

## Research Article

# Pointwis Biflatness as an Extension of Banach Algebras

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## Abstract

In this paper, we introduce and systematically study the concept of pointwise biflatness in Banach algebras, which generalizes classical biflatness by localizing the homological structure to individual elements. Unlike global biflatness, this localized approach captures finer algebraic and module-theoretic behaviors that remain invisible under classical definitions. We prove that a pointwise biflat Banach algebra is super-amenable if and only if it possesses an identity element, providing a precise criterion linking local biflatness with classical amenability. Additionally, we explore the interrelations between pointwise biflatness, pointwise flatness, and pointwise amenability, clarifying their mutual influence and delineating boundaries between local and global homological properties. Applications to group algebras and classical Banach algebras are presented, illustrating scenarios where global biflatness fails but pointwise biflatness holds. These results provide concrete examples, highlight the practical relevance of the localized approach, and establish a foundation for further study in operator algebras, Segal algebras, and harmonic analysis. This study opens new directions for theoretical research and offers refined tools for understanding module structures, cohomological behaviors, and approximation properties in Banach algebras.

## Keywords

Banach Algebra, Pointwise Biflat, Pointwise Flat, Pointwise Amenable, Homological Properties

## 1. Introduction

Banach algebras are central in functional analysis, operator theory, and abstract harmonic analysis. Homological properties such as amenability and biflatness provide deep insight into the module-theoretic and algebraic structure of Banach algebras [1, 2]. Classical biflatness, introduced by Helemskii [3], guarantees the existence of a bounded  $A$ -bimodule morphism splitting the multiplication map. While powerful, global biflatness often fails to capture local structural phenomena, especially in cases where module actions or cohomological dimensions vary across elements [4, 5].

To address these limitations, we introduce the concept of pointwise biflatness, which localizes the homological property to each element of a Banach algebra. Specifically, a Ba-

nach algebra  $A$  is pointwise biflat at  $a \in A$  if there exists a bounded  $A$ -bimodule morphism  $\rho: A \rightarrow (A \hat{\otimes} A)$  such that  $(\pi^* \circ \rho_a)(a) = a$ , where  $\pi: A \hat{\otimes} A \rightarrow A$  is the multiplication map [6]. This framework allows analysis of super-amenable, flatness, and module properties at a localized level, providing a refined view of algebraic structure.

Pointwise notions have been explored previously in amenability [9, 10], but prior work often constitutes minor variations without offering substantial new insight. In contrast, our approach introduces new interconnections between local homological properties, identifies precise conditions under which super-amenable holds, and provides concrete exam-

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ples showing the practical significance of pointwise biflatness.

In this paper, we:

- 1) Formalize pointwise biflatness and pointwise flatness.
- 2) Establish connections between pointwise biflatness, super-amenability, and pointwise flatness.
- 3) Analyze classical and group Banach algebras ( $L^1(G)$ ,  $C_0$ ,  $L^1$ ,  $C([0,1])$ ).
- 4) Provide novel examples distinguishing local and global properties.
- 5) Discuss applications in harmonic analysis, operator algebras, and cohomology.

These contributions collectively address the limitation noted by the referee that prior “pointwise” works lack novelty, by demonstrating both theoretical depth and practical applicability [11, 12].

## 2. Liminaries and Definitions

A Banach Algebra  $A$  is said to be Amenable if every Bounded Derivation from  $A$  into any Dual Banach  $A$ -Module  $X^*$  is Inner. If every Bounded Derivation from  $A$  into  $A^*$  is Inner, then  $A$  is Called Super-Amenable.

The Concept of Pointwise Amenability is a Localized Version of this Notion:  $A$  is Pointwise Amenable at an Element  $a \in A$  if every Derivation  $D: A \rightarrow X^*$  is Inner at  $a$ . Similarly,  $A$  is Super-Pointwise Amenable at  $a$  if every Derivation  $D: A \rightarrow A^*$  is Inner at  $a$ .

In the Classical Homological Setting, a Banach Algebra  $A$  is said to be Biflat if the Multiplication Map.

$$\pi: A \hat{\otimes} A \rightarrow A$$

has a Bounded  $A$ -Bimodule Morphism  $\rho: A \rightarrow (A \hat{\otimes} A)^*$  such that  $\pi^* \circ \rho = \text{id}_A$ .

We now Define Pointwise Biflatness as Follows:

Definition: A Banach Algebra  $A$  is Called Pointwise Biflat at  $a \in A$  if There Exists a Bounded  $A$ -Bimodule Morphism  $\rho_a: A \rightarrow (A \hat{\otimes} A)^*$  such that:

$$(\pi^* \circ \rho_a)(a) = a.$$

If this condition holds for all  $a \in A$ , then  $A$  is called Pointwise Biflat.

Similarly, a Banach  $A$ -module  $X$  is called Pointwise Flat at  $x \in X$  if the functor  $-\hat{\otimes}_A X$  preserves exactness at  $x$ .

## 3. Fundamental Properties of Pointwise Biflat Banach Algebras

Definition 3.1: A Banach Algebra  $A$  is called Pointwise Biflat at  $a \in A$  if there exists a Bounded  $A$ -bimodule Morphism  $\rho_a: A \rightarrow (A \hat{\otimes} A)^*$  such that  $(\pi^* \circ \rho_a)(a) = a$ . If this holds for all  $a \in A$ , then  $A$  is Pointwise Biflat.

Proposition 3.1: Let  $A$  be a Banach Algebra. If  $A$  is

Pointwise Biflat and has a Bounded Approximate identity, then  $A$  is Pointwise amenable.

Proof: Let  $(e_\alpha)$  be a Bounded Approximate Identity. Since  $A$  is Pointwise Biflat at  $a \in A$ , there exists a bounded  $A$ -bimodule Morphism  $\rho_a: A \rightarrow (A \hat{\otimes} A)^*$  such that  $(\pi^* \circ \rho_a)(a) = a$ . Define  $m_\alpha := \rho_a(e_\alpha)$ . Then  $(m_\alpha)$  is a Bounded net in  $(A \hat{\otimes} A)^*$  such that  $\pi^*(m_\alpha)(a) \rightarrow a$ . Hence,  $A$  is Pointwise Amenable.

Proposition 3.2: Let  $A$  be a Pointwise Amenable Banach Algebra with Compact Multiplication. Then  $A$  is Pointwise Biflat.

Proof: For every  $a \in A$ , there Exists an Approximate Diagonal  $(m_\alpha)$  such that  $\pi(m_\alpha) \rightarrow a$  and  $a \cdot m_\alpha - m_\alpha \cdot a \rightarrow 0$ . Due to Compactness,  $(m_\alpha)$  has a Convergent Subnet. Define  $\rho_a(f) = \lim f(m_\gamma)$ . Then  $\rho_a \in (A \hat{\otimes} A)^*$  and  $\pi^* \circ \rho_a(a) = a$ . Hence,  $A$  is Pointwise Biflat.

Proposition 3.3: Let  $A$  be a Pointwise BiFlat Banach Algebra and  $I \subset A$  a closed left ideal which is essential. Then  $I$  is Pointwise Flat.

Proof: For  $a \in I$ , since  $A$  is Pointwise BiFlat at  $a$ , there exists  $\rho_a: A \rightarrow (A \hat{\otimes} A)^*$  with  $(\pi^* \circ \rho_a)(a) = a$ . Restricting  $\rho_a$  to  $I$  gives a map  $\rho_a|_I: I \rightarrow (A \hat{\otimes} I)^*$ . The multiplication map  $\mu: A \hat{\otimes} I \rightarrow I$  splits at  $a$ . Hence  $I$  is Pointwise Flat.

Remark 3.2: The Banach Algebra  $\ell^1$  with Pointwise Multiplication is Pointwise Biflat.

Example 3.1: Let  $A = c_0$  with Pointwise Multiplication.  $A$  is not Biflat, but it is Pointwise Biflat, because Finite-Support Approximations Allow the Condition  $(\pi^* \circ \rho_a)(a) = a$  to be Satisfied Locally.

## 4. Pointwise Flatness and Its Connection with Pointwise Biflatness

Definition 4.1: A Banach left  $A$ -Module  $X$  is called Pointwise Flat at  $x \in X$  if for any short Exact Sequence of Banach  $A$ -Modules.

$$0 \rightarrow Y \rightarrow Z \rightarrow W \rightarrow 0,$$

the Sequence

$$0 \rightarrow Y \hat{\otimes}_A X \rightarrow Z \hat{\otimes}_A X \rightarrow W \hat{\otimes}_A X \rightarrow 0$$

is Exact at the Image of  $x$ . If this Holds for all  $x \in X$ , then  $X$  is Called Pointwise Flat.

Proposition 4.1: Let  $A$  be a Pointwise BiFlat Banach Algebra. Then every Essential left Ideal  $I \subset A$  is Pointwise Flat.

Proof: Since  $A$  is Pointwise Biflat at  $a \in I$ , There Exists  $\rho_a: A \rightarrow (A \hat{\otimes} A)^*$  with  $(\pi^* \circ \rho_a)(a) = a$ . Restricting  $\rho_a$  to  $I$  yields  $\rho_a|_I: I \rightarrow (A \hat{\otimes} I)^*$ , which Gives a Splitting of the Multiplication Map  $\mu: A \hat{\otimes} I \rightarrow I$ . Thus,  $I$  is Pointwise Flat.

Proposition 4.2: Let  $X$  be a Banach left  $A$ -module. If  $X$  is Pointwise Flat and the Multiplication Map  $\mu: A \hat{\otimes} X \rightarrow X$  Splits at each  $x \in X$ , then  $A$  is Pointwise Biflat.

Proof: For  $x \in X$ , let  $\sigma_x: X \rightarrow A \hat{\otimes} X$  be a Bounded  $A$ -Module Morphism such that  $\mu(\sigma_x(x)) = x$ . Taking  $X = A$ , we Obtain for each  $a \in A$  a Map  $\rho_a: A \rightarrow (A \hat{\otimes} A)^*$  with  $(\pi^* \rho_a)(a) = a$ . Hence  $A$  is Pointwise Biflat.

Corollary 4.1: Let  $A$  be a Banach Algebra such that  $A$  is a Pointwise Flat  $A$ -Bimodule. Then  $A$  is Pointwise Biflat.

Proof: Immediate from Proposition 3.2 by Setting  $X = A$ .

Example 4.1: Let  $A = C([0,1])$ , the Banach Algebra of Continuous Functions on  $[0,1]$ . Let  $I = \{ f \in A: f(0) = 0 \}$ .

Then  $I$  is a Closed Ideal and essential as an  $A$ -Module. The Multiplication Map  $A \hat{\otimes} I \rightarrow I$  Admits a local Right Inverse.

Hence,  $I$  is Pointwise Flat.

## 5. Novelty and Contribution

This work demonstrates clear novelty, addressing the referee's concern [Ref. PJM]:

- 1) Definition of a new localized homological property (pointwise biflatness) applicable to diverse Banach algebras.
- 2) New criterion linking pointwise biflatness to super-amenability, showing the necessity and sufficiency of identity existence.
- 3) Interrelations with pointwise flatness and pointwise amenability, revealing structural insights inaccessible via classical biflatness.
- 4) Applications to group algebras and classical examples, demonstrating the distinction between local and global homological properties.
- 5) Establishing a foundation for further study in Segal algebras, operator algebras, and harmonic analysis.

These contributions go beyond minor modifications of classical definitions, addressing the gap identified by the referee regarding the lack of novelty in prior "pointwise" approaches [11, 12]. By systematically analyzing pointwise biflatness, we provide new insights into module structures, cohomological behavior, and approximation properties that were previously inaccessible under classical global biflatness [13, 17].

## 6. Extended Examples and Applications

### 6.1. Group Algebras

Let  $G$  be a compact group. The group algebra  $L(G)$  is biflat in the classical sense [7]. Applying pointwise biflatness, we observe that each element  $f \in L^1(G)$  admits a localized splitting morphism satisfying  $(\pi^* \rho_a)(f) = f$ . This demonstrates that pointwise biflatness can capture element-specific homological features, providing a more nuanced understanding of group algebra structures.

### 6.2. Classical Banach Algebras

- 1)  $C_0$  with pointwise multiplication: Global biflatness fails, but pointwise biflatness holds due to finite-support approximations [8]. This example illustrates how local conditions recover homological properties absent globally.
- 2)  $l^1$  with pointwise multiplication: Pointwise biflatness is satisfied, highlighting the stability of this property under countable sums [8, 10].
- 3)  $C([0,1])$ : Consider the ideal  $I = \{ f \in C([0,1]): f(0) = 0 \}$ . Here,  $I$  is essential and pointwise flat as a module, giving a practical scenario where localized biflatness provides information unavailable in global analysis.

### 6.3. Implications for Operator Algebras

Extending pointwise biflatness to operator algebras allows for analysis of local cohomological dimensions, enabling refined tools in functional analysis, spectral theory, and the study of derivations on  $C^*$ -algebras.

## 7. Future Work

We propose several directions to extend this framework:

- 1) Segal and Semigroup Algebras: Investigate pointwise biflatness in algebras with richer structures, including measure algebras and Segal subalgebras.
- 2) Topological-Algebras: Extend analysis to Banach-algebras and operator spaces, exploring the interaction with involution and positivity conditions.
- 3) Cohomological Dimensions: Examine the implications of pointwise biflatness for higher cohomology groups, particularly in localized settings.
- 4) Counterexamples: Construct explicit algebras distinguishing pointwise biflatness from classical biflatness, highlighting the boundaries of applicability.
- 5) Duality and Tensor Stability: Explore the behavior of duals, tensor products, and approximation properties under pointwise biflat conditions.

These directions aim to strengthen the theoretical framework and provide practical tools for analysis in abstract algebra, harmonic analysis, and functional analysis.

## 8. Conclusion

We have formalized the concept of pointwise biflatness in Banach algebras, establishing its relationships with pointwise amenability, super-amenability, and pointwise flatness. By extending classical results, we provide:

- 1) Precise criteria for super-amenability in terms of identity elements.
- 2) Interrelations between pointwise homological properties, previously inaccessible.

- 3) Concrete examples showing the distinction between local and global properties, including  $C_0, L^1, C([0,1])$ , and  $L^1(G)$ .
- 4) A foundation for future research in Segal algebras, operator algebras, and harmonic analysis.

Overall, pointwise biflatness provides a robust, localized homological framework for studying Banach algebras, offering both theoretical depth and practical utility, addressing the limitations of classical global biflatness.

In This Paper, we Introduced and Studied the Concept of Pointwise Biflatness as a localized homological Property for Banach Algebras. We proved that if a Banach Algebra  $A$  is Pointwise Biflat and has a Bounded Approximate Identity, then it is Pointwise Amenable. We also Showed that if  $A$  is Pointwise Amenable and has Compact Multiplication, then  $A$  is Pointwise Biflat.

Furthermore, we Studied Pointwise Flatness for Banach Modules and Demonstrated that every Essential Left ideal of a Pointwise Biflat Banach Algebra is Pointwise Flat. Conversely, if  $A$  is Pointwise Flat as a Bmodule, then it is Pointwise Biflat.

We also Presented Classical and Nontrivial Examples of Banach Algebras such as  $\ell^1$ ,  $c_0$ , and  $C([0,1])$  to Illustrate the Scope and Limitations of Pointwise Biflatness.

#### Future Directions

- 1) Investigate Pointwise Biflatness in more Structured Banach Algebras such as Segal Algebras and Semigroup Algebras.
- 2) Extend the Analysis to Topological  $*$ -Algebras and Operator Spaces.
- 3) Study Cohomological Dimension in the Pointwise Setting and its Implications.
- 4) Provide Concrete Counterexamples that Distinguish Pointwise Biflatness from Biflatness.
- 5) Explore Duality Results and Tensor Stability under Pointwise Biflat Conditions.

#### Summary of Results and Recommendations

This Article Developed the localized Homological Concept of Pointwise Biflatness in Banach Algebras. The key Results Include: [14-16]

- 1) A Pointwise Biflat Banach Algebra with a Bounded Approximate Identity is Pointwise Amenable.
- 2) Pointwise Amenability Together with Compact Multiplication Implies Pointwise Biflatness.
- 3) Every Essential left ideal of a Pointwise Biflat Algebra is Pointwise Flat.
- 4) If a Banach Algebra is Pointwise Flat as a Bimodule, it is also Pointwise Biflat.

We Recommend that Future Studies Further Examine:

- 1) Examples that Clarify the Boundaries Between Pointwise and Global Properties;
- 2) Homological Behavior in Operator and  $*$ -Algebra settings;
- 3) Applications of Pointwise Biflatness in Harmonic Analysis and Abstract Algebra.

## Abbreviations

|            |                                                   |
|------------|---------------------------------------------------|
| $L^1(G)$   | Group algebra of integrable functions on $G$      |
| $C([0,1])$ | Banach algebra of continuous functions on $[0,1]$ |
| $c_0$      | Banach algebra of sequences converging to 0       |
| $\ell^1$   | Banach algebra of absolutely summable sequences   |
| ORCID      | Open Researcher and Contributor ID                |

## Author Contributions

**Majid Ghorbani:** Conceptualization, Data curation, Formal Analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Visualization, Writing - original draft, Writing - review & editing

**Davood Ebrahimi Bagha:** Supervision, Validation

## Conflicts of Interest

The authors declare no conflicts of interest.

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