prime number, then by setting $G_n = (p + \sqrt{p})^n\mathbb{N}$, $g = (G_n)_{n\in\mathbb{N}\cup\{\infty\}}$ is a quasi-filtration of the semiring B with $G_\infty = (0)$.

3.2. Some Classes of Quasi-filtrations

Definition 3.2. The set $QF(B)$ of quasi-filtrations of the semiring B is ordered by the relation: $h = (H_n) \leq g = (G_n)$ if $H_n \subseteq G_n$ for all $n \geq 0$.

(1) Let I be a submonoid of a semiring B and A be a sub-semiring of B such that $IA \subseteq I$. The quasi-filtration $g = g_I = (I^n)_{n\in\mathbb{N}\cup\{\infty\}}$ where $I^0 = A$ and $I^\infty = \{0\}$, is called the *I-adic quasi-filtration* of B.

(2) A quasi-filtration $g = (G_n)_{n\in\mathbb{N}\cup\{\infty\}}$ of the semiring B is said to be *AP* if there exists a sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that: (i) $G_{mk_n} \subseteq G_m^n, \forall m, n \in \mathbb{N}^*$ (ii) $\lim_{n\to\infty} \dfrac{k_n}{n} = 1$.

(3) A quasi-filtration $g = (G_n)_{n\in\mathbb{N}\cup\{\infty\}}$ of the semiring B is said to be *strongly AP* if there exists $m \in \mathbb{N}^*$ such that: $(G_m)^{n+1} \subseteq G_{mn+j} \subseteq (G_m)^n, \forall n \geq 0$ and $\forall j \in \{0;...;m-1\}$.

(4) Let λ be a positive real number. We denote by $\{\lambda\}$ the smallest integer greater than or equal to λ, and for any quasi-filtration $g = (G_n)_{n\in\mathbb{N}\cup\{\infty\}}$ of the semiring B, we consider the quasi-filtration $g^{(\lambda)} = (G_{\{n\lambda\}})_{n\in\mathbb{N}\cup\{\infty\}}$.

(5) A quasi-filtration $g = (G_n)_{n\in\mathbb{N}\cup\{\infty\}}$ of the semiring B is said to be *regular* if there exists $k \in \mathbb{N}^*$ such that: $g^{(k)} = g_{G_k}$.

Proposition 3.1. Let $g = (G_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be a quasi-filtration of the semiring B. Then we have: (1) If g is strongly AP, then g is AP. (2) If g is strongly AP, then g is regular.

Proof. (1) By taking k = m, we get the result. (2) The proof is the same as that of Proposition 3.3.1 in [1].

3.3. Operations on the Set of Quasi-filtrations of a Semiring

Definition 3.3. Let $h = (H_n)_{n\in\mathbb{N}\cup\{\infty\}}$ and $g = (G_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be two quasi-filtrations of a semiring B. (1) The *sum* of the quasi-filtrations h and g is the quasi-filtration $h + g = (\sum_{k=0}^{n} H_k G_{n-k})_{n\in\mathbb{N}\cup\{\infty\}}$ of B, with $(h+g)_\infty = \{0\}$.

(2) The *product* of h and g is the quasi-filtration of B defined by: $hg = (H_n G_n)_{n\in\mathbb{N}\cup\{\infty\}}$.

4. Study of $\overline{w}_f(g)$

Definition 4.1. Let $f = (F_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be a quasi-filtration of the semiring B and J be a submonoid of B such that $JF_0 \subseteq J$. We set: $w_f(J) = \inf\{r \in \mathbb{N} | F_r \subseteq J\}$ and if the set $\{r \in \mathbb{N} | F_r \subseteq J\}$ is empty, we set $w_f(J) = +\infty$. Let $x \in B$, we set $w_f(x) = w_f(xF_0)$.

Proposition 4.1. Let $f = (F_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be a quasi-filtration of the semiring B and K be a submonoid of B such that $KF_0 \subseteq K$. Then we have: $w_f(K) \leq \inf_{x\in K} w_f(x)$

Proof. Let $m = \inf_{x\in K} w_f(x)$. We can assume $m < +\infty$. Then there exists $x \in K$ such that $w_f(x) = m$, and we have $F_m \subseteq xF_0 \subseteq K$. Hence $w_f(K) \leq m = \inf_{z\in K} w_f(z)$.

Definition 4.2. Let $f = (F_n)_{n\in\mathbb{N}\cup\{\infty\}}$ and $g = (G_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be quasi-filtrations of the semiring B such that $F_0 = G_0 = A$. We set $\overline{w}_f(g) = \lim_{n\to\infty} \frac{1}{n} w_f(G_n)$ if this limit exists. If $g = f_J$ is the J-adic quasi-filtration, then we set $\overline{w}_f(f_J) = \overline{w}_f(J)$. If $g = f_J$ and $f = f_I$, we set $\overline{w}_f(g) = \overline{w}_I(J)$.

Proposition 4.2. Let $f = (F_n)_{n\in\mathbb{N}\cup\{\infty\}}$ and $g = (G_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be quasi-filtrations of the semiring B such that $F_0 = G_0 = A$. The following assertions are equivalent: i) $\overline{w}_f(g)$ exists in $\overline{\mathbb{R}}$. ii) $\forall\lambda \in \mathbb{R}_+^*, \overline{w}_{f^{(\lambda)}}(g)$ exists in $\overline{\mathbb{R}}$. iii) $\exists\lambda \in \mathbb{R}_+^*, \overline{w}_{f^{(\lambda)}}(g)$ exists in $\overline{\mathbb{R}}$. If one of these assertions holds, we have $\lambda\overline{w}_{f^{(\lambda)}}(g) = \overline{w}_f(g)$.

Proof. Let us show that $w_f(G_n) = +\infty$ if and only if $w_{f^{(\lambda)}}(G_n) = +\infty, \forall\lambda \in \mathbb{R}_+^*$. Indeed, let $w_{f^{(\lambda)}}(G_n) = +\infty$. Then by definition $\beta = \{r \in \mathbb{N} | F_{\{r\lambda\}} \subseteq G_n\} = \emptyset$. Suppose that $w_f(G_n) = p \in \mathbb{N}$. Then there exists $s \in \mathbb{N}$ such that $s\lambda \geq p$, hence $\{s\lambda\} \geq p$ and $F_{\{s\lambda\}} \subseteq F_p \subseteq G_n$. We then obtain $\beta \neq \emptyset$, which is absurd. Thus $w_f(G_n) = +\infty$. Conversely, if $s = w_{f^{(\lambda)}}(G_n) < +\infty$, then $F_{\{s\lambda\}} \subseteq G_n$. Consequently, $w_f(G_n) \leq \{s\lambda\} < +\infty$. We thus deduce that $w_f(G_n) \in \mathbb{N}$ if and only if $w_{f^{(\lambda)}}(G_n) \in \mathbb{N}$.

Let us show that i) implies ii). If there exists $p \in \mathbb{N}$ such that $w_f(G_p) = +\infty$, then $\forall k \geq p$, we have $w_f(G_k) = +\infty$ because the sequence $w_n = w_f(G_n)$ is increasing. From the above, we thus have $w_{f^{(\lambda)}}(G_k) = +\infty$ for all $k \geq p$, hence ii).

Suppose now that $w_f(G_n) \in \mathbb{N}$ for all $n \in \mathbb{N}$. Then $w_{f^{(\lambda)}}(G_n) \in \mathbb{N}$ for all $n \in \mathbb{N}$. Let $r = w_{f^{(\lambda)}}(G_n)$; then $F_{\{r\lambda\}} \subseteq G_n$ and $F_{\{(r-1)\lambda\}} \not\subseteq G_n$. We then have: $\lambda(r-1) \leq \{\lambda(r-1)\} < w_f(G_n) \leq \{\lambda r\} < \lambda r + 1$ then $\lambda w_{f^{(\lambda)}}(G_n) - \lambda \leq w_f(G_n) \leq \lambda w_{f^{(\lambda)}}(G_n) - \lambda + 1$. Equation (4.1) implies $w_f(G_n) - \lambda \leq \lambda w_{f^{(\lambda)}}(G_n) - \lambda + 1$, hence:

$$\frac{\lambda w_{f^{(\lambda)}}(G_n) - \lambda}{n} \leq \frac{w_f(G_n)}{n} \leq \frac{\lambda w_{f^{(\lambda)}}(G_n) + 1}{n} \tag{1}$$

As $n \to +\infty$, the first and last terms of the inequalities tend to $\overline{w}_f(g)$, and we deduce the existence of $\overline{W}_{f^{(\lambda)}}(g)$ with: $\overline{w}_f(g) = \lambda\overline{w}_{f^{(\lambda)}}(g)$.

Thus i) $\implies$ ii). That ii) implies iii) is evident.

Let us show that iii) implies i). If $\overline{w}_{f^{(\lambda)}}(G_n) = +\infty$, then $w_f(G_n) = +\infty$ and we have the result. Suppose therefore that $w_{f^{(\lambda)}}(G_n) \in \mathbb{N}$ for all $n \in \mathbb{N}$. Then according to (1), we have: $\frac{\lambda w_{f^{(\lambda)}}(G_n) - \lambda}{n} \leq \frac{w_f(G_n)}{n} \leq \frac{\lambda w_{f^{(\lambda)}}(G_n) + 1}{n}$. As $n \to +\infty$, the first and last terms of the inequalities tend to $\lambda\overline{w}_{f^{(\lambda)}}(g)$, so we deduce that $\overline{w}_f(g)$ exists in $\overline{\mathbb{R}}$ and that: $\lambda\overline{w}_{f^{(\lambda)}}(g) = \overline{w}_f(g)$. Hence the result.

Proposition 4.3. If $f = (F_n)_{n\in\mathbb{N}\cup\{\infty\}}$ is a regular quasi-filtration of B and J is a submonoid of B such that $JF_0 \subseteq J$, then $w_f(J)$ exists in $\overline{\mathbb{R}}$.

Proof. Let $w_{f_k}(J) = w_k(J)$. We remark that $w_1(J^n) \leq nw_1(J)$ for all $n \in \mathbb{N}$. Since f is a regular quasi-filtration, there exists $k \in \mathbb{N}^*$ such that $f^{(k)} = f_{F_k}$. Thus, it suffices to show that $\overline{w}_{f^{(k)}}(J)$ exists in $\overline{\mathbb{R}}$ (Proposition 4.4). Let the sequence $(w_n)_{n\in\mathbb{N}}$ be defined by: $w_n = \frac{1}{n} w_{f^{(k)}}(J^n) = \frac{1}{n} w_{F_k}(J^n)$ (since $f^{(k)} = f_{F_k}$). Let $\overline{w}$ (resp. $\underline{w}$) be the limit superior (resp. inferior) of the sequence w_n. We need to show that $\overline{w} = \underline{w}$. If $\overline{w} = +\infty$ or $\underline{w} = 0$, it is clear. We therefore assume that $\overline{w}$ is finite and $\underline{w} > 0$. By definition: (1) $\forall\epsilon > 0, \forall i \in \mathbb{N}, \exists j > i; w_j < \overline{w} + \epsilon$. (2) $\forall\epsilon > 0, \forall i \in \mathbb{N}, \exists j \geq i; w_j > \underline{w} - \epsilon$ (if $\underline{w} < +\infty$). (3) $\forall N \in \mathbb{N}, \forall i \in \mathbb{N}, \exists j > i; w_j > N$ (if $\underline{w} = +\infty$). Let us fix $\epsilon > 0$ and choose an index i large enough such that $\frac{1}{i} < \frac{\epsilon}{\overline{w} + \epsilon}$. From (1), there exists $j \geq i$ such that $w_j \leq \overline{w} + \epsilon$. There exists $s \geq ij$ such that $w_s > \underline{w} - \epsilon$ (resp. N if $\underline{w} = +\infty$). Let us divide s by j; $s = lj + q$ with $0 \leq q < j$. Since for any submonoid I of B, $w_I(J^n) \leq nw_I(J), \forall n \in \mathbb{N}$, then $\underline{w} - \epsilon \leq \frac{1}{lj+q} w_{F_k}(J^{lj+q}) \leq \frac{1}{lj+q} w_{F_k}(J^{lj}) \leq$

$\dfrac{1}{lj+q}lw_{F_k}(J^j)$.

Proposition 4.4. Let $f = (F_n)_{n\in\mathbb{N}}$ be a regular quasi-filtration and $g = (G_n)_{n\in\mathbb{N}}$ be a quasi-filtration of the semiring A such that $F_0 = G_0 = A$. Then we have: i) $\lim_{k\to+\infty}\frac{1}{k}\overline{w}_f(G_k)$ exists in $\overline{\mathbb{R}}$ and $\lim_{n\to\infty}\frac{1}{n}\overline{w}_f(G_n) \leq \lim_{k\to+\infty}\frac{1}{k}\overline{w}_f(G_k)$. ii) If g is an AP quasi-filtration, $\overline{w}_f(g)$ exists in $\overline{\mathbb{R}}$ and we have $\overline{w}_f(g) = \lim_{n\to+\infty}\frac{1}{n}\overline{w}_f(G_n)$.

Proof. Since f is a regular quasi-filtration, $\overline{w}_f(G_n)$ exists in $\overline{\mathbb{R}}$, $\forall n \in \mathbb{N}$ (Proposition 4.5). Let us prove i). a) First, let's show that $\lim_{k\to+\infty}\frac{1}{k}\overline{w}_f(G_k)$ exists in $\overline{\mathbb{R}}$. Indeed, for all $(k,n) \in \mathbb{N}^2$, assume $n \geq k$. By Euclidean division, we have $n = q_nk + r_n$, $0 \leq r_n < k$. And $\frac{1}{m}\overline{w}_f(G_m^n) \leq (q_n+1)\frac{1}{(q_n+1)m}\overline{w}_f(G_k^{(q_n+1)m})$. because $G_n = G_{q_nk+r_n} \supseteq G_{q_nk+k} = G_{(q_n+1)k} \supseteq G_k^{q_n+1}$. Hence, as $m \to +\infty$, $\overline{w}_f(G_n) \leq (q_n+1)\overline{w}_f(G_k)$. Consequently, $\frac{1}{n}\overline{w}_f(G_n) \leq \frac{1}{n}(q_n+1)\overline{w}_f(G_k)$. And as $n\to+\infty$, we have $\lim_{n\to\infty}\frac{1}{n}\overline{w}_f(G_n) \leq \frac{1}{k}\overline{w}_f(G_k), \forall k \in \mathbb{N}^*$. Thus the sequence $\frac{1}{k}\overline{w}_f(G_k)$ converges. b) Now let's show that $\lim_{n\to\infty}\frac{1}{n}\overline{w}_f(G_n) \leq \lim_{k\to\infty}\frac{1}{k}\overline{w}_f(G_k)$. As before, let $n = q_nk + r_n$ with $0 \leq r_n < k$. Then we have: $\frac{1}{n}\overline{w}_f(G_n) \leq \frac{q_n+1}{n}\frac{\overline{w}_f(G_k^{q_n+1})}{q_n+1}$, hence $\lim_{n\to\infty}\frac{1}{n}\overline{w}_f(G_n) \leq \frac{1}{k}\overline{w}_f(G_k), \forall k \in \mathbb{N}^*$, and we have the result. Let's prove ii). It suffices to show that $\lim_{n\to\infty}\frac{1}{n}\overline{w}_f(G_n) \leq \lim_{n\to\infty}\frac{1}{n}\overline{w}_f(G_n)$ and use i). Since g is an AP quasi-filtration, there exists a sequence k_n of positive integers such that for all $n,m \in \mathbb{N}, G_{k_n,m} \subseteq G_m^n$ with $\frac{k_n}{n} \to 1$ as $n\to+\infty$. By Euclidean division, we have $k_nq_m \leq m = k_nq_m + r_m$ with $0 \leq r_m < k_n$ and $G_m \subseteq G_{k_nq_m} \subseteq G_{q_m}^{k_n}$, which implies $\frac{1}{q_m}\overline{w}_f(G_{q_m}^{k_n}) \leq \frac{1}{q_m}\overline{w}_f(G_m)$. As $m\to+\infty$, we have $\frac{1}{k_n}\overline{w}_f(G_n) \leq \lim_{m\to\infty}\frac{1}{m}\overline{w}_f(G_m)$, and since $\frac{n}{k_n} \to 1$ as $n\to+\infty$, then $\lim_{n\to\infty}\frac{1}{n}\overline{w}_f(G_n) \leq \lim_{m\to\infty}\frac{1}{m}\overline{w}_f(G_m)$.

Proposition 4.5. Let $f = (F_n)_{n\in\mathbb{N}\cup\{\infty\}}$ and $g = (G_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be quasi-filtrations of the semiring A and λ be a positive real number. Then the following assertions are equivalent: i) $\overline{w}_f(g)$ exists. ii) $\overline{w}_{f^{(\lambda)}}(g)$ exists. iii) $\overline{w}_f(g^{(\lambda)})$ exists. Under these conditions, we have: $\overline{w}_f(g) = \lambda\overline{w}_{f^{(\lambda)}}(g) = \dfrac{\overline{w}_f(g^{(\lambda)})}{\lambda}$.

Proof. The proof is analogous to that of Proposition 2.3 in [12].

Theorem 4.1. Let $f = (F_n)_{n\in\mathbb{N}\cup\{\infty\}}$ and $g = (G_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be quasi-filtrations of the semiring B such that $F_0 = G_0 = A$ and f is a strongly AP quasi-filtration. Then the sequences $(\frac{1}{n}\overline{w}_f(G_n))_{n\in\mathbb{N}^*}$ and $(n\overline{w}_{F_n}(g))_{n\in\mathbb{N}^*}$ converge in $\overline{\mathbb{R}}_+$ and we have: i) $\overline{w}_f(g) = \lim_{n\to\infty}\frac{1}{n}\overline{w}_f(G_n) = \inf_{n\in\mathbb{N}^*}\frac{1}{n}\overline{w}_f(G_n)$. ii) $\overline{w}_f(g) = \lim_{n\to\infty} n\overline{w}_{F_n}(g)$.

Proof. Since f is strongly AP, f is an AP quasi-filtration, so there exists a sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that $F_{k_nm} \subseteq F_m^n$ for all n, m with $\lim_{n\to\infty}\frac{k_n}{n} = 1$. i) We know from Proposition 4.5 that $\overline{w}_f(G_n)$ exists in $\overline{\mathbb{R}}$ for every integer $n \geq 0$. Furthermore, since the sequence $(\overline{w}_f(G_n))_{n\in\mathbb{N}}$ is increasing, we can assume that $\overline{w}_f(G_n) < +\infty$ for every integer n. Since $\overline{w}_f(J) \leq k\overline{w}_f(J)$ for any submonoid J of B such that $JA \subseteq J$, we obtain $\lim_{n\to\infty}\overline{w}_f(G_n) \leq \overline{w}_f(g)$. On the other hand, for any integers n and m, we have $G_m^n \subseteq G_{nm}$, hence $\frac{\overline{w}_f(G_m^n)}{n} \geq \frac{\overline{w}_f(G_{nm})}{nm}$. As $m\to\infty$, we get $\frac{\overline{w}_f(G_n)}{n} \geq \overline{w}_f(g)$ for all n. It follows that the sequence $(\frac{1}{n}\overline{w}_f(G_n))_{n\in\mathbb{N}^*}$ converges to $\overline{w}_f(g)$. Moreover, $\overline{w}_f(g) = \inf_{n\in\mathbb{N}^*}\frac{1}{n}\overline{w}_f(G_n)$.

ii) Let us first show that $\overline{w}_f(g) = \lim_{n\to\infty} n\overline{w}_{F_n}(g)$. We can assume that $k_n \geq n$ for all $n \geq 1$. Then if $g = (G_n)$, the sequence of inequalities $f_{F_{k_n}} \leq f^{(k_n)} \leq f_{F_n}$ gives us $\overline{w}_{F_{k_n}}(g) \leq \overline{w}_{f^{(k_n)}}(g) = \frac{1}{k_n}\overline{w}_f(g) \leq \overline{w}_{F_n}(g)$. (*) Then, by multiplying the terms of (*) by k_n and taking the limit inferior, we obtain: $\liminf k_n\overline{w}_{F_n}(g) \leq \overline{w}_f(g) \leq \limsup n\overline{w}_{F_n}(g)$ and $\limsup n\overline{w}_{F_n}(g) \leq \limsup k_n\overline{w}_{F_{k_n}}(g)$ because $k_n \geq n$ for all n. We have thus shown that $\overline{w}_f(g) = \lim_{n\to\infty} n\overline{w}_{F_n}(g)$.

Now let us show that $\liminf n\overline{w}_{F_n}(g) \leq \overline{w}_f(g)$. Suppose that $\overline{w}_f(g) \in \mathbb{R}_+$ and let us take two integers $p \geq 1$ and $q \geq 1$ such that $\overline{w}_f(g) < \frac{p}{q}$. Then $\overline{w}_f(g^{(q)}) < p$ (Proposition 4.7) and there thus exists an integer $N \geq 1$ such that for all $m \geq N$, we have $\overline{w}_f(G_{mq}) \leq mp$. Consequently, for any integer $m \geq N$, we have $F_{mp} \subseteq G_{mq}$. Let n be an integer greater than or equal to 1 and q_m be the integer part of $\frac{m}{n}$. We then have the sequence of inclusions $F_{(q_m+1)p} \subseteq F_{n(q_m+1)p} \subseteq F_{mp} \subseteq G_{mq}$, which gives the inequality $\overline{w}_{F_n}(G_{mq}) \leq \frac{p(q_m+1)}{mq}$. By taking the limit as m tends to $+\infty$, we obtain $\limsup \overline{w}_{F_n}(g) \leq \frac{1}{n}\frac{p}{q}$, that is, $\limsup n\overline{w}_{F_n}(g) \leq \frac{p}{q}$. Hence $\limsup n\overline{w}_{F_n}(g) \leq \overline{w}_f(g)$. We deduce from (4.2) and (4.3) that $\overline{w}_f(g) = \lim_{n\to\infty} n\overline{w}_{F_n}(g)$.

ORCID

0009-0000-5895-2264 (Kouadjo Pierre Brou)

0009-0009-1464-5049 (Edjabrou Ulrich Blanchard Kablam)

0009-0008-7475-2156 (Grah Gelin Bogui)

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] K. P. Brou and E. D. Béché, *Deep Classification of a Generalization of Ring Filtration in Commutative Algebra*, International Journal of Algebra, Vol. 19 (2025), no. 2, pp. 79-88. https://doi.org/10.12988/ija.2025.91962

[2] E. D. Akeke, P. K. Ayegnon, and H. Dichi, *Extension of Samuel Number to Bifiltrations*, Southeast Asian Bulletin of Mathematics, Vol. 45 (2021), pp. 143-153.

[3] E. D. Akeke, S. Ouattara, and P. Ayegnon, *Another generalized Samuel number $\hat{b}f(g)$ on a semi-ring*, JP J. Algebra Number Theory Appl., Vol. 36 (2015), no. 2, pp. 123-139.

[4] E. D. Akeke and P. Ayegnon, *Some aspects of generalized Samuel numbers and quasi graduations on a semi-ring*, Pioneer J. Algebra Number Theory Appl., Vol. 5 (2013), no. 1, pp. 17-28.

[5] Y. M. Diagana and S. Ouattara, *Extension of the Samuel numbers of two distinctive types to quasi-gradings on a semi-ring*, Afr. Mathématiques. Ann. (AFMA), Vol. 2 (2011), pp. 119-128.

[6] S. Ouattara, E. D. Akeke, and P. Ayegnon, *Generalized Samuel numbers $\bar{\nu}_\varphi(\theta)$ and $\bar{\omega}_\varphi(\theta)$ on a semi-module*, Afr. Math. Ann. (AFMA), Vol. 2 (2011), pp. 175-189.

[7] S. Ouattara, E. D. Akeke, and P. Ayegnon, *Another generalized Samuel number $\hat{a}_f(g)$ on a semi-ring*, JP J. Algebra Number Theory Appl., Vol. 19 (2010), no. 2, pp. 185-201.

[8] M. Lejeune-Jalabert and B. Teissier, *Clôture intégrale des idéaux et équisingularité*, Ann. Fac. Sci. Toulouse Math., Vol. 17 (2008), pp. 781-859.

[9] P. Ayegnon, *Filtrations on a set and Samuel numbers*, Afr. Mat., Sér. III, Vol. 16 (2005), pp. 139-144.

[10] P. Ayegnon and H. Dichi, *Extensions to filtrations of the Samuel numbers associated to ideals*, Communication in Algebra, Vol. 22 (1994), no. 9, pp. 3249-3263.

[11] P. Ayegnon and D. Sangare, *Generalized Samuel numbers and AP-filtrations*, Journal of Pure and Applied Algebra, Vol. 65 (1990), pp. 1-13.

[12] D. Sangare, *On various generalizations of the formula $\bar{v}_{f_n} = \frac{1}{n}\bar{v}_f$ to pseudo-valuations associated with a filtration*, Department of Mathematics, University of Abidjan, Ivory Coast, (1984).

[13] J. W. Petro, *Some results in the theory in pseudo-valuations*, Ph.D. dissertation, State University of Iowa, Iowa City, (1961).

[14] D. Rees, *Variations associated with a local ring (1)*, Proc. London Math. Soc., Series 3, Vol. 5 (1955), pp. 107-128.