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Research Article

On the Conditions of Ensuring Uniform Almost Periodic Functions of the Class of Entire Functions

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Abstract

The study of almost periodic functions occupies an important place in functional analysis and the theory of differential equations, beginning with the classical works of H. Bohr, A. S. Besikovich and B. M. Levitan. Almost periodic functions, being a generalization of periodic functions, are characterized by the fact that they retain their structure under shifts, without being strictly periodic. On the other hand, entire functions are functions of a complex variable that are analytic in the entire complex plane. Their behavior, especially their growth and the location of their zeros, is studied in detail in the theory of functions of a complex variable. Of particular interest is the study of entire functions whose values on the real axis are almost periodic in the sense of Bohr. The question of under what conditions an entire function takes on values on the real axis that form a uniformly almost periodic function is a non-trivial problem at the intersection of function theory and spectral analysis. Such conditions can be formulated through the properties of the spectrum of the function, through the conditions on the coefficients of the Fourier series, and also through the growth properties of the function itself. These functions find application in spectral theory, quantum mechanics, oscillation theory, and other areas of mathematics and physics. In this section, we study the problems of approximation of functions $f(x) \in B$ by entire functions of finite degree with arbitrary Fourier exponents. We establish necessary and sufficient conditions for functions $f(x) \in B$ to belong to the class of entire functions of bounded degree.

Keywords

Almost Periodic Functions, Fourier Series, Spectrum of a Function, Entire Functions of Finite Degree, Trigonometric Polynomials, Best Uniform Approximation

1. Introduction

Almost periodic functions play an important role in various areas of analysis, especially in the theory of differential equations, spectral theory, and signal theory. They were classically introduced by G. Bohr in the context of functions defined and continuous on the entire number line, possessing the property of approximate periodicity. Subsequently, the concept of almost periodicity was significantly expanded in the works of such mathematicians as S. Besikovich, A. Stefanov, G. Weyl and others, including to spaces of entire and analytic functions.

Of particular interest is the study of conditions under which an entire function (that is, a function that is analytic in the entire complex plane) is uniformly almost periodic. This requires considering the behavior of the function not only on

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the real axis, but also in a broader context - for example, in strips of the complex plane, where its periodic or quasiperiodic properties manifest themselves.

Uniform almost periodicity in the classical sense means that with any degree of accuracy one can find a "substitute period" that will bring the function closer to itself over the entire domain of definition. However, in the case of entire functions this requirement must be adapted: it is important to take into account their growth (e.g. the order and type of the function), the distribution of zeros, and the structure of the spectrum (set of frequencies).

2. Manuscript Formatting

By B_σ $(\sigma > 0)$ we denote the class of bounded entire functions $g_\sigma(x)$ of degree no greater than σ on the entire real axis. S. N. Bernstein [1] established that among the functions from the class B_σ that realize the best approximation of the 2π-periodic function f(x) on the entire real axis, there is a trigonometric polynomial of degree no greater than σ. The results of this work are analogous to some results of works [7-16] for the class of uniform almost-periodic Bohr functions.

Аналоги теоремы С. Н. Бернштейна для функций $f(x) \in B$ получены в работах Е. А. Бредихиной [2, 3]. Она доказала, что если функция $f(x) \in B$ с рядом Фурье

$$f(x) \sim \sum_k A_k e^{i\lambda_k \beta x},$$

where λ_k are rational numbers and β is a real number, then among the functions $g_\sigma(x) \in B_\sigma$ $(\sigma > 0)$, for which

$$\sup_{-\infty<x<\infty} |f(x) - g_\sigma(x)| = A_\sigma(f), \tag{1}$$

there exists a trigonometric polynomial of degree $\leq \sigma$.

The space of uniform almost periodic functions, denoted by B, is the closure of the set of trigonometric polynomials

$$T_k(x) = \sum_{k=1}^{n} A_k e^{i\lambda_k x},$$

where A_k are the Fourier coefficients, λ_k is the spectrum of the function $f(x) \in B$, with the norm

$$\|f(x)\|_B = \sup_{x\in R} |f(x)|.$$

Let a function $f(x) \in B$ with an arbitrary spectrum $\{\lambda_k\}$, have a Fourier series

$$f(x) \sim \sum_{k=-\infty}^{\infty} A_k e^{i\lambda_k x}, \tag{2}$$

where the Fourier coefficients $\{A_k\}$ are defined as follows

$$A_k = M\{f(x)e^{-i\lambda_k x}\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x) e^{-i\lambda_k x} dx.$$

Consider the following problem. Let the function $f(x) \in B$. What are the necessary and sufficient conditions for this function to belong to the class B_σ. The answer to this question is given by the following.

Theorem 1 [7]. In order for $f(x) \in B$ to belong to the class of entire functions B_σ, it is necessary and sufficient that the following conditions be satisfied:

$$|\lambda_k| \leq \sigma. \tag{3}$$

Proof. Sufficiency. Consider the function

$$f_{a,b}(x) = \int_{-\infty}^{\infty} f(x+t) \varphi_{a,b}(t) dt,$$

where

$$\varphi_{a,b}(t) = \frac{2}{\pi(a-b)t^2} \cdot \sin\frac{a-b}{2} t \sin\frac{b+a}{2} t.$$

It is easy to show that the function $f_{a,b}(x)$ is continuous and almost periodic. Indeed, since [6] (see [6], p. 76)

$$\int_{-\infty}^{\infty} |\varphi_{a,b}(t)| dt \leq A + B \ln\frac{a+b}{b-a},$$

where A and B are constants, then

$$|f_{a,b}(x+\tau) - f_{a,b}(x)| \leq \int_{-\infty}^{\infty} |f(x+t+\tau) - f(x+t)| \cdot |\varphi_{a,b}(t)| dt \leq$$

$$\leq \sup_{-\infty<x<\infty} |f(x+t+\tau) - f(x+t)| \left(A + B \ln\frac{a+b}{b-a}\right).$$

This implies the continuity and almost periodicity of the function $f_{a,b}(x) \in B$.

Let $a = \lambda, b = 2\lambda$. Then

$$f_{a,b}(x) = \frac{2}{\lambda\pi} \int_{-\infty}^{\infty} f(t) \frac{\sin\frac{(t-x)\lambda}{2} \sin\frac{3\lambda(t-x)}{2}}{(t-x)^2} dt,$$

or

$$f_{a,b}(x) \sim \sum_{k=-\infty}^{\infty} A_k \exp(i\lambda_k x).$$

The reason for this is the exceptionality [4] (see [4], p. 73), will be

$$f_{a,b}(x) \to f(x).$$

Necessity. Let the function $f(x) \in B$ belong to the class B_σ and have a Fourier series of the form (2). Then, for any natural r, the derivative of order r will be

$$f^{(r)}(x) \sim \sum_{k=-\infty}^{\infty} (i\lambda_k)^r A_k \exp(i\lambda_x x).$$

Using S. N. Bernstein's inequality, we arrive at

$$\left|f^{(r)}(x)\right| \le \sigma^r \cdot \sup_x |f(x)| = \sigma^r \cdot C.$$

Means

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\left|f^{(r)}(x)\right|^2 dx \le \sigma^{2r}\cdot C^2.$$

From here, applying Bessel's inequalities, we get

$$|\lambda_k|^{2r}|A_k|^2 \le \sigma^{2r}\cdot C^2 (r = 1, 2, \dots),$$

or

$$|A_k| \le C\cdot\left(\frac{\sigma}{|\lambda_k|}\right)^r.$$

Therefore, for $|\lambda_k| > \sigma$ the coefficients $A_k = 0$. Theorem 1 is proven.

It is known [5] that from the set of functions $f(x) \in B_\sigma$, which perform the best approximation of functions $f(x) \in L_p[-\pi,\pi]$, on the entire real axis, one can find a trigonometric polynomial of degree $\le \sigma$.

This statement is based on the fact that if $g_\sigma(f;x) \in B_\sigma$ and

$$\sup_{x\in R}|f(x) - g_\sigma(f;x)| = A_\sigma(f),$$

then the following relations are valid uniformly for all $x \in R$

$$\left|f(x) - \frac{1}{2n+1}\sum_{k=-n}^{n} g_\sigma(x+2k\pi)\right| \le A_\sigma(f), \tag{4}$$

$$\lim_{n\to\infty}\frac{1}{2n+1}\sum_{k=-n}^{n}\{g_\sigma(x+2k\pi+2\pi) - g_\sigma(x+2k\pi)\} = 0, \tag{5}$$

where $A_\sigma(f)$ is the best approximation of order σ.

This result can be obtained if (4) and (5) are replaced by the relations

$$\left|\Phi_n(x) - Q_{\sigma,N,n}(x)\right| \le A_\sigma(f),$$

where $\sigma > 0; N, n$ are any natural numbers,

$$\Phi_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)F_n(t)dt = \frac{1}{2\pi N}\int_{-2\pi N}^{2\pi N} f(x+t)F_n(t)dt,$$

$$Q_{\sigma,N,n}(x) = \frac{1}{2\pi N}\int_{-2\pi N}^{2\pi N} g_\sigma(x+t)F_n(t)dt,$$

$$F_n(t) = \frac{\sin^2(n+1)\frac{t}{2}}{2(n+1)\sin^2\frac{t}{2}},$$

and for all fixed n uniformly in $x \in R$

$$\lim_{N\to\infty}\{Q_{\sigma,N,n}(x+2\pi) - Q_{\sigma,N,n}(x)\} = 0.$$

When establishing such results for functions f(x)∈B, a number of problems arise. First, additional conditions are imposed on the smoothness of the functions, and second, the Fourier indices of such functions can lie everywhere densely, i.e. (see [6], [8-16]):

$$\lambda_0 = 0;\ \lambda_{-k} = -\lambda_k, |\lambda_k| > |\lambda_{k+1}|, (k = 1,2,\dots), \lim_{k\to\infty}|\lambda_k| = 0;$$

$$\lambda_0 = 0;\ \lambda_{-k} = -\lambda_k, |\lambda_k| < |\lambda_{k+1}|, (k = 1,2,\dots), \lim_{k\to\infty}|\lambda_k| = \infty.$$

Similar results of S. N. Bernstein for uniform almost-periodic Bohr functions were first established in the works of E. A. Bredikhina [2, 3]. She proved that if $f(x) \in B$ and has a corresponding Fourier series with spectrum at infinity, then from the sets of functions $f(x) \in B_\sigma$ one can find a trigonometric polynomial of degree $\le \sigma$.

Let us formulate the main result of this section, which is also an analogue of S. N. Bernstein's theorem for uniform almost-periodic functions with arbitrary Fourier exponents $\{\lambda_k\}$ $(k = 0, \pm 1, \pm 2, \dots)$.

Теорема 2 [7]. If $f(x) \in B$ and

$$A_\sigma(f) = \sup_{x\in R}|f(x) - g_\sigma(f;x)| (\sigma > 0)$$

the best uniform approximation of order σ, then for any $\varepsilon > 0$ there is a finite trigonometric polynomial

$$P_\sigma(x) = \sum_{k=1}^{n} b_k e^{i\lambda_k x},$$

for which the estimate holds uniformly in x

$$|f(x) - P_\sigma(x)| \le A_\sigma(f) + \varepsilon,$$

where $\varepsilon = \varepsilon(N,\delta)(\delta < \pi)$.

Proof. Suppose that the set of numbers β{β_k } is a basis for the set $\Lambda\{\lambda_k\}$ and $\beta\{\beta_k\} \subset \Lambda\{\lambda_k\}$ (see [10], p. 99).

Consider the Bochner-Fejér sums that correspond to functions $f(x) \in B$

$$P_N^{(m)}(f;x) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x+t)K_N^{(m)}(t)dt =$$

$$= \sum_{|\nu_k|\le n_k}\left(1 - \frac{|\nu_1|}{n_1}\right)\dots\left(1 - \frac{|\nu_r|}{n_r}\right) exp\left[i\left(\frac{\nu_1}{m!}\beta_1 + \dots + \frac{\nu_r}{m!}\beta_r\right)\right]$$

$$\cdot A\left(\frac{\nu_1}{m!}\beta_1 + \dots + \frac{\nu_r}{m!}\beta_r\right) (k = 1, 2, \dots, r),$$

where $A\left(\frac{\nu_1}{m!}\beta_1 + \dots + \frac{\nu_r}{m!}\beta_r\right)$ Fourier coefficients of the function $f(x)$,

$$\lambda_\nu = \frac{\nu_1}{m!}\beta_1 + \dots + \frac{\nu_r}{m!}\beta_r,$$

$$K_N^{(m)}(t) = \prod_{j=1}^{r} K_{n_j}\left(\frac{\beta_j t}{m!}\right),$$

$$K_{n_j}\left(\frac{\beta_j t}{m!}\right) = \sum_{|\nu|\le n_j}\left(1-\frac{|\nu|}{n_j}\right) exp\left[-i\left(\frac{\nu}{m!}\beta_j\right)\right].$$

It is known [10] (see [10], p. 103) that $K_N^{(m)}(t)$ are positive quantities for which

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} K_N^{(m)}(t)\,dt = 1.$$

Next, we consider the entire function $g_\sigma(x) \in B_\sigma$, satisfying relation (3) and the function

$$F_{N,T}(x) = \frac{1}{2T}\int_{-T}^{T} f(x+t)K_N^{(m)}(t)dt,$$

for which uniformly in $x \in R$

$$\lim_{T\to\infty} F_{N,T}(x) = P_N^{(m)}(f;x),$$

and also

$$Q_{\sigma,N,n}(x) = \frac{1}{2T}\int_{-T}^{T} g_\sigma(x+t)K_N^{(m)}(t)dt,$$

where $g_\sigma(x) \in B_\sigma$ and uniformly in $x \in R$

$$|g_\sigma(x)| \le C, |f(x) - g_\sigma(x)| \le A_\sigma(f).$$

By virtue of the Bochner-Fejér theorem, for $\boldsymbol{\varepsilon} > 0$ there exists a number N such that

$$\left|f(x) - P_N^{(m)}(f;x)\right| \le \varepsilon. \tag{6}$$

Next, since

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} K_N^{(m)}(t)dt = 1,$$

that

$$\left|F_{N,T}(x) - Q_{\sigma,T,N}(x)\right| \le A_\sigma(f)\cdot\frac{1}{2T}\int_{-T}^{T} K_N^{(m)}(t)dt \tag{7}$$

and when $T \to \infty$ the right-hand side in (7) is equal to the value of A_$A_\sigma(f)$.

Since $g_\sigma(x) \in B_\sigma$ and и $|g_\sigma(x)| \le C$, then for a fixed $\sigma > 0$ each of $Q_{\sigma,T,N}(x)$ is an entire function of degree no greater than σ as $T \to \infty$

$$\left|Q_{\sigma,T,N}(x)\right| \le C\cdot\frac{1}{2T}\int_{-T}^{T} K_N^{(m)}(t)dt \le C_1,$$

uniformly in x, T and N.

Next, it can be shown that the set of functions $\{Q_{\sigma,T,N}(x)\}$ is uniformly continuous on $(-\infty,\infty)$. This means that for a fixed number N, there is a sequence of numbers $\{T_l\}$, for which

$$\lim_{T_l\to\infty} Q_{\sigma,T_l,N}(x) = Q_{\sigma,N}(x)$$

uniformly on any finite segment of the real axis. Since $Q_{\sigma,N}(x) \in B_\sigma$, then by (7) uniformly in x and N we have

$$\left|P_N^{(m)}(x) - Q_{\sigma,N}(x)\right| \le A_\sigma(f). \tag{8}$$

Now we can show that the function $Q_{\sigma,N}(x) \in B$. Choosing an arbitrary number τ, we estimate the following difference

$$Q_{\sigma,T,N}(x+\tau) - Q_{\sigma,T,N}(x) =$$

$$= \frac{1}{2T}\int_{-T}^{T} g_\sigma(x+t+\tau)K_N^{(m)}(t)dt - \frac{1}{2T}\int_{-T}^{T} g_\sigma(x+t)K_N^{(m)}(t)dt =$$

$$= \frac{1}{2T}\int_{-T}^{T} g_\sigma(x+t+\tau)\left\{K_N^{(m)}(t) - K_N^{(m)}(t+\tau)\right\}dt +$$

$$+\frac{1}{2T}\int_{-T}^{T} g_\sigma(x+t+\tau)K_N^{(m)}(t+\tau)dt - \frac{1}{2T}\int_{-T}^{T} g_\sigma(x+t)K_N^{(m)}(t)dt =$$

$$= \frac{1}{2T}\int_{-T}^{T} g_\sigma(x+t+\tau)\left\{K_N^{(m)}(t) - K_N^{(m)}(t+\tau)\right\}dt +$$

$$+\frac{1}{2T}\int_{T}^{T+\tau} g_\sigma(x+u)K_N^{(m)}(u)du + \frac{1}{2T}\int_{-T+\tau}^{T} g_\sigma(x+u)K_N^{(m)}(u)du =$$

$$= I_1(T) + I_2(T) + I_3(T). \tag{9}$$

It was shown above that $|g_\sigma(x)| \le C$ uniformly in $x \in R$, so for fixed numbers τ and N

$$\lim_{T\to\infty} I_2(T) = 0, \lim_{T\to\infty} I_3(T) = 0. \tag{10}$$

Let $f(x) \in B$ have spectrum $\Lambda\{\lambda_k\}$. Then [6] (see [6], p. 104) for any integers N, n_k and $\delta > 0$ $(\delta < \pi)$ one can find a positive number $\varepsilon = \varepsilon(N,\delta)$ such that in each interval of length ε there is at least one number τ that satisfies the inequality

$$|\lambda_k\tau - 2\pi n_k| < \delta \ (k = 1, 2, \dots, N).$$

So, for τ, taking ε - the almost-period of the function $f(x)$, we estimate $I_1(T)$

$$|I_1(T)| \le C \cdot \frac{1}{2T}\int_{-T}^{T}\left|K_N^{(m)}(t+\tau) - K_N^{(m)}(t)\right|dt \le$$

$$\le C \cdot \sum_{|\nu_k| \le n_k}\left|exp\left[-i\left(\frac{\nu_1}{m!}\beta_1 + \dots + \frac{\nu_r}{m!}\beta_r\right)\right] - 1\right|\left(1 - \frac{|\nu_1|}{n_1}\right)\dots\left(1 - \frac{|\nu_r|}{n_r}\right) \le$$

$$\le C \cdot S_n \le C \cdot N \cdot \delta\ (k = 1, 2, \dots, r),$$

where

$$S_n = 2\sum_{\nu=1}^{N}\left|sin\frac{\lambda_\nu \tau}{2}\right|, \lambda_\nu = \frac{\nu_1}{m!}\beta_1 + \dots + \frac{\nu_r}{m!}\beta_r,$$

and N is the number of terms.

Assuming $\delta = \frac{\varepsilon}{C \cdot N}$, we will have

$$|I_1(T)| \le \varepsilon C N \delta = CN\frac{\varepsilon}{CN} = \varepsilon. \qquad (11)$$

Substituting (10) and (11) into (9) we finally obtain that

$$\lim_{T \to \infty}\left|Q_{\sigma,T,N}(x+\tau) - Q_{\sigma,T,N}(x)\right| \le \varepsilon.$$

This means the function $Q_{\sigma,N}(x) \in B$.

Let us prove that the function $Q_{\sigma,N}(x)$ is a finite trigonometric polynomial sum with spectrum satisfying the condition $|\lambda_k| \le \sigma$ from the set $\Lambda\{\lambda_k\}\ (k = 1, 2, \dots, N)$. Indeed,

$$Q_{\sigma,N}(x) = \lim_{T_l \to \infty} Q_{\sigma,T_l,N}(x) = \lim_{T_l \to \infty}\frac{1}{2T_l}\int_{-T_l}^{T_l} g_\sigma(x+t)K_N^{(m)}(t)dt =$$

$$= \sum_{|\nu_k| \le n_k} C\left(\frac{\nu_1}{m!}\beta_1 + \dots + \frac{\nu_r}{m!}\beta_r\right)\prod_{k=1}^{r}\left(1 - \frac{|\nu_k|}{n_k}\right) exp\left(i\frac{\nu_k}{m!}\beta_k\right)(k = \overline{1,r}),$$

where

$$C\left(\frac{\nu_1}{m!}\beta_1 + \dots + \frac{\nu_r}{m!}\beta_r\right) =$$

$$= \lim_{T_l \to \infty}\frac{1}{2T_l}\int_{-T_l}^{T_l} g_\sigma(x+t)exp\left[-i\left(\frac{\nu_1}{m!}\beta_1 + \dots + \frac{\nu_r}{m!}\beta_r\right)(x+t)\right]dt =$$

$$= \lim_{T_l \to \infty}\frac{1}{2T_l}\int_{-T_l+x}^{T_l+x} g_\sigma(u)exp\left[-i\left(\frac{\nu_1}{m!}\beta_1 + \dots + \frac{\nu_r}{m!}\beta_r\right)u\right]du =$$

$$= \lim_{T_l \to \infty}\frac{1}{2T_l}\int_{-T_l+x}^{T_l+x} g_\sigma(u)exp(-i\lambda_\nu u)du.$$

The condition $|\lambda_k| \le \sigma$ follows from Theorem 1. Setting N such that inequality (6) is satisfied, we estimate the difference

$$\left|f(x) - Q_{\sigma,N}(x)\right| \le$$

$$\le \left|f(x) - P_N^{(m)}(x)\right| + \left|P_N^{(m)}(x) - Q_{\sigma,N}(x)\right| = \varepsilon + A_\sigma(f).$$

By virtue of (6) and (8) we obtain the statement of Theorem 2.

Obtaining further results essentially relies on the following concepts [6] (see [6], pp. 47-48).

Definition 1. A family of functions $f(x)$ is called equi-bounded if there exists a number $A > 0$ such that for all functions of this family the inequality $|f(x)| < A$ holds.

Definition 2. A family of functions f(x) is called equicontinuous if for each $\varepsilon > 0$ it is possible to find $\delta = \delta(\varepsilon)$ such that for $|x' - x''| < \delta$ the inequality $|f(x'') - f(x')| < \varepsilon$ holds for all functions in the family.

Definition 3. A family of functions f(x) is called equi-almost-periodic if for each $\varepsilon > 0$ it is possible to specify $l = l(\varepsilon)$ such that in each interval of length l there is a common -almost-period for all functions of the family.

Theorem 3. Let $f(x) \in B$. Then among the functions $g_\sigma(x) \in B_\sigma$, there is a function $Q_\sigma(f; x) \in B$ with a Fourier series

$$\sum_{|\lambda_k| \le \sigma} A_k e^{i\lambda_k x}.$$

Proof. By Theorem 2, the set of polynomials $\{Q_{\sigma,N}(x)\}$ in the inequality

$$\left|P_N^{(m)}(x) - Q_{\sigma,N}(x)\right| \le A_\sigma(f),$$

is compact. Therefore, we can select from it a subsequence $\{Q_{\sigma,N_k}(x)\}$ that converges uniformly on any segment with the corresponding limit function $f_{N_k}(x)$. Therefore, by [4] (see [4] p. 48) $f_{N_k}(x) \in B$. This means that as $N_k \to \infty$ the function $f_{N_k}(x)$ converges to $f(x)$ uniformly for all x. It follows that $f(x) \in B$ with spectrum $\{\lambda_k\}(k = 0, \pm 1, \pm 2, \dots)$. Theorem 3 is proved.

Let $f(x)$ be a uniformly continuous and bounded on $(-\infty, \infty)$ function and $A_\sigma(f)$ be the best approximations of the function $f(x)$ by entire functions of degree $\le \sigma$, that is,

$$A_\sigma(f) = \inf_{g(x) \in B_\sigma}\left\{\sup_{-\infty < x < \infty}|f(x) - g(x)|\right\}.$$

S. N. Bernstein proved (see [3], pp. 371-373) that for uniform continuity on the entire real axis of a bounded tion $f(x)$, it is necessary and sufficient that

$$A_\sigma(f) \le C\omega\left(f; \frac{1}{\sigma}\right), \qquad (12)$$

where C is some constant and

$$\omega(f;\delta)=\sup_{|x_1-x_2|\le\delta}|f(x_1)-f(x_2)|$$

the modulus of continuity of the functions $f(x)$. Estimate (12) is obtained by passing to the limit as n→∞ from the inequality

$$A_\sigma(f;\delta)\le E_n(f;\delta)\left(\sigma=\frac{n}{\delta}\right),$$

where $E_n(f;\delta)$ is the best approximation of the function $f(x)$ on the interval $[-\sigma,\sigma]$ by polynomials of degree n.

Next, for functions $f(x)\in B$ we will find a function $g(x)\in B_\sigma$ that satisfies condition (12).

Theorem 4. Let $f(x)\in B$ and for $z=x+iy$

$$f_\sigma(z)=\int_{-\infty}^{\infty}f(u)\psi_\sigma(u-z)du,$$

where

$$\psi_\sigma(z)=\frac{1}{\pi\sigma^3}\left(\frac{\sin\frac{\sigma z}{4}}{z}\right)^4.$$

Then$f_\sigma(z)$ belongs to the class of entire functions B_σ and the following holds:

$$sup_{x\in R}|f(x)-f_\sigma(z)|<C\omega\left(f;\frac{1}{\sigma}\right), \tag{13}$$

C is an absolute constant.

Proof. It is easy to prove that $f_\sigma(z)$is an entire function of degree $\le\sigma$. Consider the integral

$$\int_{-\infty}^{\infty}|\psi_\sigma(u-z)|du\le\le\int_{|u-z|\le2}|\psi_\sigma(u-z)|du+\int_{|u-z|>2}|\psi_\sigma(u-z)|du. \tag{14}$$

Because

$$\left|\frac{sinz}{z}\right|\le\sum_{k=0}^{\infty}\frac{|z|^{2k}}{(2k+1)!}<\sum_{k=0}^{\infty}\frac{|z|^{2k}}{(2k)!}<e^{|z|},$$

that

$$\int_{|u-z|\le2}|\psi_\sigma(u-z)|du=\frac{1}{\pi\sigma^3}\int_{|u-z|\le2}\left|\frac{sin^4\frac{(u-z)\sigma}{4}}{\sigma^3(u-z)^3}\right|du=\frac{1}{\pi\sigma^3}\int_{|u-z|\le2}\left|\frac{sin^4\frac{(u-z)\sigma}{4}}{\left(\frac{u-z}{4}\sigma\right)^4}\cdot\frac{(u-z)\sigma}{4^4}\right|du\le C\frac{\sigma}{\pi}e^{2\sigma}(|z|+2), \tag{15}$$

where C is an independent constant.

Estimation of the second integral on the right-hand side of (14) is carried out using the following inequality

$$|sinz|\le\frac{1}{2}(|e^{-y}|+|e^y|)<\sum_{k=0}^{\infty}\frac{|y|^k}{k!}=e^{|y|}(z=x+iy).$$

Hence,

$$\int_{|u-z|>2}|\psi_\sigma(u-z)|du=\frac{1}{\pi\sigma^3}\int_{|u-z|>2}\left|\frac{sin^4\frac{(u-z)\sigma}{4}}{\sigma^3(u-z)^3}\right|du\le\frac{e^{\sigma y}}{\pi\sigma^3}\left(\int_{|u-x|>1}\frac{du}{(u-x)^4}+\int_{\substack{|u-x|>1\\y^2>3}}\frac{du}{y^4}\right)\le C\frac{1}{\pi\sigma^3}e^{\sigma|z|}. \tag{16}$$

Substituting (15) and (16) into (14), we obtain

$$\int_{-\infty}^{\infty}\psi_\sigma(u-z)du<C\left(\frac{\sigma}{\pi}e^{2\sigma}(|z|+2)+\frac{1}{\pi\sigma^3}e^{\sigma|z|}\right).$$

It follows that $f_\sigma(z)$ is an entire function.

Let the function be given

$$\psi_\sigma(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi_\sigma(t)\,e^{-itu}dt,$$

where

$$\varphi_\sigma(t)=\begin{cases}1-\frac{6t^2}{\sigma^2}+\frac{6|t|^3}{\sigma^3},|t|<\frac{\sigma}{2},\\ 2\left(1-\frac{|t|}{\sigma}\right)^3,\frac{\sigma}{2}\le|t|<\sigma,\\ 0,|t|\ge\sigma.\end{cases}$$

Then

$$\int_{-\infty}^{\infty}\psi_\sigma(u)du\le\frac{3\sigma}{4\pi}\int_0^{\frac{4}{\sigma}}du+\frac{2}{\pi\sigma^3}\int_{\frac{4}{\sigma}}^{\infty}\frac{du}{u^4}\le C\cdot\frac{1}{\pi}, \tag{17}$$

$$\sigma\int_0^{\infty}u\psi_\sigma(u)du\le\frac{3\sigma^2}{4\pi}\int_0^{\frac{4}{\sigma}}udu+\frac{2}{\pi\sigma^2}\int_{\frac{4}{\sigma}}^{\infty}\frac{du}{u^3}\le C\cdot\frac{1}{\pi}. \tag{18}$$

Using the Fourier inversion formula [6] (see [6], p. 77), we find

$$\int_{-\infty}^{\infty}\psi_\sigma(u)\exp(itu)du=\varphi_\sigma(t). \tag{19}$$

At $t=0$ φ_$\varphi_\sigma(0)=1$, then (19) takes the form

$$\int_{-\infty}^{\infty}\psi_\sigma(u)du=1. \tag{20}$$

That's why,

$$f_\sigma(x)=\int_{-\infty}^{\infty}f(x)\psi_\sigma(u)du. \tag{21}$$

Multiplying both sides of (20) by $f(x)$ and subtracting the resulting equality from (21), we obtain

$$|f(x)-f_\sigma(x)|\le\int_{-\infty}^{\infty}|f(x)-f(x+u)|\psi_\sigma(u)du\le$$

$$\le C\omega(f;\sigma^{-1})\int_0^{\infty}(1+\sigma u)\psi_\sigma(u)du.$$

Due to (17), (18) and the last inequality, estimate (13) follows. Theorem 4 is proved.

Along with Theorem 4, we prove that if the function $f(x) \in B$, then $f_\sigma(x) \in B$.

Theorem 5. If $f(x) \in B$ with Fourier series

$$f(x) \sim \sum_{k=-\infty}^{\infty} A_k e^{i\lambda_k x},$$

then $f_\sigma(x) \in B$ *and the function* $f_\sigma(x)$ *has a Fourier series*

$$f_\sigma(x) \sim \sum_{|\lambda_k|<\sigma} A_k \varphi_\sigma(\lambda_k) e^{-i\lambda_k x}. \quad (22)$$

Proof. Let the function $f_\sigma(x)$ be given in the form

$$f_\sigma(x) = \int_{-\infty}^{\infty} f(x+u)\psi_\sigma(u)du.$$

It can be shown that this function is almost periodic. Due to inequality (17), we have

$$|f_\sigma(x+\tau) - f_\sigma(x)| \le \int_{-\infty}^{\infty} |f(x+u+\tau) - f(x+u)| \psi_\sigma(u)du \le$$

$$\le C \sup_{x \in R} |f(x+\tau) - f(x)|,$$

where C is an absolute constant. It follows from the latter that $f_\sigma(x) \in B$.

Let B_λ be the Fourier coefficient of the function $f_\sigma(x)$, with spectrum λ. Then, using the strengthened mean value theorem [4] (see [4], pp. 55-56), using relation (19), we obtain

$$B_\lambda = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_\sigma(x) \exp(-i\lambda x)dx =$$

$$= \int_{-\infty}^{\infty} \psi_\sigma(u) \exp(i\lambda u) \left[\lim_{T\to\infty} \frac{1}{2T} \int_{-T+u}^{T+u} f(x) \exp(-i\lambda x)dx\right] du = A_\lambda \varphi_\sigma(\lambda),$$

where A_λ is the Fourier coefficient of the function $f(x)$, which corresponds to the spectrum of λ. By virtue of Theorem 1, condition (2) is satisfied for the spectrum of the function $f_\sigma(x)$, and since

$A_\lambda = 0$ при $|\lambda| \ge \sigma$; $A_\lambda = 0$ при $\lambda \ne \lambda_k$, то $B_{\lambda_k} = A_{\lambda_k} \varphi_\sigma(\lambda_k)$.

From this follows relation (22) and Theorem 5 is proven.

3. Materials and Methods

The study is based on the developed concepts of the theory of almost periodic functions, especially in the sense of Bohr and Boscovich. The following spaces are considered as the main mathematical model:

AP is the space of almost periodic functions in the sense of Bohr;

B_p is the space of functions in the sense of Boscovich, where $p \ge 1$;

W_p, S_p are spaces in the sense of Weyl and Stepanov.

4. Results

The results of the work are new, obtained by the author independently and consist of the following: the conditions for belonging of entire functions of limited degree to the class of almost periodic functions are proved.

5. Discussion

Let us consider the conditions of uniform almost periodicity of functions belonging to the class of entire functions. We will limit ourselves to functions of one variable, although some of the reasoning can be generalized to functions of several variables.

6. Conclusions

Uniform almost periodicity of entire functions is ensured by a number of conditions, among which the main ones are the local boundedness of the function, its analytical structure, and the nature of its spectrum. Understanding these conditions is important for working with functions that can be used in number theory, analytic number theory, and other areas of mathematics where such functions occur.

Author Contributions

Talbakzoda Farhodjon Mahmadsho is the sole author. The author read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Bernstein S. N. Collected Works, Vol. II, Publishing House of the USSR Academy of Sciences, Moscow: 1954, pp. 371-375. (Journal Articles)

[2] Bredikhina E. A. On the Approximation of Almost-Periodic Functions. Siberian Math. J., 1964, Vol. 5, No. 4, pp. 768-773. (Journal Articles)

[3] Bredikhina E. A. On S. N. Bernstein's Theorem on the Best Approximation of Continuous Functions by Entire Functions of a Given Degree. Izvestiya Universiteta Mathematica, 1961, No. 6, pp. 3-7. (Journal Articles)

[4] Bohr, G. Almost Periodic Functions. Moscow: LIBROKOM, 2009, 128 p. (Books)

[5] Timan, M. F., Khasanov, Yu. Kh. On Approximations of Almost Periodic Functions by Entire Functions. Izvestiya vuzov. Mat. 2011, No. 12, pp. 64-70. (Books)

[6] Levitan, B. M. Almost Periodic Functions. Moscow-Leningrad: Gostekhizdat, 1953, 396 p. (Books)

[7] Talbakov F. M. On the absolute convergence of Fourier series of uniform almost periodic functions and some issues of their approximation / Dissertation for the degree of candidate of physical and mathematical sciences // National Academy of Sciences of Tajikistan Institute of Mathematics named after A. Juraev. Dushanbe, 2021, 396 p. (Dissertations)

[8] Talbakov F. M. Analogue of S. N. Bernstein's theorem on the best approximation of almost periodic functions [Text] / Yu. Kh. Khasanov, F. M. Talbakov // DAN RT. - 2016. - V.59. - No.1-2. - P.11-18. (Journal Articles)

[9] Talbakov F. M. On the Absolute Convergence of Double Fourier Series of Uniform Almost Periodic Functions in a Uniform Metric [Text] / F. M. Talbakov // Russian Mathematics. - 2023. Vol. 67. - No. 4. - Pp. 56-65. (Journal Articles)

[10] Talbakov F. M. On the Approximation of Uniform Almost Periodic Functions by Certain Sums and Integrals [Text] / F. M. Talbakov // Bulletin of the Tajik National University. Series of natural sciences. - 2024. - № 2. - C. 19-27. (Journal Articles)

[11] Talbakov F. M. On Sufficient Condition for Absolute Convergence Fourier Series of Bezikovichs Almost-Periodic Functions [Text] / Yu. Kh. Khasanov and F. M. Talbakov // ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2024, Vol. 45, No. 6, pp. 2743-2752. c Pleiades Publishing, Ltd., 2024. https://doi.org/10.1134/S199508022460290X

[12] Talbakov F. M. On the Absolute Convergence of Fourier Series of Almost Periodic Functions [Text] / Yu. Kh. Khasanov, F. M. Talbakov// Russian Mathematics. 2024, Vol. 68, No. 4, pp. 60-71. https://doi.org/10.3103/S1066369X24700282

[13] Farkhodzhon Makhmadshevich Talbakov About Absolute Convergence of Fourier Series of Almost Periodic Functions [Text] / F. M. Talbakov // Pure and Applied Mathematics Journal 2024, Vol. 13, No. 3, pp. 36-43. (Journal Articles) https://doi.org/10.11648/j.pamj.202420241303.11

[14] Talbakov Farkhodzhon Makhmadshevich. Approximation of Uniform Almost Periodic Functions by Inclusions and Integrals [Text] / F. M. Talbakov // Beyond Signals - Exploring Revolutionary Fourier Transform Applications. Pure and Applied Mathematics Journal 2025, pp. 36-43 https://dx.doi.org/10.5772/intechopen.1008170

[15] Talbakzoda F. M. On sufficient conditions for the absolute convergence of Fourier series with small gaps [Text] / F. M. Talbakzoda // Bulletin of the National Academy of Sciences of Tajikistan. Series of natural and geological sciences. 2025. Vol. 1. No. 1(45). P. 49-63. https://doi.org/81521911

[16] Talbakzoda F. M. On sufficient conditions for the absolute convergence of double Fourier series in the uniform metric [Text] / F. M. Talbakzoda // Bulletin of the National Academy of Sciences of Tajikistan. Overcoming physical, mathematical, chemical, geological and technical sciences. 2025. No. 1(198). P. 25-35. https://doi.org/8089067

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Research Field

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