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Research Article

An Efficient Numerical Solution Method of the Bagley-Torvik Equation with Damping Term by Equivalent Transform and Series Approximation

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Abstract

In this paper, we proposed a new approximate solution method of the Bagley- Torvik fractional order differential equation with damping term. The solution of this equation is of great practical interest and has been studied extensively. In general, the solution of the Bagley-Torvik equation is known to be computationally expensive and the process is complicated. However, the new proposed method significantly reduced the computational effort while ensuring the accuracy of the solution in a novel way. Since 1.5 order derivative of absolutely continuous a function on some interval in the Caputo's sense is equal to the temperature gradient at the boundary of the one-dimensional heat conduction problem in the semi-infinite interval with 1 order derivative of the function as boundary conditions, we transformed the given fractional differential equation into a general differential equation. Then, according to the characteristics of the obtained equation, we used the variables separation and Fourier series approximation. So the proposed method transforms the Bagley-Torvik fractional order differential equation into an integer order differential equation. Prior to the main content, we have given and proved two definitions and two theorems as preliminaries. And the convergence analysis of this method is discussed. And then we solved two different example problems for the cases with and without damping by using various methods including proposed method and verified that the computational effort is significantly small and the accuracy is guaranteed through comparison of the results with other methods. So the effectiveness of the proposed method is analyzed.

Keywords

Bagley-Torvik Equation, Fractional Order Differential Equation, Fourier Series Approximation

1. Introduction

This paper proposed the method to decrease the computational time of numerical solution of Bagley- Torvik equation. Fractional order calculus has been used in many fields, such as automation engineering, signal processing, viscoelastic mechanics, because it can accurately model the real world compared to integer-order calculus. [1, 2, 21, 22] Especially, the viscoelastic model lying between the elastic models based on Hook's model and the damping model based on the Newton model can be accurately described by fractional order calculus. [3] The Bagley-Torvik equation is obtained when modeling the motion of a steel plate immersed in a Newtonian fluid. [4] Due to the great practical significance

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of this equation, Studies on analytical and numerical solutions of this equation have been carried out in many literatures. [5-7] In the study by Z. H. Wang et al. [5], a method for finding the general solution of the homogeneous Bagley-Torvik equation using α-exponential functions was proposed, but no general solution of the nonhomogeneous Bagley-Torvik equation with damping terms was proposed. In the study by Denghao Pang et al. [6], the analytical solution of the Bagley-Torvik equation without damping terms is obtained using the Prabhakar function and the Wiman function. However, the integration of the Mittag- Leffler function is included in the analytical solution. In the study by Yuji Liu. [7], also the analytical solution without damping was obtained using Picard iterative technique using Schauder's fixed point theorem, and the integration of the Mittag-Leffler function as in the study by Denghao Pang et al. [6] is included in the analytical solution.

In the study by Rajarama Mohan Jena et al. [8], an analytical solution method based on the Sumudu transformation method is proposed, but this method does not consider the damping term and has the disadvantage of being difficult to find the inverse Sumudu transformation.

In addition, there are the Adomian decomposition method [9], the homotopy perturbation method [10], the evolu- tionary computational intelligence method [11], the Laplace transform method [12], the Galerkin method [13], the reproducing kernel method [14], etc. But the computational cost is huge and all the damping terms are not considered. In addition to the above methods, one of the important methods for solving the Bagley-Torvik fractional differential equation is the solution based on the approximation of the basis function, and the advantage of this method is that the given linear fractional-order differential equation is trans- formed into a linear algebraic equation. For example, methods using Taylor polynomials [15], Bessel polynomials [16], Haar wavelets [17], Fermat polynomials [18], Chebyshev polynomials [19], Legendre polynomials [20], etc. have been proposed.

These methods have the advantage of being applicable to general linear fractional differential equations and can be used for linear fractional identification. [21, 22] Since fractional differential equations are directly transformed into algebraic equations, they have the disadvantage of excessive computational effort when the number of collision points increases.

We aim to propose a solution of the Bagley-Torvik fractional differential equation with damping terms, and transform the given fractional differential equation into a general differential equation, unlike the previous methods, since the 1.5-order fractional derivative in the Caputo's sense of the function $y(t)$ is equal to the temperature gradient at the boundary of the one-dimensional heat conduction problem in the semi-infinite interval with boundary conditions. Then, according to the characteristics of the problem obtained, an approximate solution method using the variables separation and Fourier series approximation is newly proposed.

The advantage of this method is its significantly lower computational cost compared to the conventional methods.

The paper consists of five sections. Section 2 contains the basic conception, Section 3 contains proposed solution method, Section 4 presents the simulation analysis, and Section 5 gives a conclusion.

2. Preliminary Knowledge

Definition 1([1])

The Riemann Liouville's fractional integral ${}^{RL}I_t^{\alpha}$ of the order α of the function $f(t)\in C[0,b]$ is defined as

$$ {}^{RL}I_t^{\alpha} f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds \tag{1}$$

Where $\alpha>0$, and $\Gamma(\cdot)$ is the gamma function.

Since the integer differential operator D^n is the left inverse operator of the integer integral operator I^n, Riemann-Liouville's fractional order integral is defined as follows.

$$\begin{aligned} {}^{RL}D_t^{\alpha} f(t) &= {}^{RL}I_t^{-\alpha} f(t)=D^nI^n\ {}^{RL}I_t^{-\alpha} f(t) = D^n\ {}^{RL}I_t^{n-\alpha} f(t) \\ &= \frac{d^n}{dt^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{-\alpha+n-1} f(\tau)d\tau\right], \quad (n-1\le\alpha<n\ ,\ n\in N) \end{aligned} \tag{2}$$

Definition 2([1])

The Caputo's fractional derivative ${}^{C}D_t^{\alpha}$ of the order α of the function $f(t)\in C[t_0,b]$ is defined as;

$$ {}^{C}D_t^{\alpha} f(t)= {}^{RL}I_t^{n-\alpha}\ D^n f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds \tag{3}$$

$$(n-1<\alpha<n\ ,\ n\in N)$$

By (1) and (3), the relation between the Riemann-Liouville's fractional integral and the Caputo's fractional derivative can be written as

$$ {}^{RL}I_t^{\alpha}\left({}^{C}D_t^{\alpha} f(t)\right) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0+)\frac{t^k}{k!}, \quad (n-1 \le \alpha < n) \tag{4} $$

We consider the Bagley-Torvik fractional differential equation with Caputo's fractional derivative. For simplicity of notation, we denote the Caputo's fractional derivative operator ${}^{C}D_t^{\alpha}$ by D^{α}.

Theorem 1([24])

For initial condition

$$ u(x,0) = 0 \tag{5} $$

and boundary condition

$$ u(0+,t) = \dot{y}(t), \quad \dot{y}(0) = 0 \tag{6} $$

$$ \lim_{x \to +\infty} u(x,t) = 0 \tag{7} $$

the one-dimensional heat conduction equation in the semi-infinite interval $[0,\infty)$ with the above initial and boundary conditions is;

$$ \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} \tag{8} $$

At any time $t \in R^{+}$ of its solution $u(x,t)$, the temperature gradient at boundary $x = 0+$ is equal to $D^{1.5}y(t)$. i.e.

$$ \lim_{x \to 0+} \frac{\partial u(x,t)}{\partial x} = -\frac{1}{\Gamma(0.5)} \int_0^t \frac{\ddot{y}(\tau)}{(t-\tau)^{0.5}} d\tau \tag{9} $$

The proof about this is given in Appendix A.

3. Main Result

3.1. Bagley-Torvik Equation with Damping Term

Suppose that a steel plate of mass M immersed in a Newtonian fluid of density ρ be suspended by a zero mass spring and a damper of damping coefficient c, and vibrate under the action of external $f(t)$ and resistive forces F_R. In the figure, the steel plate is assumed to be fixed at one end of the spring and one end of the damper and not affected by the fluid flow.

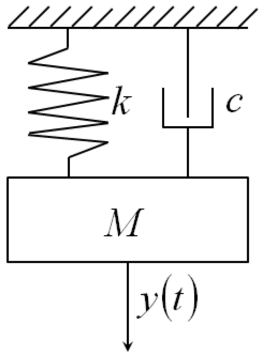

Figure 1. *The steel plate of mass M immersed in a Newtonian fluid.*

The elastic force acting on the steel plate and the damping resistance are respectively $-ky(t), -c\,\dot{y}(t)$ and the resistance force acting on the mechanism is:

$$ F_{\mathrm{R}} = -2s\sqrt{\mu\rho}\,\frac{1}{\sqrt{\pi}} \int_{-\infty}^{t} \frac{\ddot{y}(\tau)}{(t-\tau)^{1/2}} d\tau $$

According to Newton's second law, the equation of motion of a steel plate is:

$$ M\,\ddot{y}(t) = F_{\mathrm{R}} - ky(t) - c\dot{y}(t) + f(t) $$

Thus

$$ M\,\ddot{y}(t) + 2s\sqrt{\mu\rho}\,\frac{1}{\sqrt{\pi}} \int_{-\infty}^{t} \frac{\ddot{y}(\tau)}{(t-\tau)^{1/2}} d\tau + c\dot{y}(t) + ky(t) = f(t) \tag{10} $$

When $t \le 0$, if $y = 0,\ \dot{y}(t) = 0,\ \ddot{y}(t) = 0,\ f(t) = 0$, the vibration equation of the steel plate is as follows:

$$ M\,\ddot{y}(t) + 2s\sqrt{\mu\rho}\,\frac{1}{\sqrt{\pi}} \int_{0}^{t} \frac{\ddot{y}(\tau)}{(t-\tau)^{1/2}} d\tau + c\dot{y}(t) + ky(t) = f(t) $$

Considering $\Gamma(1/2) = \sqrt{\pi}$ and Definition 2, the above expression becomes as follows:

$$ M\ddot{y}(t) + mD^{3/2}y(t) + c\dot{y}(t) + ky(t) = f(t), \quad t > 0 \tag{11} $$

Here, $m = 2S\sqrt{\mu\rho}$, S is the surface area of the steel plate and μ is the viscous drag coefficient. This equation is called the Bagley-Torvik equation with damping term.

By Theorem 1, the fractional equation (11) is transformed into an equivalent integer-order differential equation as follows:

$$M\ \ddot{y}(t) - m\frac{\partial u(x,t)}{\partial x}\Big|_{x=0+} + c\dot{y}(t) + ky(t) = f(t) \tag{12}$$

The differential equation (12) is called the equivalent transformed integer-order Bagley-Torvik differential equation.

3.2. A Variables Separation Method for Equivalently Transformed Integer-order Bagley- Torvik Differential Equation

We propose a variables separation method for the solution of differential equations (12).

If we assume as follows for sufficiently large parameter ($L > 0$),

$$u(L,t) \approx 0 \tag{13}$$

We can consider the problem of finding the solution of the one-dimensional heat transfer equation (12) with boundary conditions (6) and (13) instead of the problem shown in Theorem 1.

We find the solution $u(x,t)$ of the heat conduction equation (8) in the form of

$$u(x,t) = A(x,t) + B(x,t) \tag{14}$$

If in Eq. (14), we suppose as follows

$$\begin{cases} A(0+,t) = 0 \\ A(L,t) = 0 \end{cases} \tag{15}$$

By boundary condition (6), (13) $B(x,t)$ must satisfy the following condition.

$$\begin{cases} B(0+,t) = \dot{y}(t) \\ B(L,t) = 0 \end{cases} \tag{16}$$

Then we choose the function $B(x,t)$ satisfying condition (16) as follows:

$$B(x,t) = \left(1 - \frac{x}{L}\right)\dot{y}(t) \tag{17}$$

Then Eq. (14), i.e. the solution $u(x,t)$ of the heat conduction equation (8) is written as:

$$u(x,t) = A(x,t) + \left(1 - \frac{x}{L}\right)\dot{y}(t) \tag{18}$$

Substituting Eq. (18) into Eq. (8), we have

$$\frac{\partial A(x,t)}{\partial t} + \left(1 - \frac{x}{L}\right)\ddot{y}(t) = \frac{\partial^2 A(x,t)}{\partial x^2} \tag{19}$$

Considering boundary conditions (15), the solution of the partial differential equation (19) can be found in the form of Fourier series as follows:

$$A(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\frac{n\pi x}{L} \tag{20}$$

Then Eq. (18) becomes as follows:

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\frac{n\pi x}{L} + \left(1 - \frac{x}{L}\right)\dot{y}(t) \tag{21}$$

Then, we have

$$\frac{\partial u(x,t)}{\partial x}\Big|_{x=0+} = \sum_{n=1}^{\infty}\frac{n\pi}{L}a_n(t) - \frac{1}{L}\dot{y}(t) \tag{22}$$

So the equivalent transformed Bagley-Torvik equation (12) with viscous term becomes as follows.

$$M\ \ddot{y}(t) - m\left(\sum_{n=1}^{\infty}\frac{n\pi}{L}a_n(t) - \frac{1}{L}\dot{y}(t)\right) + c\dot{y}(t) + ky(t) = f(t)$$

Or

$$M\ \ddot{y}(t) + \left(\frac{m}{L} + c\right)\dot{y}(t) + ky(t) - m\sum_{n=1}^{\infty}\frac{n\pi}{L}a_n(t) = f(t) \tag{23}$$

Let us now promise the following variables:

$$\dot{y}(t) = z(t) \tag{24}$$

And, by truncating the Fourier series to the N^{th} term in Eq. (23), and considering Eq. (24), we have

$$M\,\dot{z}(t)+\left(\frac{m}{L}+c\right)z(t)+ky(t)-m\sum_{n=1}^{N}\frac{n\pi}{L}a_n(t)\approx f(t) \quad (25)$$

To obtain the relation between $y(t)$ and $a(t)$ in (25), we approximate the function $1-x/L$ of (21) by a Fourier series.

For this, in the interval $[-L,L]$

$$g(x)=\begin{cases}1-\dfrac{x}{L}, & (0<x\le L)\\ 0\ , & (x=0)\\ -1+\dfrac{x}{L}, & (-L\le x<0)\end{cases}$$

We can find the Fourier series of an odd function $g(x)$ with period $2L$ defined as above as follows.

$$g(x)=\sum_{n=1}^{\infty}\alpha_n\sin\frac{n\pi x}{L} \quad (26)$$

Where

$$\alpha_n=\frac{1}{L}\int_{-L}^{L}g(x)\sin\frac{n\pi x}{L}dx=\frac{2}{L}\int_0^L\left(1-\frac{x}{L}\right)\sin\frac{n\pi x}{L}dx=\frac{2}{n\pi} \quad (27)$$

Thus

$$1-\frac{x}{L}=\sum_{n=1}^{+\infty}\frac{2}{n\pi}\sin\frac{n\pi x}{L},\quad (0<x\le L) \quad (28)$$

Comparing the Fourier coefficients of both sides in the above equation, the following result is obtained.

$$\dot{a}_n(t)=-\frac{n^2\pi^2}{L^2}a_n(t)-\frac{2}{n\pi}\ddot{y}(t),\quad (n=1,2,\cdots) \quad (29)$$

And also we can also obtain a relation between the initial conditions of the function $y(t)$ and $a(t)$ by the initial condition (5).

$$\sum_{n=1}^{\infty}a_n(0)\sin\frac{n\pi x}{L}=-\sum_{n=1}^{\infty}\alpha_n\dot{y}(0)\sin\frac{n\pi x}{L}$$

Comparing the Fourier series coefficients in the above equation, we have

$$a_n(0)=-\frac{2}{n\pi}\dot{y}(0) \quad (30)$$

This shows that all initial conditions for $a(t)$ in Eq. (25) can be expressed by $\dot{y}(0)$.

For Eq. (25), we define the following parameter vector:

$$\Theta_N=[y,z,a_1,a_2,\cdots,a_N]^T,\Phi_N=[0,f,0,0,\cdots,0]^T$$

Then, using Eqs. (24), (25) and (29), we can write

$$E_N\dot{\Theta}_N\approx F_N\Theta_N+\Phi_N \quad (31)$$

Here

$$E=\begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & M & 0 & \cdots & 0\\ 0 & 2/\pi & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 2/(N\pi) & 0 & \cdots & 1\end{bmatrix}$$

$$F=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ -k & -m/L-c & \pi m/L & \cdots & N\pi m/L\\ 0 & 0 & -\pi^2/L^2 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & -N^2\pi^2/L^2\end{bmatrix}$$

When $M\ne0$, $\det(E)=M\ne0$ and

$$E^{-1}=\begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & 1/M & 0 & \cdots & 0\\ 0 & -2/(M\pi) & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & -2/(NM\pi) & 0 & \cdots & 1\end{bmatrix} \quad (32)$$

So the differential equation (31) can be written as follows.

$$\dot{\Theta}_N\approx E_N^{-1}F_N\Theta_N+E_N^{-1}\Phi_N \quad (33)$$

Thus, from Peano's theorem [23] on the unique existence of solutions of ordinary differential equations, the solution of the initial value problem (33) is unique. Then the first component $y(t)$ of solution Θ_N of the differential equation (33) is the solution of the Bagley-Torvik equation with the viscous term. Using Eq. (30), the initial condition for the differential equation (33) can be obtained as follows:

$$\Theta(0)=[y(0),\dot{y}(0),-\frac{2}{\pi}\dot{y}(0),-\frac{2}{2\pi}\dot{y}(0),\cdots,-\frac{2}{N\pi}\dot{y}(0)]^T$$

3.3. Convergence Analysis

Let us define the following variables:

$$\Theta=\left[y,z,a_1,\cdots,a_N,a_{N+1},\cdots\right]^T=\left[\Theta_N^T,\Theta_e^T\right]^T$$
$$\Phi=\left[0,f,0,0,\cdots\right]^T=\left[\Phi_N^T,0^T\right]^T$$

Then Eq. (31) can be written as

$$\begin{bmatrix}E_N & 0\\ 0 & \mathrm{I}\end{bmatrix}\begin{bmatrix}\dot{\Theta}_N\\ \dot{\Theta}_e\end{bmatrix}=\begin{bmatrix}F_N & X\\ 0 & Y\end{bmatrix}\begin{bmatrix}\Theta_N\\ \Theta_e\end{bmatrix}+\begin{bmatrix}\Phi_N\\ 0\end{bmatrix} \tag{34}$$

where I is an infinite dimensional unit matrix and the matrices n and v are respectively

$$[X]_{ij}=\begin{cases}\dfrac{(N+j)m\pi}{L},(i=2)\\ 0 \quad ,\ (i\neq 2)\end{cases},\qquad [Y]_{ij}=\begin{cases}-\dfrac{(N+i)^2\pi^2}{L^2},(i=j)\\ 0 \quad ,(i\neq j)\end{cases} \tag{35}$$

Since $|E_N|\neq 0$, Eq. (34) can be written as

$$\begin{cases}\dot{\Theta}_N=E_N^{-1}F_N\Theta_N+E_N^{-1}\Phi_N+E_N^{-1}X\Theta_e\\ \dot{\Theta}_e=Y\Theta_e\end{cases} \tag{36}$$

Comparing Eqs. (33) and (36), the first expression of Eq. (36) is the exact solution.

We prove the convergence of the approximate solution by Eq. (33) through the following theorem.

Theorem 2

For $\forall t\geq 0$, the approximate solution Θ_N of Eq. (33) corresponding to the order of cut N and the exact solution Θ_N^* of Eq. (36) are obtained by the following relation:

(I) $\lim_{N\to+\infty} a_{N+i}(t)=0,\ \lim_{t\to+\infty} a_{N+i}(t)=0,(i=1,2,\cdots)$

(II) $\lim_{N\to+\infty}\left\|\Theta_N^*-\Theta_N\right\|_2=0$

Proof

The relation (I) is proved by considering the initial condition (30) and using the solution of the second expression in Eq. (36) as follows.

$$a_{N+i}(t)=-\frac{2\dot{y}(0)}{(N+i)\pi}e^{-\frac{(N+i)^2}{L^2}\pi^2 t},(i=1,2,\cdots) \tag{37}$$

Let us prove the relation (II).

The exact solution Θ_N^* based on the first expression of Eq. (36) is obtained as follows:

$$\Theta_N^*(t)=e^{E_N^{-1}F_N t}\Theta_N(0)+\int_0^t e^{E_N^{-1}F_N(t-\tau)}\left[E_N^{-1}\Phi_N(\tau)+E_N^{-1}X\Theta_e(\tau)\right]d\tau \tag{38}$$

Thus, using Eqs. (32), (36), and (37), the norm of the error vector between the approximate solution Θ_N of Eq. (33) and the exact solution Θ_N^* of Eq. (36) becomes as follows:

$$\begin{aligned}\left\|\Theta_N^*-\Theta_N\right\|_2&=\left\|\int_0^t e^{E_N^{-1}F_N(t-\tau)}E_N^{-1}X\Theta_e(\tau)d\tau\right\|_2\leq\int_0^t\left\|e^{E_N^{-1}F_N(t-\tau)}E_N^{-1}X\Theta_e(\tau)\right\|_2 d\tau\\ &\leq\int_0^t Qe^{\gamma(t-\tau)}\left\|E_N^{-1}X\Theta_e(\tau)\right\|_2 d\tau\\ &=Q\frac{2m\dot{y}(0)}{LM}\int_0^t e^{\gamma(t-\tau)}\left(\sum_{i=1}^{+\infty}e^{-\left(\frac{N+i}{L}\right)^2\tau}\right)\left\|\left[0,1,-\frac{2}{\pi},\cdots-\frac{2}{N\pi}\right]^T\right\|_2 d\tau\\ &\leq Q\frac{2m\dot{y}(0)}{LM}\int_0^t e^{\gamma(t-\tau)}\frac{1}{1-e^{-\frac{1}{L^2}\tau}}\frac{\sqrt{N+2}}{e^{\frac{N+1}{L^2}\tau}}d\tau\xrightarrow{N\to+\infty}0\end{aligned}$$

Here $\gamma=\max\left\{\operatorname{Re}\lambda:\lambda\in\sigma\left(E_N^{-1}F_N\right)\right\}$ and $Q\geq 1$ is a certain number satisfying $\left\|e^{E_N^{-1}F_N t}\right\|\leq Qe^{\gamma t}$.

End of Proof

4. Simulation Analysis

Example 1

First, we compared proposed method of solution of the

Bagley-Torvik fractional differential equation with damping terms with the Chebyshev collocation method [19] and the homotopy perturbation method [10] that are the methods based on polynomial basis functions through the following example.

$$\ddot{y}(t)+0.3D_t^{3/2}y(t)+0.2\dot{y}(t)+2y(t)=2\cos t \tag{39}$$

The initial condition is $y(0)=2,\ y'(0)=2.5$. The time step is given by $h=10^{-3}s$. The simulation time interval is given as $[0,40]$.

The calculation result is shown in Figure 2.

As shown in Figure 2, the proposed method has almost the same result as the Chebyshev collocation method. As pointed out in [19], the Chebyshev collocation method is more accurate than the homotopy perturb- bation method.

Therefore, it is shown that the proposed method has almost the same accuracy as those based on polyno- mial basis functions. To show the effectiveness of our method, we compare the computational time according to step number.

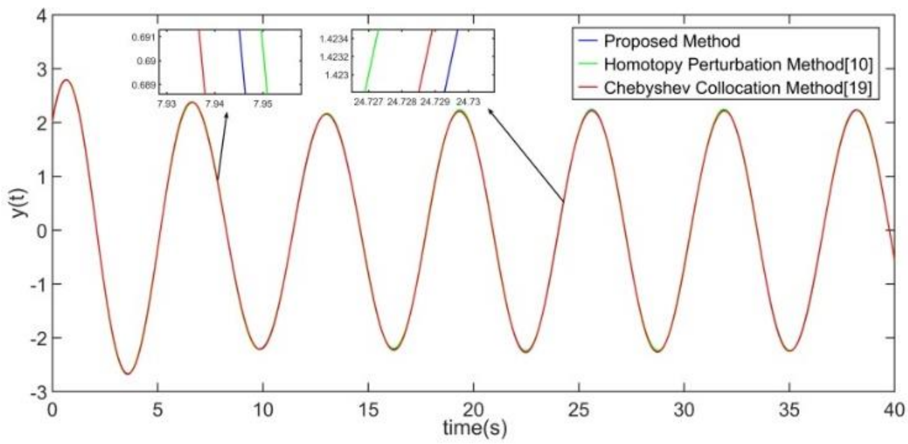

Figure 2. Comparison of the computational accuracy of the proposed method with the Homotopy perturbation method and the Chebyshev collocation method.

The fountion used in the simulation is Processor: 11th Gen Intel(R) Core(TM)i9-11900F@2.5GHz(16 CPUs), ~2.5GHz, RAM: 32GB.

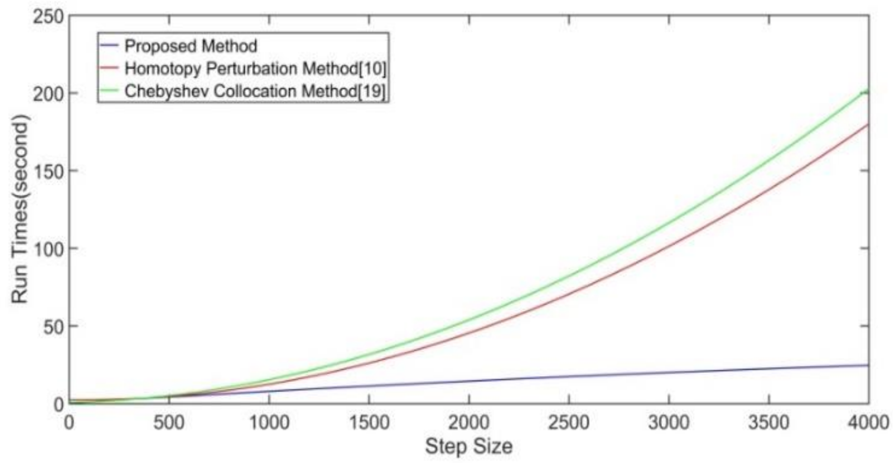

Figure 3. Comparison of the computational time of the proposed method with the homotopy perturbation method and the Chebyshev collocation method.

The curves shown in Figure 3 were obtained by collecting the computational times of each 50 steps and by polynomial interpolation of these points. As shown in Figure 3, the pro- posed method has much less computational time than the homotopy perturbation method and the Chebyshev collocation method. Unlike Figure 2, the homotopy method has less computational accuracy but less computational time than the Chebyshev collocation method.

Table 1 gives the computational time according to step number quantitatively.

Table 1. *Comparison of the computational time of three methods.*

Method	**Step Number**					
	500	**1000**	**1500**	**2000**	**3000**	**4000**
Proposed method	5.0730s	7.9572	11.3718	14.6043	20.0623	24.5993
Homotopy method	5.1233s	12.6603	26.2513	45.9833	101.6672	178.8723
Chebyshev method	5.2315s	15.3425	31.9521	54.5471	115.6518	201.6393

Example 2

Our method can be applied even in the absence of damping. In the past, many methods have been considered for the cases without damping.

To this case the analytical exact solution [5] and the wavelet method [17] are compared with our proposed method. As an example of simulation, this Vagley-Torvik equation without damping term was used in the study by S. Saha Ray. [17].

$$\ddot{y}(t)+0.5D_t^{3/2}y(t)+0.5y(t)=f(t) \qquad (40)$$

Where $f(t)=\begin{cases}8, & (0\le t\le 1)\\ 0, & (t>1)\end{cases}$, the initial condition is $y(0)=\dot{y}(0)=0$ The simulation time interval is given by $[0,20]$. The results are shown in Figure 4. The time step is given as $h=10^{-2}s$.

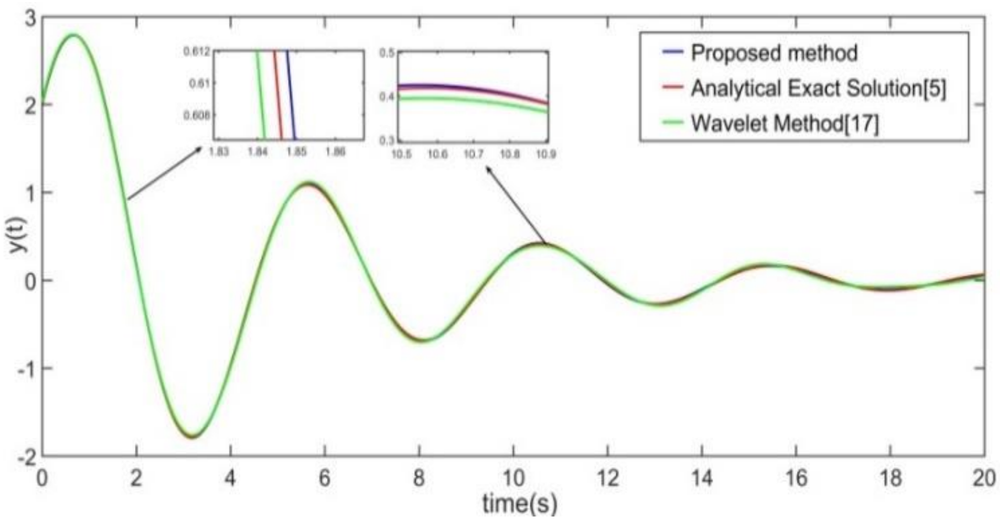

Figure 4. *Comparison of accuracy of the proposed method with analytical exact solution and wavelet method.*

To evaluate the accuracy of the proposed method, we used the mean square error (MSE) between the exact solution $y(t)$ and approximate solutions $\hat{y}(t)$.

$$MSE=\frac{1}{N}\sqrt{\sum_{k=1}^{N}\left(y(k)-\hat{y}(k)\right)^2} \qquad (41)$$

Table 2 shows the MSE according to the total step number when the simulation time interval is constant ($[0,20]$).

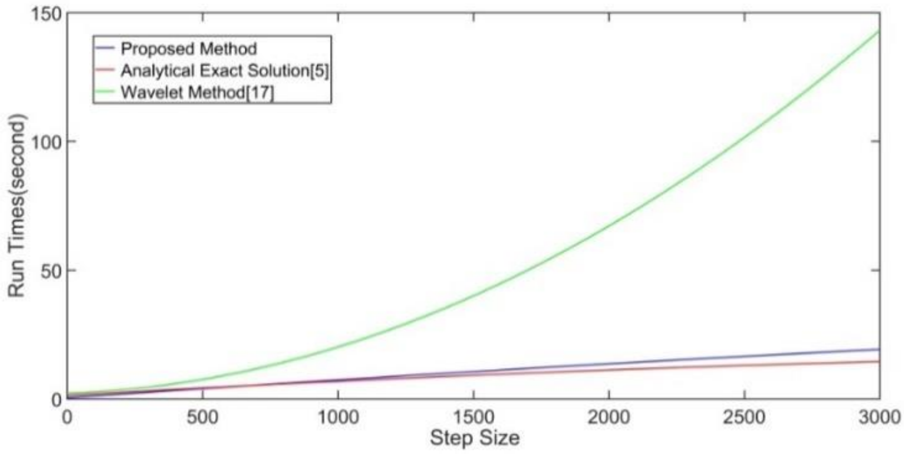

Figure 5. *Comparison of computational time between the proposed method, analytical method and wavelet method.*

Table 2. *MSE according to step number.*

Method	Step Number				
	1000	1500	2000	2500	3000
Proposed method	0.0236	0.0201	0.0173	0.0103	0.0093
Wavelet method	0.0421	0.0366	0.0278	0.0208	0.0163

As shown in Figure 4 and Table 2, the proposed method is more accurate than the wavelet method. Since the method using basis functions such as wavelet method uses inverse matrix of matrix with dimensions equal to step number, these methods have the disadvantage of limiting step number in computation by computer.

However, proposed method has the advantage of not being limited to this kind of problem.

Figure 5 compares the computational time between the proposed method, the analytical method and the wavelet method. In Figure 5, the computational time of the proposed method is not significantly different than that of the analytical method, but the wavelet method still has the disadvantage of large computational time.

Since the methods based on basis functions, such as wavelet methods, are almost the same computational time as wavelet methods, the proposed method is superior to those based on basis functions.

5. Conclusion

In this paper, we studied the efficient solution of the Bagley-Torvik fractional differential equation with damping terms transformed into integer-order differential equations. The advantage of the proposed method over the previous solution methods for the Bagley-Torvik equation is the low computational load. This was compared and evaluated in computer computational time through several simulation examples. A small computational load can satisfy the requirements of the necessary computational time for the vibration control of a rigid body immersed in a Newtonian fluid and are expected to have a wide range of applications.

We also compared the accuracy of the proposed method over the previous methods based on the exact solution of the Bagley-Torvik equation without damping terms which is possible for analytical solution.

Appendix

Proof of Theorem 1

We prove using Laplace transform.

First, the Laplace transform of both sides of Eq. (8) and using the initial condition (5) can be obtained as follows:

$$sU(x,s) = \frac{\partial^2 U(x,s)}{\partial^2 x}$$

The solution of this equation by the boundary condition (7) is

$$U(x,s) = c_1 e^{-\sqrt{s}x}$$

Then

$$\frac{\partial U(x,s)}{\partial x} = -\frac{1}{\sqrt{s}} sU(x,s) = -\mathcal{L}\left\{\frac{1}{\Gamma(0.5)t^{1/2}}\right\}\mathcal{L}\left\{\frac{\partial u(x,t)}{\partial t}\right\}$$

$$\frac{\partial u(x,t)}{\partial x} = -\frac{1}{\Gamma(0.5)t^{1/2}} * \frac{\partial u(x,t)}{\partial t} = -\frac{1}{\Gamma(0.5)}\int_0^t \frac{\frac{\partial u(x,\tau)}{\partial \tau}}{(t-\tau)^{1/2}} d\tau$$

By the boundary condition (6), the above equation is obtained as follows:

$$\lim_{x\to 0+} \frac{\partial u(x,t)}{\partial x} = -\frac{1}{\Gamma(0.5)}\int_0^t \frac{\frac{\partial u(0+,\tau)}{\partial \tau}}{(t-\tau)^{1/2}} d\tau = -\frac{1}{\Gamma(0.5)}\int_0^t \frac{\ddot{y}(\tau)}{(t-\tau)^{1/2}} d\tau$$

End of proof

Abbreviations

MSE	Mean Square Error

Author Contributions

Myong Hyok Sin: Supervision, Conceptualization, Methodology, Writing-editing, Software

Ryu Song Jo: Data curation, Validation, Resources

Data Availability Statement

No new data were created for this study.

Conflicts of Interest

The authors states no conflicts of interest.

References

[1] Diethelm, K. The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type, Springer Science & Business Media, 2010.

[2] H. Rojas, C. Cortes. Denoising of measured lightning electric field signals using adaptive filters in the fractional Fourier domain, Measurement 55 (2014) 616-626.

[3] R. Bagley, R. Calico. Fractional order state equations for the control of viscoelastically damped structures, J. Guidance Control Dynam. 14 (1991) 304-311.

[4] Torvik, P. J., and Bagley, R. L. On the Appearance of the Fractional Derivative in the Behavior of Real Materials, Journal of Applied Mechanics, 51(2) (1984) 294-298.

[5] Z. H. Wang, X. Wang, General solution of the Bagley-Torvik equation with fractional-order derivative, Commun Nonlinear Sci Numer Simulat 15 (2010) 1279-1285. https://doi.org/10.1016/j.cnsns.2009.05.069

[6] Denghao Pang, Wei Jiang, Jun Du, Azmat Ullah Khan Niazi, Analytical solution of the generalized Bagley-Torvik equation. (2019) 2019: 207. https://doi.org/10.1186/s13662-019-2082-8

[7] Yuji Liu, Boundary value problems for impulsive Bagley-Torvik models involving the Riemann-Liouville fractional derivatives, S ão Paulo J. Math. Sci. (2017) 11: 148-188. https://doi.org/10.1007/s40863-016-0054-4

[8] Rajarama Mohan Jena, S. Chakraverty, Analytical solution of Bagley - Torvik equations using Sumudu transformation method, SN Applied Sciences (2019) 1: 246, https://doi.org/10.1007/s42452-019-0259-0

[9] S. Saha Ray, R. K. Bera, Analytical solution of the Bagley Torvik equation by Adomian decomposition method, Applied Mathematics and Computation 168 (2005) 398-410. https://doi.org/10.1016/j.amc.2004.09.006

[10] M Zolfaghari, R Ghaderi, A SheikholEslami, A Ranjbar, S H Hosseinnia, S Momani, J Sadati, Application of the enhanced homotopy perturbation method to solve the fractional-order Bagley-Torvik differential equation, Phys. Scr. T136 (2009) 014032 (7 pp). https://doi.org/10.1088/0031-8949/2009/T136/014032

[11] Muhammad Asif Zahoor Raja, Junaid Ali Khan, Ijaz Mansoor Qureshi, Solution of Fractional Order System of Bagley-Torvik Equation Using Evolutionary Computational Intelligence, Mathematical Problems in Engineering, Volume 2011, Article ID 675075, 18 pages, https://doi.org/10.1155/2011/675075

[12] Azhar Ali Zafar, Grzegorz Kudra, Jan Awrejcewicz, An Investigation of Fractional Bagley-Torvik Equation, Entropy2020, 22, 28; https://doi.org/10.3390/e22010028

[13] Jian Chen, A fast multiscale Galerkin algorithm for solving boundary value problem of the fractional Bagley-Torvik equation, Boundary Value Problems, (2020) 2020: 91 https://doi.org/10.1186/s13661-020-01391-8

[14] Mehmet Giyas Sakar, Onur Saldır, Ali Akgu, A Novel Technique for Fractional Bagley-Torvik Equation, Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci. https://doi.org/10.1007/s40010-018-0488-4

[15] Mustafa Gülsu, Yalçın Öztürk, Ayşe Anapali, Numerical solution of the fractional Bagley-Torvik equation arising in fluid mechanics, International Journal of Computer Mathematics. http://dx.doi.org/10.1080/00207160.2015.1099633

[16] Suayip Yüzba sı, Numerical solution of the Bagley-Torvik equation by the Bessel collocation method, Mathematical Method in the Applied Sciences, 2012, https://doi.org/10.1002/mma.2588

[17] S. Saha Ray, On Haar wavelet operational matrix of general order and its application for the numerical solution of fractional Bagley Torvik equation, Applied Mathematics and Computation 218 (2012) 5239-5248. https://doi.org/10.1016/j.amc.2011.11.007

[18] Youssri H Youssri, A new operational matrix of Caputo fractional derivatives of Fermat polynomials: an application for solving the Bagley-Torvik equation, Advances in Difference Equations,(2017) 2017: 73, https://doi.org/10.1186/s13662-017-1123-4

[19] Vijay Saw, Sushil Kumar, Numerical Solution of Fraction Bagley-Torvik Boundary Value Problem Based on Chebyshev Collocation Method, Int. J. Appl. Comput. Math. (2019) 5: 68. https://doi.org/10.1007/s40819-019-0653-8

[20] Saeed Althubiti, Abdelaziz Mennouni, An Effective Projection Method for Solving a Coupled System of Fractional-Order Bagley-Torvik Equations via Fractional Shifted Legendre Polynomials, Symmetry 2022, 14, 1514. https://doi.org/10.3390/sym14081514

[21] Myong-Hyok Sin, Chol Min Sin, Song Ji, Su-Yon Kim, Yun-Hui Kang, Identification of fractional-order systems with both nonzero initial conditions and unknown time delays based on block pulse functions, Mechanical Systems and Signal Processing 169 (2022). https://doi.org/10.1016/j.ymssp.2021.108646

[22] Myong-Hyok Sin, Chol Min Sin, Hyang-Yong Kim, Yong-Min An, Kum-Song Jang, Parameter identification of fractional-order systems with time delays based on a hybrid of orthonormal Bernoulli polynomials and block pulse functions, Nonlinear Dyn (2024) 112: 15109-15132. https://doi.org/10.1007/s11071-024-09703-8

[23] Hartmut Logemann, Eugene P. Ryan, Ordinary Differential Equations: Analysis, Qualitative Theory and Control, (2014) springer, http://www.springer.com/series/3423

[24] P. J. Torvik, R. L. Bagley, On the Appearance of the Fractional Derivative in the Behavior of Real Materials, Journal of Applied Mechanics, 1984; 51(2), pp. 294-298.