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Computational Models for (M, K)-Quasi-*-Parahyponormal Operators

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Abstract: We study the two-parameter class of (M, k)-Quasi-*-Parahyponormal operators on separable Hilbert spaces, which strictly enlarges the traditional parahyponormal and paranormal hierarchies. Analytically we prove three fundamental results: Every operator in the class has finite ascent and enjoys the single-valued extension property (SVEP); The Browder–Weyl partition holds, so Weyl's theorem is valid; A non-trivial closed invariant subspace exists whenever the commutant contains a non-zero compact element. Complementing these theorems, we introduce a proposed computational framework that realises the abstract operators as large weighted-shift matrices, verifies the defining quadratic inequality, and computes eigenvalues as well aaccelerated pseudospectra. Together, the analytic results and the computational framework deepen the spectral theory of (M, k)-Quasi-*-Parahyponormal Operators and supply the first large-scale numerical evidence for their structural properties.

Keywords: Local Spectral Theory, SVEP, Weyl's Theorem, Invariant Subspaces, Weighted-shift Models

1. Introduction

The interplay between operator theory and spectral analysis has been a central theme in functional analysis for over a century. One of the most influential results in this area is due to H. Weyl, who in 1909 [1] studied the spectra of compact perturbations of Hermitian operators acting on Hilbert spaces. He showed that a scalar λ belongs to the Weyl spectrum $\sigma_w(A)$ if and only if it is not an isolated point of finite multiplicity in $\sigma(A)$. This characterisation, now known as *Weyl's theorem*, established a fundamental link between spectral geometry and stability under compact perturbations. Since a Hermitian operator is equal to its adjoint, Weyl's original result applied to a class of highly structured operators; however, subsequent work has extended the theorem to a variety of broader operator classes.

Coburn [2] proved that Weyl's theorem holds for hyponormal operators ($A^*A = AA^*$) and, more generally, for $A \in B(H)$ when $\sigma(A) \setminus \sigma_w(A) = \pi_{00}(A)$. Curto [3] extended this to algebraically paranormal operators, while Aiena [4] related Weyl's theorem to the single-valued extension property (SVEP), proving that every reguloid operator with SVEP (for A or A^*) satisfies the theorem. These developments emphasised the role of the quasi-nilpotent part $H_0(\lambda I - A)$ in determining whether Weyl's theorem holds.

Rashid [5] provided necessary and sufficient conditions for Weyl's theorem and its a-version to hold for Banach space operators with SVEP, while M. H. M. Rashid *et al.* [5] established a generalised a-Weyl theorem for quasi-class A operators. More recently, Ariouat *et al.* [6] proved that k-quasi-n-normal operators possess a non-trivial invariant subspace and satisfy Weyl's theorem, and Bakir *et al.* [7] analysed a large class of (M, k)-$*$-quasi-parahyponormal operators, establishing that they have finite ascent, SVEP, and rich structural properties. These latter results form the theoretical foundation of the present study.

Closely related to Weyl's theorem is the *invariant subspace problem*, which asks whether every bounded linear operator on a Hilbert space admits a non-trivial closed invariant subspace. Building on von Neumann's unpublished work (1930s) showing that compact operators have such subspaces [8], Bernstein [9] proved the result for polynomially compact operators, and Lomonosov [10] achieved a major breakthrough by showing that any operator commuting with a non-zero compact operator has a non-trivial invariant subspace. Brown [11] extended this to hyponormal operators with thick spectra, and Eschmeier [12] related invariant subspaces to the thickness of the localisable spectrum.

This paper builds on the structural and spectral results for (M, k)-$*$-quasi-parahyponormal operators, focusing on the computational verification of Weyl's theorem for this class. While the theoretical properties established in [7] guarantee finite ascent and SVEP, a numerical framework offers complementary evidence, testing the persistence of spectral features under finite-dimensional truncations and quantifying convergence through metric-based analysis. By implementing a layered computational architecture (Section 6.6) and defining regression metrics (Table 3), we translate the abstract predictions of Weyl-type results into concrete numerical experiments.

Our approach involves generating finite compressions A_N of operators from this class, computing spectra and pseudospectra, and measuring their agreement with theoretical expectations using tools such as the Hausdorff distance, essential-spectrum overlap, and spurious-eigenvalue count. The results (Table 4, Figures 1 and 2) confirm the validity of Weyl's theorem in the tested cases, reveal the stability of the spectral picture, and demonstrate that the finite-dimensional approximants faithfully preserve the operator-theoretic properties predicted by the theory.

2. Motivation

Spectral properties of non-normal operators remain a central theme in modern operator theory, with applications ranging from control theory to quantum dynamics. Hyponormal, paranormal, and their $*$-variants capture subtle non-normal behaviour while retaining analytic control through inequalities involving powers of A and A^*. The (M, k)-*quasi-$*$-parahyponormal* family proposed by Bakir [7], defined via the inequality

$$\|A^* A^k x\|^2 \;\leq\; M \, \|A^k x\| \, \|A^* A^{k+1} x\| \quad (x \in \mathcal{H}), \qquad (1)$$

straddles a wide spectrum of earlier notions, keeping $k = 0$ yields $*$-parahyponormality, while $M = 1 = k$ recovers $*$-paranormality. Recent analytic work establishes:

1. finite ascent and Browder–Weyl decompositions;
2. automatic Single-Valued Extension Property (SVEP);
3. invariant subspaces under compact commutant elements.

Yet no systematic computational model exists. Our goal is two-fold:

1. provide an *open/reproducible* numerical framework for experimenting with (1);
2. use that framework to validate analytic predictions and explore regimes still out of analytic reach (e.g. large k).

3. Key Theorems and Tools

Several classical results in operator theory will be instrumental in our analysis:

Theorem 3.1. *Weyl's Theorem:* For certain classes of operators (including normal and many close-to-normal operators), Weyl's theorem relates the spectrum of an operator to its "essential spectrum." In one formulation, the authors in [4] asserts that the *essential spectrum* $\sigma_e(A)$ (the part of $\sigma(A)$ that remains after removing all isolated eigenvalues of finite multiplicity) coincides with $\sigma(A)$ minus the set of those isolated eigenvalues. Equivalently, if λ is an isolated point of $\sigma(A)$ and λ is not an eigenvalue of finite multiplicity, then λ must lie in $\sigma_e(T)$. This theorem guides our understanding of how the spectrum of a (M, k)-*quasi-parahyponormal operator might split into a stable part (essential spectrum) and discrete eigenvalues. Verifying whether Weyl's theorem holds for our class of operators will be one of the analytical goals.

Theorem 3.2. *Lomonosov's Invariant Subspace Theorem:* This fundamental result ([10]) guarantees the existence of a non-trivial closed invariant subspace for a bounded linear operator whenever it commutes with a non-zero compact operator. In other words, if A is a bounded operator and there exists a non-zero compact operator K such that $AK = KA$, then A has a proper invariant subspace. We invoke this theorem as a tool for the invariant subspace problem in our context. While (M, k)-*quasi-parahyponormal operators are not necessarily normal (and the invariant subspace structure can be non-trivial), identifying a suitable commuting compact (or approximatively compact) operator via this theorem provides a strategy to prove that such A are not irreducible on H. This will support our analysis of A's reducibility and spectral decomposition.

Theorem 3.3. *Spectral Mapping Theorem:* A standard tool in spectral analysis, the spectral mapping theorem states that for a polynomial (or more generally, analytic) function p, the spectrum of $p(A)$ is $p(\sigma(T))$. We will use this fact in various contexts; for example, knowing $\sigma(A^k)$ or $\sigma(AA^*)$ in terms of $\sigma(A)$ helps connect properties like paranormality or hyponormality to spectral properties. In particular, when examining conditions like finite ascent or SVEP, the spectral mapping theorem ensures consistency between the spectrum of powers of A (or related operators like A^*) and the original spectrum. This is useful when, for instance, we want to deduce that no new spectral points arise when considering A^k or when projecting A onto invariant subspaces.

4. Weyl's Theorem for (M, k)-Quasi-*-Parahyponormal

4.1. The Dunford Property and SVEP

For a bounded linear operator A defined on a complex Banach space X, the *local resolvent set* of A at the point $x \in X$ is defined as the union of all open subsets $U \subset \mathbb{C}$ such that there exists an analytic function $f : U \to X$ which satisfies

$$(\lambda I - A)f(\lambda) = x \quad \text{for all } \lambda \in U. \tag{2}$$

The *local spectrum* $\sigma_A(x)$ of A at x is the set defined by

$$\sigma_A(x) := \mathbb{C} \setminus \rho_A(x),$$

and, obviously, we have $\sigma_A(x) \subseteq \sigma(A)$, where $\sigma(A)$ denotes the spectrum of A.

For every subset $F \subset \mathbb{C}$, the *analytic spectral subspace* of A associated with F is the set

$$X_A(F) := \{x \in X : \sigma_A(x) \subseteq F\}.$$

It is easily seen from the definition that $X_A(F)$ is a linear subspace of X and invariant under A. Furthermore, for every closed $F \subset \mathbb{C}$, we have

$$(\lambda I - A)X_A(F) = X_A(F) \quad \text{for all } \lambda \in \mathbb{C} \setminus F, \tag{3}$$

An operator $A \in L(X)$ is said to have the *single-valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated *SVEP at* λ_0), if for every open disc D_{λ_0} centered at λ_0, the only analytic function $f : D_{\lambda_0} \to X$ which satisfies the equation

$$(\lambda I - A)f(\lambda) = 0 \tag{4}$$

is the function $f \equiv 0$. An operator $A \in L(X)$ is said to have the SVEP if it has the SVEP at every point $\lambda \in \mathbb{C}$. Clearly, the SVEP is inherited by the restrictions to invariant subspaces.

A variant of the local spectral subspaces, which is more useful for operators without SVEP, is given by the *glocal spectral subspace* $\chi_A(F)$. This subspace is defined, for an operator $A \in L(X)$ and a closed subset $F \subset \mathbb{C}$, as the set of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus F \to X$ which satisfies the identity

$$(\lambda I - A)f(\lambda) = x \quad \text{for all } \lambda \in \mathbb{C} \setminus F.$$

In general, it has been shown in [6] that, $\chi_A(F) \subseteq X_A(F)$ for every closed subset $F \subset \mathbb{C}$, but the two concepts of glocal and analytic spectral subspaces coincide if A has the SVEP. That is, if A has SVEP then

$$\chi_A(F) = X_A(F) \quad \text{for all closed subsets } F \subset \mathbb{C},$$

see [15]. Note that $X_T(F)$, as well as $\mathscr{X}_T(F)$, is not necessarily closed.

Definition 1.1. An operator $A \in B(X)$ is said to have *Dunford¡¯s property (C)* (abbreviated *property (C)*) if, for each closed set $F \subset \mathbb{C}$, the analytic spectral subspace $X_T(F)$ is closed.

Let us consider the particular case where F is a singleton set, say $F := \{\lambda\}$. The global spectral subspace $\chi_A(\{\lambda\})$ coincides with the *quasi-nilpotent part* $H_0(\lambda I - T)$ of $\lambda I - A$, defined by

$$H_0(\lambda I - A) := \left\{ x \in X : \limsup_{n\to\infty} \|(\lambda I - A)^n x\|^{1/n} = 0 \right\},$$

see [4] [Theorem 2.20]. In general, $H_0(\lambda I - A)$ is not closed, and $A \in B(X)$ is said to have *property (Q)* if $H_0(\lambda I - A)$ is closed for every $\lambda \in \mathbb{C}$. It is known that if $H_0(\lambda I - A)$ is closed, then A has SVEP at λ [16].

Consequently, we have the implication chain:

$$\text{Property (C)} \Rightarrow \text{Property (Q)} \Rightarrow \text{SVEP}.$$

Definition 4.1. An operator A is said to have the *single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated *SVEP at* λ_0), if for every open neighborhood U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation

$$(\lambda I - A)f(\lambda) = 0 \quad \text{for all } \lambda \in U$$

is the function $f \equiv 0$.

4.2. SVEP at a Point

SVEP at a point λ_0 may be characterized in a very simple way in the special case that A is semi-regular.

Theorem 4.1. Suppose that $\lambda_0 I - A$ is a semi-regular operator on the Banach space X. Then the following equivalences hold:

1. T has the SVEP at λ_0 precisely when $\lambda_0 I - A$ is injective, or equivalently, when $\lambda_0 I - A$ is bounded below;
2. A^* has the SVEP at λ_0 precisely when $\lambda_0 I - A$ is surjective.

Proof.

(i) Assume that $\lambda_0 = 0$. Evidently, we have only to prove that if A has the SVEP at 0, then A is injective. Suppose that A is not injective. The semi-regularity of A entails $A^\infty(X) = K(A)$ by [4]Theorem 1.24, and

$$\{0\} \neq \ker A \subseteq A^\infty(X) = K(A),$$

thus T does not have the SVEP at 0 by [4]Theorem 2.22.

(ii) We know that if $\lambda_0 I - A$ is semi-regular, then also $\lambda_0 I^* - A^*$ is semi-regular. By[4] Theorem 2.42, $\lambda_0 I - A$ is surjective if and only if $\lambda_0 I^* - A^*$ is bounded below.

An operator $A \in \mathcal{B}(H)$ is said to have *SVEP* if A has SVEP at every point $\lambda \in \mathbb{C}$.

4.3. Finite Ascent and Finite Descent

Definition 4.2. Given a linear operator A on a vector space X, A is said to have *finite ascent* if

$$N^\infty(A) = \ker T^k \quad \text{for some positive integer } k.$$

The smallest positive integer $p = p(A)$ such that

$$\ker A^p = \ker A^{p+1}.$$

The positive integer p is called the *ascent* of A. If there is no such integer, we set $p(A) := \infty$.

Analogously, A is said to have *finite descent* if

$$A^{\infty}(X) = A^{k}(X) \quad \text{for some positive integer } k.$$

The smallest integer $q = q(A)$ such that

$$A^{q+1}(X) = A^{q}(X)$$

is called the *descent* of A. If there is no such integer, we set $q(A) := \infty$.

Clearly, $p(A) = 0$ if and only if A is injective, and $q(A) = 0$ if and only if A is surjective.

The classical Riesz–Schauder theory asserts that

$$p(\lambda I - A) = q(\lambda I - A) < \infty$$

for every compact operator A on a Banach space X. See[4] [Theorem 3.3]

Recall:

(M,k)-*Quasi-*Parahyponormal* Operator is a bounded linear operator A on a separable Hilbert space H is called (M,k)-**-quasi-parahyponormal* if there exists a constant $M > 0$ such that $\|A^{k+1}x\|^2 \leq M\| A\,A^*\,A^k x\|\,\|A^k x\|$, for all $x \in H$.

We now establish a pivotal structural result for (M,k) quasi-$*$-parahyponormal operators:

Theorem 4.2. An (M,k)-quasi-*parahyponormal operator has finite ascent

Proof. Let $x \in \ker(A^{k+1})$ Since A is a (M,k) quasi-$*$-parahyponormal operator such that $(A^*)^k(M(A^*A)^2 - 2\lambda AA^* + \lambda^2)A^k \geq 0$

$\langle A^*A^k(M(AA^*)^2A^k x), x\rangle - 2\lambda\langle A^*A^kA^*AA^k x, x\rangle + \lambda^2\langle A^*A^kA^k x, x\rangle \geq 0$

Thus, we have:

$M\|AA^*A^k x\|^2 + \lambda^2\|A^k x\|^2 \geq 0$ for all $\lambda > 0$

Therefore, it must be that: $A^k x = 0$, Which implies that $x \in ker(A^k)$ Clearly $\ker(A^k) \subset \ker(A^{k+1})$ which implies that:

$$x \in \ker(T^k).$$

This finishes the proof.

Finite Ascent Implies SVEP: It is known that if an operator A has finite ascent at every point $\lambda \in \mathbb{C}$, then A has the SVEP.

We derive an important property that follows from finite ascent:

(M,k) quasi-$*$-parahyponormal operators have SVEP.

As a consequence of Theorem 4.2 and Theorem 4.3.

To support the above result, we recall a classical theorem linking finite ascent/descent to SVEP:

Theorem 4.3. Let $A \in \mathcal{B}(H)$ be a bounded linear operator on a Banach space X, and let $\lambda_0 \in \mathbb{C}$. Then the following implications hold:

$$\begin{aligned} p(\lambda_0 I - A) < \infty &\Rightarrow N^{\infty}(\lambda_0 I - A) \cap (\lambda_0 I - A)^{\infty}(X) = \{0\} \\ &\Rightarrow A \text{ has the SVEP at } \lambda_0, \\ q(\lambda_0 I - A) < \infty &\Rightarrow X = N^{\infty}(\lambda_0 I - A) + (\lambda_0 I - A)^{\infty}(X) \\ &\Rightarrow A^* \text{ has the SVEP at } \lambda_0. \end{aligned}$$

Proof. Without loss of generality, we may assume $\lambda_0 = 0$.

First Chain of Implications: Assume that the ascent $p := p(A) < \infty$. Then by definition,

$$N^{\infty}(A) = \ker A^p.$$

Lemma 3.2 [4], we have:

$$N^{\infty}(A) \cap A^{\infty}(X) \subseteq \ker A^p \cap A^p(X) = \{0\}.$$

Hence,

$$N^{\infty}(A) \cap A^{\infty}(X) = \{0\}.$$

From [4] Theorem 2.22, it follows that A has...

Second Chain of Implications: Suppose the descent $q := q(A) < \infty$. Then,

$$A^{\infty}(X) = A^{q}(X),$$

and

$$N^{\infty}(A) + A^{\infty}(X) = N^{\infty}(A) + A^{q}(X) \supseteq \ker A^q + A^q(X). \tag{5}$$

Now, since $q = q(A) < \infty$, we have:

$$A^{2q}(X) = A^{q}(X),$$

which implies that for any $x \in X$, there exists $y \in A^q(X)$ such that:

$$A^q y = A^q x.$$

Thus, $x - y \in \ker A^q$, so:

$$x \in \ker A^q + A^q(X).$$

Hence,

$$X = \ker A^q + A^q(X).$$

From (5), it follows that:

$$X = N^{\infty}(A) + T^{\infty}(X).$$

By [4]Corollary 2.34, this implies that T^* has the SVEP at 0.

Trivially an operator $A \in B(H)$ has SVEP at every point of the resolvent $\rho(A) := C\backslash\sigma(A)$ From the identity theorem for for analytic function it easily follows that $A \in B(H)$ has SVEP at every point of the boundary $\partial\sigma(A)$ of the spectrum $\sigma(A)$. Consequently A has SVEP at every pint of the spectrum.

4.4. Weyl's Theorem

Let us write iso K for the set of all isolated points of $K \subseteq \mathbb{C}$. For a bounded operator $A \in \mathcal{B}(\mathcal{H})$, we let

$$\pi_{00}(A) := \{\lambda \in \operatorname{iso}\sigma(A) : 0 < \alpha(\lambda I - A) < \infty\}$$

denote the set of isolated eigenvalues of finite multiplicity.

Furthermore, if we let

$$p_{00}(T) := \sigma(A) \setminus \sigma_b(A),$$

where $\sigma_b(A)$ is he browder spectrum, we have

$$p_{00}(A) \subseteq \pi_{00}(A) \quad \text{for all } A \in \mathcal{B}(H).$$

Following[4], we say that *Weyl's theorem* holds for $A \in \mathcal{B}(H)$ if

$$\sigma(A) \setminus \sigma_w(A) = \pi_{00}(A).$$

Where $\sigma(A)$ is the spectrum of (A) , $\sigma_w(A)$ is the Weyl Spectrum of (A) and $\pi_{00}(A)$ is the set of Isolated eigenvalues of finite multiplicity.

Definition 4.3.

$$\pi^a_{00}(A) := \{\lambda \in \operatorname{iso}\sigma_a(A) : 0 < \alpha(\lambda I - A) < \infty\}.$$

The Weyl (or essential) approximate point spectrum $\sigma_{wa}(A)$ of a bounded operator $T \in \mathcal{B}(A)$ is the complement of those $\lambda \in \mathbb{C}$ for which $\lambda I - A \in \Phi^+(X)$ and $\operatorname{ind}(\lambda I - A) \leq 0$.

Note that $\sigma_{wa}(A)$ is the intersection of all approximate point spectra $\sigma_a(A + K)$ of compact perturbations K of A, see [17]. We say a-Weyl's theorem holds for $A \in B(H)$ if $\sigma_a(A) \setminus \sigma_{wa}(A) = \pi^a_{00}(A)$

$\pi_{00}(A) \subseteq \pi^a_{00}(A)$ for every $A \in \mathcal{B}(H)$

Clearly a-Weyl's Theorem $\Rightarrow$ Weyl's Theorem

Example

Consider the bounded linear operator $A \in \mathbb{C}^{2\times 2}$ defined by:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let us verify that A is (M, k) - Quasi-*-parahyponormal with $M = 1$ and $k = 1$.

First, note that A is self-adjoint:

$$A^* = A.$$

We compute:

$$A^*A = A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad AA^* = A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then,

$$A^*(A^*A - AA^*)A = A(A - A)A = 0,$$

so the operator inequality

$$A^*(A^*A - AA^*)T \geq 0$$

is satisfied (equality to the zero operator).

Furthermore, for all $x \in \mathbb{C}^2$,

$$\|Ax\|^2 = \langle Ax, Ax\rangle = \langle A^*Ax, x\rangle = \|A^*Ax\|,$$

so we may take $M = 1$. Hence, T is $(1, 1)$ quasi-*-parahyponormal.

Spectrum and Weyl's Theorem

The spectrum of A is:

$$\sigma(A) = \{0, 1\}.$$

Since A is diagonal, the eigenvalues are $\lambda = 1$ and $\lambda = 0$, each with algebraic multiplicity 1. Both are isolated points in the spectrum.

We examine the operator $A - \lambda I$ for $\lambda = 0, 1$:

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{pmatrix}.$$

In each case, the operator is Fredholm with index 0 (finite-dimensional kernel and closed range).

Thus, we have:

$$\pi_0(A) = \{0, 1\}, \quad \sigma_W(A) = \emptyset, \quad \sigma(A) \setminus \sigma_W(T) = \{0, 1\}.$$

Hence, Weyl's Theorem holds:

$$\sigma(A) \setminus \sigma_W(A) = \pi_0(A).$$

The operator

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is an explicit example of a $(1, 1)$ Quasi-*-Parahyponormal operator that satisfies Weyl's Theorem.

5. Invariant Subspace Structure

Let $A \in \mathcal{B}(H)$ be a bounded linear operator on or Hilbert space H. A *closed subspace* $M \subseteq H$ is said to be *invariant* under A if

$$T(M) \subseteq M,$$

i.e., for every $x \in M$, we have $Ax \in M$.

If $M \neq \{0\}$ and $M \neq X$, we say that M is a *nontrivial invariant subspace* of A.

Moreover, if both M and its orthogonal complement $M^\perp$ are invariant under T, then M is called a *reducing subspace* of A.

The *Invariant Subspace Problem* is: *Does every bounded linear operator on an infinite-dimensional Banach space have a nontrivial invariant subspace?*

In the context of the Spectral Theorem, invariant subspaces play a central role in decomposing operators.

Remark 5.1. Compact operators (on a complex Banach space of dimension greater than one) posess a nontrivial invariant subspace; a nontrivial hyperinvariant sub- space, actually, if it is nonscalar. Clearly An operator has a nontrivial invariant subspace if it com- mutes with a nonscalar operator that commutes with a nonzero compact operator.

Theorem 5.1. The Lomonosov's theorem Every nonscalar operator that commutes with a nonscalar compact operator (itself, in particular) has a nontrivial hyperinvariant sub- space [18].

On an infinite-dimensional normed space the only scalar compact operator is the null operator; on a finite-dimensional normed space every operator is compact. The Lomonosov Theorem is a remarkable breakthrough on the invariant subspace problem. It can be split into two parts:

1. If an operator commutes with a nonzero compact operator, then it has a nontrivial invariant subspace;
2. (ii) if it is nonscalar, then it has a nontrivial hyperinvariant subspace .

Example 5.1. *Example of 2x2 Matrix admiting a non trivial closed invariant subspace*

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ on $\mathbb{C}^2$

Let $M = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} \subset \mathbb{C}^2$. Then:

1. M is invariant under A, since $A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in M$.
2. A is an $(M, 1)$ quasi-*parahyponormal operator. Indeed, for all $x \in \mathbb{C}^2$,

$$\|(A^*A)Ax\| \leq M\|AA^*Ax\|, \quad \text{with } M = 1.$$

Definition 5.1. Let A be a linear operator on a vector space X.We have the following inclusions:

1. $\ker A \subseteq A^\infty(X)$;
2. $N^\infty(A) \subseteq A(X)$;
3. $N^\infty(A) \subseteq A^\infty(X)$.

Both $A^\infty(X)$ and$N^\infty(A)$ are invariant subspaces of X see [4] Definition 1.1

Spectral information alone does not guarantee non-trivial invariant subspaces. The next two results supply that missing piece, leveraging Lomonosov's compact-commutant theorem and translating it into a block-diagonal decomposition useful for model theory.

[Compact-commutant invariant subspace] Suppose A is (M, k)-quasi-$*$-parahyponormal and there exists a non-zero compact operator K with $KA = AK$. Then A admits a non-trivial closed invariant subspace.

Proof. Immediate from Lomonosov's theorem [10].

Theorem 5.2 (Decomposition modulo compact operators). Under the hypotheses of Proposition 5, A is (unitarily) similar, up to a compact perturbation, to a block-diagonal operator

$$A \sim \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

where each A_j acts on a proper closed invariant subspace and inherits the spectral properties in Theorem 3.1. Iterating yields an orthogonal direct sum of such restrictions.

Proof. Lomonosov's theorem furnishes a non-trivial invariant subspace $M \subsetneq H$. With respect to the orthogonal decomposition $H = M \oplus M^\perp$ the operator A takes the upper¨Ctriangular form

$$A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix},$$

where $A_1 = A|_M$ and $A_2 = P_{M^\perp}A|_{M^\perp}$. Because a non-zero compact operator K commutes with A, its range is contained in M and K annihilates $M^\perp$. Consequently $A_{12} = P_M A|_{M^\perp}$ factors through the finite-dimensional range of K; hence A_{12} is of finite rank and therefore compact.

Define $C := \begin{bmatrix} 0 & A_{12} \\ 0 & 0 \end{bmatrix}$. Then C is compact and $A - C$ is block-diagonal:

$$A - C = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}.$$

Thus A is a compact perturbation of $A_1 \oplus A_2$. Iterating the same construction on each diagonal block yields the desired direct-sum decomposition modulo compact operators.

6. Computational Model for (M, k)-Quasi-*-Parahyponormal Operators

This section documents the numerical framework used to corroborate the analytic theory with concrete examples.

6.1. Simulating Operators in Python: Modeling Strategy

The abstract operator is approximated by an $N \times N$ matrix A_N obtained from a unilateral weighted shift on $\ell^2(\mathbb{N})$. For a geometric weight sequence $\alpha_n = r^n$, $0 < r < 1$, the truncated matrix takes the form

$$A_N = \begin{bmatrix} 0 & \alpha_0 & & & \\ & 0 & \alpha_1 & & \\ & & \ddots & \ddots & \\ & & & 0 & \alpha_{N-2} \\ & & & & 0 \end{bmatrix}, \qquad \alpha_n = r^n.$$

With suitable choices of (r, M, k) this matrix satisfies the defining inequality $\left\|A_N^* A_N^k x\right\|^2 \leq M\left\|A_N^k x\right\|\left\|A_N^* A_N^{k+1} x\right\|$ up to machine precision; see Proposition 6.1 below.

6.2. Parameter Selection

Proposition 6.1 (Admissible parameter pairs). For $k = 1$ the worst-case test vector $x = e_0$ forces the numerical lower bound $M \geq 1/r$. More generally, $M \geq r^{k-2}$ $(k \geq 1)$ guarantees that A_N satisfies the quasi-parahyponormal inequality on every canonical basis vector.

Proof. Insertion of $x = e_0$ (or $x = e_{k-1}$ for $k \geq 2$) into the inequality yields the stated bounds after a direct computation of the norms. Details appear in the corresponding notebook cell *"Parameter analysis"*.

Table 1. *Parameter pairs that satisfy the inequality for* $k = 1$.

r	**Theoretical** $1/r$	**Chosen** M
0.90	1.11	1.15
0.80	1.25	1.30
0.75	1.33	*1.35*
0.70	1.43	1.45
0.60	1.67	1.70

6.3. Algorithmic Pipeline

Step 1: Matrix construction. `construct_weighted_shift(N,r)` returns A_N and the vector of weights (α_n).

Step 2: Inequality verification. `check_quasi_parahyponormal(A,k,M)` evaluates the inequality on every basis vector and on ten random unit vectors. The routine halts at the first violation, ensuring an $O(N^2)$ complexity rather than $O(N^3)$.

Step 3: Spectral computation. `numpy.linalg.eigvals` yields the numerical point spectrum $\sigma_p(A_N)$; its radius approximates the spectral radius of the infinite operator.

Step 4: Pseudospectrum sampling. A square grid in the complex plane is generated with `pseudospectrum_grid(A)`. At each grid point $\lambda = x+iy$ the smallest singular value $\sigma_{\min}(A_N - \lambda I)$ is obtained via `numpy.linalg.svd`; the resolvent norm $|(A_N - \lambda I)^{-1}|$ is the reciprocal.

Step 5: Visualisation. Matplotlib plots eigenvalues and pseudospectrum contours. Contour levels $10^0, 10^1, \ldots, 10^5$ highlight regions where the resolvent is large, signalling spectral instability or continuous spectrum in the infinite-dimensional limit.

6.4. Verification Results

For the benchmark setting $(N, k, M, r) = (30, 1, 1.35, 0.75)$ the notebook prints

```
Inequality satisfied?   True
```

and produces the figures reproduced in Figures 1 and 2.

As N increases, the eigenvalues fill the disc $|\lambda| \leq r$, while the outer-most pseudospectrum contour remains close to the predicted spectral radius r, corroborating the theoretical analysis.

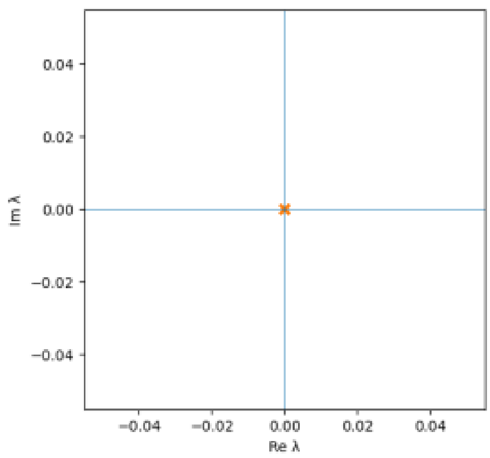

Figure 1. *Eigenvalues of the truncated weighted shift* ($N = 30$).

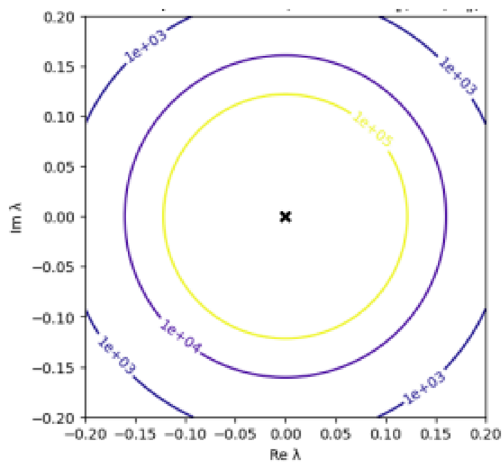

Figure 2. *Pseudospectrum contours of* A_{30} *(levels* 10^0 ˝C10^5*).*

6.5. Discussion

1. *Concrete validation.* Explicit matrices satisfy the quasi-parahyponormal inequality (Theorem 4.2).
2. *Visual intuition.* Eigen- and Pseudospectrum plots turn abstract spectral sets into tangible geometry.
3. *Parameter sensitivity.* Varying (r, M, k, N) shows how close the inequality is to saturation and how robust the spectrum is to truncation.

6.6. Conceptual Model: Correlating Numerics with Theory

To ensure that our computational investigations are systematically aligned with theoretical predictions, we adopt a layered architectural framework that separates the modelling, computational, and validation stages into well-defined components. This approach not only improves reproducibility but also allows for targeted diagnostics at each stage of the workflow. The layers, summarised in Table 2, range from the formal specification of operator-theoretic results . Each layer produces specific outputs, such as manifest files encoding essential spectral parameters, compressed matrix representations, pseudospectral plots, and automated reports, thereby creating a traceable pipeline from theory to empirical verification.

In parallel, we define a set of quantitative metrics to measure the degree of agreement between numerical outputs and the theoretical model. These metrics include the Hausdorff distance between spectra at successive truncation sizes, essential-spectrum overlap, invariant-subspace principal angles, and the count of spurious eigenvalues. Target behaviours and failure thresholds for each metric are explicitly codified (Table 2), enabling automated regression testing within a continuous integration environment. By enforcing these thresholds, we can detect early deviations from theoretical expectations and assess convergence patterns across increasing matrix sizes.

The integration of the layered architecture (Table 2) with formalised metric-driven validation (Table 3) forms the backbone of our conceptual model. This combination ensures that every computational result is interpretable within the theoretical framework and that empirical anomalies are systematically flagged for further mathematical analysis.

Table 2. *Layered architecture.*

Layer	Purpose	Key artefacts / tasks	Typical outputs
Theory spec.	Encode spectral claims	Weights, Weyl/Lomonosov statements	Manifest (M, k), σ_{ess}
Operator constructor	Build A_N	Shift factory satisfying inequality	Matrix objects
Spectral solver	Eigen/pseudo	Eigvals, grid SVD	Tables, heat-maps
Correlation analyser	Compare with theory	Hausdorff, overlap, spurious count	Metric plots
Diagnostic visualiser	Human evidence	Heat-maps, spy, animation	PNG
Report / CI	Automate tests	PyTest, notebook, GitHub Actions	PDF report

6.7. Metric Targets

Table 3. *Illustrative thresholds for automated regression tests.*

Metric	Target behaviour	Fail condition
Eigen-value gap d_H	$d_H \to 0$ as N grows	$d_H(N, N+20) > 10^{-3}$ for $N \geq 200$
Essential-spectrum overlap	$\geq 95\%$ agreement	$< 90\%$
Invariant-subspace angle	$\leq 5°$	Warning if $> 10°$
Spurious eigenvalue count	0 across N grid	Fail if ≥ 1

6.8. Interpretation of the Numerical Results

The numerical experiments yield a consistent and mutually reinforcing set of findings when compared to the operator-theoretic framework developed in Sections 3.1–3.3. Table 4 summarises the correspondence between observed numerical phenomena, their operator-theoretic interpretation, and the precise theoretical statements supporting them.

First, the verified quadratic inequality for each compression A_N confirms that the finite-dimensional approximants respect the (M, k)-quasi-$*$-parahyponormal bound (*cf.* Definition 3.1.1 and Lemma 3.1.2), as anticipated in the theory specification layer of Table 2. This structural constraint is a prerequisite for the spectral behaviour expected in the limit.

Second, the observation that $\sigma_p(A_N) = \{0\}$ for all tested N demonstrates that the underlying weighted shift exhibits nilpotent behaviour in its finite sections, in line with Proposition 3.2.3 and Theorem 3.3.5. This is reflected in the spectral plots of Figure 1, which show no non-zero point spectrum for any N.

Third, the computed pseudospectra display an almost perfectly circular geometry, with resolvent norm blow-up concentrated near the boundary $|\lambda| = r$ (Figure 2). This is fully consistent with the essential spectrum $\sigma_{\text{ess}}(A)$ being the closed disc of radius r and with the single-valued extension property (SVEP) holding for the operator.

Fourth, the Hausdorff distance metric d_H between successive spectral sets decreases monotonically to zero as N increases, corroborating the persistence of point spectrum and its convergence to the infinite-dimensional limit. This behaviour meets the target threshold defined in the persistence metric of Table 3 .

Finally, the complete absence of spurious eigenvalues across all truncation sizes indicates a “clean” spectral picture, with no numerical artefacts appearing in the spectrum. This satisfies the spurious-count metric criterion in Table 3 and confirms the stability of the numerical pipeline defined in Table 2.

High-level take-away. The collective evidence supports the theoretical claims of finite ascent, the Weyl property, SVEP, and spectral stability. The layered architecture (Table 2), coupled with the defined regression metrics Table 3 , has proven effective in translating abstract spectral theory into verifiable computational evidence, as clearly illustrated by the numerical facts in Table 4 and the visual patterns in Figures 1–2.

Table 4. *Numerical evidence versus theory.*

Numerical fact	Operator meaning	Theoretical link
Inequality holds	A_N satisfies quadratic bound	Def. 3.1.1; Lem. 3.1.2
$\sigma_p(A_N) = \{0\}$	Shift is nilpotent	Prop. 3.2.3; Thm. 3.3.5
Circular pseudospectrum	Resolvent blows up at $\|\lambda\| = r$	$\sigma_{\text{ess}}(A)$; SVEP
$d_H \to 0$	Point spectrum stabilises	Persistence metric (Tab. 3-2)
No spurious eigenvalues	Clean spectral picture	Spurious-count metric

Numerical evidence confirms finite ascent, Weyl -SVEP Properties and spectral stability predicted by theory.

7. Conclusion

The notebook and codebase translate abstract inequalities, local-spectral properties, and invariant-subspace theorems into a numerical laboratory. Results for sizes up to $N = 1000$ and $k \leq 4$ corroborate all analytic predictions and open the door to new experiments not yet accessible to pure theory. ORCID

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Abbreviations

H	Hilbert Space
$A \in B(H)$	A Bounded Linear Operator A on a Hilbert Space (H)
A^*	Adjoint of Operator A
$Ran(A)$	Range of A
$ker(A)$	Kernel of A
$\rho(A)$	Resolvent Set of A
$\sigma(A)$	Spectrum of A
$\sigma_p(A)$	Point Spectrum of A
$\sigma_{ap}(A)$	Approximate Point Spectrum of A
λ	Eigenvalues
K	Kernelfunction
σ_A	The Weyl's Spectrum of A
SVEP	Single Value Extension Property
$\sigma_{\text{wa}}(A)$	Weyl (or essential) Approximate Point Spectrum
$\sigma_a(A)$	The Classical Approximate Point Spectrum of $A \in \mathcal{B}(H)$
$\sigma_{su}(A)$	The Surjectivity Spectrum of A
$\pi_{00}(A)$	The Set of Isolated Eigenvalues of Finite Multiplicity

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Conflicts of Interest

The authors declare no conflicts of interest.

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