

Research Article

Delta-Hedging of a European Call Options in Black-Scholes Under the Replicating Portfolio Strategy

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Abstract

Investing money involves different levels of risks depending on the choice of the investment. In finance, an investor is faced with the problems of where and when to invest; ability to regularly and dynamically build his portfolio of investments; ability to find investment strategies that give profits with zero initial expenditures; proper risk management; option pricing; and many more. Delta-hedging, which involves trading financial instruments strategically, helps investors to eliminate or reduce risk associated with option trading. This can be achieved by continuous re-balancing the portfolio of the stock and option to always have after re-balancing, a total delta of zero. Practically, hedging is being done periodically. This work deals with the Delta-hedging of a European Call Options in Black-Scholes under the replicating portfolio strategy. This replicating portfolio contains stocks and money market accounts. We obtain the initial value required to build a trading strategy that produces exact payoff and has similar cash flow as that of the Call Option at any time which is the replicating portfolio. From there, we derive the delta of a European Call Option. The condition of the self-financing trading strategy is satisfied by the replicating strategy. Generally, the payoff from delta-hedging a European option depends on the stock path.

Keywords

Delta Hedging, European Call Options, Black-Scholes Model, Replicating Portfolio, Trading Strategy

1. Introduction

In finance, hedging is the method of strategically trading financial instruments with the aim of removing possible risks associated with the initial position of an investor in assets already owned. A hedging strategy can be perfect or partial. When it totally eliminates the risk in a position, we say that it is perfect, otherwise, it is partial. There are different hedging methods depending on whether the market is complete or incomplete as well as whether the model in question is a continuous-time or a discrete-time model. Furthermore, a hedge can either be static or dynamic. We have a dynamic hedge when the hedging portfolio is rebalanced through time,

otherwise, the hedge is static.

Various hedging strategies have been examined in the literature. Hardy and Wirch [1] hedged against the risk of additional capital using the Iterated Conditional Tail Expectation (ICTE), by reduction in the volatility of cash flows. Moghtadai [2] examined the application of the Iterated Conditional Value at Risk (ICVaR) in reducing the cost of the hedging portfolio with the constraint on non-positive risk measure. Moller [3, 4] derived hedging strategies for risk-minimization in life insurance products under a generalised Black-Scholes frame work. His work was in a continuous-time setting where the financial

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market contains only stocks and money market accounts. On the other hand, Moller [5] presented a similar work under the Cox, Ross and Rubinstein model in a discrete-time setting. Jaimungal [6] used the Greeks to hedge dynamically Ratchet EIAs in a variance-Gamma economy. The models were analysed by comparing them to those derived in the Black-Scholes and Heston model like the work done in Mackay [7]. The Black and Scholes model [8] is an analytic model that sets a fair market price for European-style options. The model is still used extensively in modelling fluctuations in stock prices and valuing more complicated types of financial derivatives. Bernard and Boyle [9] proposed natural hedge, which is a complementary technique that protects issuers against market volatility risk. They achieved this by building a portfolio of policies with different payoffs. Each component of risk can be treated independently depending on the hedge parameter used. This makes it easier for the investor to manipulate and achieve a desired risk exposure by rebalancing his portfolio. Gaillardetz and Lakhmiri [10] formed a replicating portfolio of shares and money market accounts for equity-linked products. They later introduced a loaded contract money market accounts for equity-linked products. Alev [11] compared optimal portfolio strategies common in financial applications in the presence of various risk measure. He applied Martingale method under the Black-Scholes model to derive solutions for the optimal terminal wealth, then obtained the optimal portfolio strategies through representation.

This work deals with the Delta hedging of a European Call Options in Black-Scholes under the Replicating Portfolio strategy. This will be carried out assuming the Black-Scholes model.

2. Basic Tools and Preliminaries

Let $S(t)$ be the stock price at any time t and $S = \{S(t) > 0\}, 0 < t < T$ be the stock price process between the time of contracting, 0 and the time of maturity, T . Then the random evolution of the stock price process S can be described as the geometric Brownian motion of the form

$$S(t) = S(0)e^{G(t)}, \quad (1)$$

$S(t)$ satisfies the stochastic differential equation

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t), \quad (2)$$

where μ and σ represent constant mean rate of return and volatility of the stock prices respectively.

By Ito calculus, the solution to this equation is

$$S(t) = S(0)\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B(t)\right] \quad (3)$$

$$\ln S(t) = \ln S(0) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B(t), \ln S(0) > 0,$$

which is a generated Brownian motion process.

The conditional distribution of $\ln S(T)$ given $S(t)$ is normal with mean

$\ln S(t) + \left(\mu - \frac{\sigma^2}{2}\right)(T - t)$ and variance $\sigma^2(T - t)$. Then $S(T)|S(t)$ is a log-normal random variable with the same parameters and

$$\begin{aligned} E[S(T)|\mathcal{F}_0] &= \exp\left[\ln S(0) + \left(\mu - \frac{\sigma^2}{2}\right)T + \frac{\sigma^2 T}{2}\right] \\ &= \exp\left[\ln S(0) + \mu T - \frac{\sigma^2 T}{2} + \frac{\sigma^2 T}{2}\right] \\ &= \exp(\ln S(0) + \mu T) \\ &= S(0)e^{\mu T}, \end{aligned}$$

where $\mathcal{F}_0$ is the filtration that represents all the information given up to time 0. The discounted price process of a stock is a martingale under the risk-neutral measure Q and a martingale process has a constant expectation over time equal to its value at time $t = 0$. Under the Q measure, we have

$$E^Q[S(T)|\mathcal{F}_0] = S(0)e^{\mu T}$$

$$E^Q[e^{-rT}S(T)|\mathcal{F}_0] = S(0)e^{(\mu-r)T}.$$

Therefore, the discounted stock price process $e^{-rT}S_T$ under the Q measure is a martingale relative to $\mathcal{F}$. If the drift μ of the stock price process S is the risk-free interest, r , that is $\mu = r$, then

$$E^Q[e^{-rT}S(T)|\mathcal{F}_0] = S(0)$$

$$S(t) = S(0)\exp\left[\left(r - \frac{\sigma^2}{2}\right)t + \sigma B(t)\right] \quad (4)$$

is the solution to the stochastic differential equation

$$dS(t) = rS(t)dt + \sigma S(t)dB(t),$$

and

$$S(T)|S(t) \sim LN\left(\ln S(t) + \left(r - \frac{\sigma^2}{2}\right)(T - t), \sigma^2(T - t)\right)$$

3. The Model

The replicating portfolio strategy involves determining the initial value required to build a trading strategy ϕ that produces exact payoff and has the same cash flow as that of the Call option at any time t between the time of contract, 0 and the expiration, T . Let this portfolio contain stocks and money market accounts. Hence, let $\phi = \{\phi_s, \phi_m\}$, be the replicating portfolio strategy where ϕ_s is the total number of shares of stocks in the portfolio and ϕ_m represents the total number of

risk-less securities in the money market account.

Let the market value of the replicating portfolio at any time t be W_t^ϕ . Therefore,

$$W_t^\phi = \phi_S(t)S(t) + \phi_m(t)m(t). \quad (5)$$

We need to derive the value of the Call option W_t^C at any time t and equate it to equation (5). Let W_t^C be the value of a Call option at time t . Let Π^C be the payoff of this Call option, then

$$\Pi^C = \max\{S(T) - K, 0\} = \begin{cases} S(T) - K & \text{if } S(T) \geq K \\ 0 & \text{otherwise,} \end{cases} \quad (6)$$

where K is the strike price.

From equation (4),

$$W_t^C = e^{-r(T-t)} \int_K^\infty \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp\left\{-\frac{\left(\ln\frac{x}{S(t)} - \left(r - \frac{\sigma^2}{2}\right)(T-t)\right)^2}{2\sigma^2(T-t)}\right\} dx - Ke^{-r(T-t)} \int_K^\infty \xi_{S(T)}(x) dx.$$

By the change of variable $\alpha = \ln\frac{x}{S(t)}$ we have

$$\begin{aligned} W_t^C &= S(t) \varphi\left(\frac{\ln\frac{S(t)}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)} \varphi\left(\frac{\ln\frac{S(t)}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\ &= S(t) \varphi(d_{1,t}) - Ke^{-r(T-t)} \varphi(d_{2,t}), \end{aligned} \quad (7)$$

where φ is the cumulative distribution function of a standard normal random variable and

$$d_{1,t} = \frac{\ln\frac{S(t)}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \quad (8)$$

$$d_{2,t} = \frac{\ln\frac{S(t)}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}. \quad (9)$$

Equation (9) can be written in terms of (8) as

$$\begin{aligned} d_{2,t} &= \frac{\ln\frac{S(t)}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{\ln\frac{S(t)}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} - \sigma\sqrt{T-t} \\ &= \frac{\ln\frac{S(t)}{K} + \left(r + \frac{\sigma^2}{2} - \sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

$$W_t^\phi = W_t^C$$

$$\phi_S(t)S(t) + \phi_m(t)m(t) = S(t) \varphi(d_{1,t}) - Ke^{-r(T-t)} \varphi(d_{2,t}), \quad (12)$$

for all $0 \leq t \leq T$.

$$W_t^C = E^Q[e^{-r(T-t)} \Pi^C | \mathcal{F}_t].$$

Let the density function of the log-normal random variable $S(T)$ be $\xi_{S(T)}$, then

$$W_t^C = E^Q[e^{-r(T-t)} \Pi^C | \mathcal{F}_t]$$

$$= e^{-r(T-t)} \int_K^\infty (x - K) \xi_{S(T)}(x) dx$$

$$= e^{-r(T-t)} \int_K^\infty x \xi_{S(T)}(x) dx - Ke^{-r(T-t)} \int_K^\infty \xi_{S(T)}(x) dx.$$

By Green's formula,

$$= \frac{\ln\frac{S(t)}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t) - \sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

$$= \frac{\ln\frac{S(t)}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} - \sigma\sqrt{T-t}.$$

Therefore,

$$d_{2,t} = d_{1,t} - \sigma\sqrt{T-t}. \quad (10)$$

At $t = 0$, equation (7) becomes

$$W_0^C = S(0) \varphi(d_{1,0}) - Ke^{-rT} \varphi(d_{2,0}), \quad (11)$$

which is the Black-Scholes pricing formula.

Equating equation (7) with equation (5) which is the market value of the replicating portfolio at time t gives

For equation (12) to hold, we must have for all $0 \leq t \leq T$,

$$\phi_S(t) = \varphi(d_{1,t}) \quad (13)$$

$$= \phi_{S,t}^\Delta. \quad (18)$$

and

Substituting equation (18) in equation (17) gives

$$\begin{aligned} \phi_m(t)m(t) &= -Ke^{-r(T-t)}\varphi(d_{2,t}) \\ &= W_t^C - S(t)\varphi(d_{1,t}) \\ &= W_t^C - \phi_S(t)S(t). \end{aligned} \quad (14)$$

Since Delta (Δ) stands for a first order risk measure that signifies how sensitive a derivative is to changes in the underlying stock price, we have

$$\Delta_t = \frac{\partial W_t}{\partial S_t}. \quad (15)$$

Let $\phi^\Delta = \{\phi_S^\Delta, \phi_m^\Delta\}$ represent the Delta-hedging strategy for the derivative security under study and W_t^Δ be the value of its Delta-hedging replicating portfolio. Therefore, this portfolio is made up of $\phi_{S,t}^\Delta$ shares of the underlying, S , and $\phi_{m,t}^\Delta$ units in the money market m at any time t . That is,

$$W_t^\Delta = \phi_{S,t}^\Delta S_t + \phi_{m,t}^\Delta m_t. \quad (16)$$

In order for W_t^Δ to have the same sensitivity to changes in the stock price as that of W_t , we have

$$\Delta_t^\Delta = \Delta_t, \quad (17)$$

where Δ_t^Δ and Δ_t represent the Delta of the Delta-hedging portfolio W_t^Δ and Delta of W_t . From equation (15) we have

$$\begin{aligned} \Delta_t^\Delta &= \frac{\partial W_t^\Delta}{\partial S_t} = \frac{\partial}{\partial S_t} (\phi_{S,t}^\Delta S_t + \phi_{m,t}^\Delta m_t) \\ &= \frac{\partial}{\partial S_t} (\phi_{S,t}^\Delta S_t) + \frac{\partial}{\partial S_t} (\phi_{m,t}^\Delta m_t). \end{aligned}$$

$\phi_{m,t}^\Delta m_t$ tends to zero since it represents the amount invested in the money market.

Therefore,

$$\Delta_t^\Delta = \frac{\partial}{\partial S_t} (\phi_{S,t}^\Delta S_t)$$

$$\Delta_t^C = \varphi(d_{1,t}) + S_t \frac{1}{\sqrt{2\pi}} e^{-\frac{d_{1,t}^2}{2}} \frac{1}{S_t \sigma \sqrt{T-t}} - Ke^{-r(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{d_{2,t}^2}{2}} \frac{1}{S_t \sigma \sqrt{T-t}}. \quad (24)$$

From equation (10), $d_{2,t} = d_{1,t} - \sigma\sqrt{T-t}$. Therefore,

Hence,

$$e^{-\frac{d_{2,t}^2}{2}} = e^{-\frac{1}{2}(d_{1,t} - \sigma\sqrt{T-t})^2} \quad (25)$$

$$(d_{1,t} - \sigma\sqrt{T-t})^2 = d_{1,t}^2 - 2d_{1,t}\sigma\sqrt{T-t} + \sigma(T-t).$$

$$\phi_{S,t}^\Delta = \Delta_t. \quad (19)$$

By fixing the amount to be invested in the money market account as the difference between W_t and the amount invested in the risky assets, we have

$$\phi_{m,t}^\Delta m_t = W_t - \Delta_t S_t. \quad (20)$$

Substituting equations (19) and (20) in equation (16) gives

$$\begin{aligned} W_t^\Delta &= \Delta_t S_t + (W_t - \Delta_t S_t) \\ &= \Delta_t S_t + W_t - \Delta_t S_t \\ &= W_t. \end{aligned} \quad (21)$$

We now derive the Delta of a European Call option at $0 \leq t \leq T$ under the Black-Scholes model with payoff, Π^C , which is the rate of change of its value, W_t^C with respect to its underlying stock price, S_t . From equation (15) and using the Call option valuation formula in equation (7), we have

$$\begin{aligned} \Delta_t^C &= \frac{\partial W_t^C}{\partial S_t} \\ &= \frac{\partial}{\partial S_t} [S_t \varphi(d_{1,t}) - Ke^{-r(T-t)} \varphi(d_{2,t})] \\ &= \varphi(d_{1,t}) \frac{\partial}{\partial S_t} S_t + S_t \frac{\partial}{\partial S_t} \varphi(d_{1,t}) - Ke^{-r(T-t)} \frac{\partial}{\partial S_t} \varphi(d_{2,t}) \\ &= \varphi(d_{1,t}) + S_t \frac{\partial}{\partial S_t} \varphi(d_{1,t}) - Ke^{-r(T-t)} \frac{\partial}{\partial S_t} \varphi(d_{2,t}). \end{aligned} \quad (22)$$

For $k = 1, 2$,

$$\begin{aligned} \frac{\partial}{\partial S_t} \varphi(d_k) &= \varphi'(d_k) \frac{\partial}{\partial S_t} d_k \\ &= \frac{\varphi(d_k)}{S_t \sigma \sqrt{T-t}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_k^2}{2}} \frac{1}{S_t \sigma \sqrt{T-t}}. \end{aligned} \quad (23)$$

Applying equation (23) in equation (22) gives

Substituting equation (25) in equation (24) and factorizing

gives

$$\Delta_t^C = \varphi(d_{1,t}) + \frac{e^{-\frac{d_{1,t}^2}{2}}}{\sqrt{2\pi\sigma\sqrt{T-t}}} \left[1 - \frac{K}{S_t} e^{-r(T-t)} e^{-\frac{1}{2}[\sigma^2(T-t) - 2d_{1,t}\sigma\sqrt{T-t}]} \right]. \quad (26)$$

From equation (8),

$$d_{1,t} = \ln \frac{S_t}{K} + \left(r + \frac{\sigma^2}{2} \right) (T - t).$$

Therefore,

$$\begin{aligned} \Delta_t^C &= \varphi(d_{1,t}) + \frac{e^{-\frac{d_{1,t}^2}{2}}}{\sqrt{2\pi\sigma\sqrt{T-t}}} \left[1 - \frac{K}{S_t} e^{-r(T-t)} e^{-\frac{1}{2}[\sigma^2(T-t) - 2\{\ln \frac{S_t}{K} + (r + \frac{\sigma^2}{2})(T-t)\}]} \right] \\ &= \varphi(d_{1,t}) + \frac{e^{-\frac{d_{1,t}^2}{2}}}{\sqrt{2\pi\sigma\sqrt{T-t}}} \left[1 - \frac{K}{S_t} e^{-r(T-t) - \frac{1}{2}\sigma^2(T-t) + \ln \frac{S_t}{K} + (r + \frac{\sigma^2}{2})(T-t)} \right] \\ &= \varphi(d_{1,t}) + \frac{e^{-\frac{d_{1,t}^2}{2}}}{\sqrt{2\pi\sigma\sqrt{T-t}}} \left[1 - \frac{K}{S_t} e^{(T-t)(-r - \frac{\sigma^2}{2} + r + \frac{\sigma^2}{2}) + \ln \frac{S_t}{K}} \right] \\ &= \varphi(d_{1,t}) + \frac{e^{-\frac{d_{1,t}^2}{2}}}{\sqrt{2\pi\sigma\sqrt{T-t}}} \left[1 - \frac{K}{S_t} e^{(T-t)(0) + \ln \frac{S_t}{K}} \right] \\ &= \varphi(d_{1,t}) + \frac{e^{-\frac{d_{1,t}^2}{2}}}{\sqrt{2\pi\sigma\sqrt{T-t}}} \left[1 - \frac{K}{S_t} e^{\ln \frac{S_t}{K}} \right] \\ &= \varphi(d_{1,t}) + \frac{e^{-\frac{d_{1,t}^2}{2}}}{\sqrt{2\pi\sigma\sqrt{T-t}}} \left[1 - \frac{K}{S_t} \left(\frac{S_t}{K} \right) \right] \\ &= \varphi(d_{1,t}) + \frac{e^{-\frac{d_{1,t}^2}{2}}}{\sqrt{2\pi\sigma\sqrt{T-t}}} [1 - 1] \\ &= \varphi(d_{1,t}) + \frac{e^{-\frac{d_{1,t}^2}{2}}}{\sqrt{2\pi\sigma\sqrt{T-t}}} [0]. \end{aligned}$$

Therefore,

$$\Delta_t^C = \varphi(d_{1,t}), \quad (27)$$

which is the Delta of a European Call option.

4. Conclusion

Equation (14) defines the amount invested in the risk-free asset. This implies that the condition of the self-financing trading strategy is satisfied by the replicating strategy. The portion of money invested in the risky asset at any time is the first derivative of the Call option price with respect to the current price of the underlying stock. Equation (21) shows that the values of the Delta-hedging portfolio W_t^Δ and W_t are always similar for all $0 \leq t \leq T$. This implies that taking

a long position in the hedging portfolio will always offset the risks of the short position in the derivative security.

From the payoff function of the Call, the higher the price of the underlying asset at maturity, the higher the payoff. Therefore, the value of the option W_t^C increases as the price of the underlying asset goes up and decreases as S_t comes down. We can say that there is a positive correlation between the Call option and its underlying. This correlation is then used to eliminate the risk associated with the randomness of the stock price. This gives rise to Delta-hedging replicating portfolio.

Abbreviations

EIA	Equity-Indexed Annuity
ICTE	Iterated Conditional Tail Expectation
ICVaR	Iterated Conditional Value at Risk

Author Contributions

Joy Ijeoma Adindu-Dick is the sole author. The author read and approved the final manuscript.

Conflicts of Interest

There are no conflicts of interest.

References

- [1] Hardy, M. and Wirch, J. (2004). The Iterated Conditional Tail Expectation. *North American Actuarial Journal* 8: 62-75.
<https://doi.org/10.1080/10920277.2004.10596147>
- [2] Moghtadai, M. (2014). Partial Hedging of Equity Linked Products in the Presence of Policyholder Surrender using Risk Measures (Master's Thesis), Concordia University, Montreal, QC.
- [3] Moller, T. (1998). Risk-Minimizing Hedging Strategies for Unit Linked Life Insurance Contracts. *ASTIN Bulletin*, 28: 17-47.
- [4] Moller, T. (2001a). Risk-Minimizing hedging Strategies for Insurance Payment Process. *Journal of Finance and Stochastic*, 5: 419-446. <https://doi.org/10.1007/PL00013529>
- [5] Moller, T. (2001b). Hedging Equity Linked Life Insurance Contracts. *North American Actuarial Journal*, 5: 79-95.
<https://doi.org/10.1080/10920277.2001.1059599>
- [6] Jaimungal, S. (2004). Pricing and Hedging Equity Indexed Annuities with Variance Gamma Deviates. Working paper, Department of Statistics, University of Toronto.
- [7] Mackay, A. (2011). Pricing and Hedging Equity Linked Products under Stochastic Volatility Models. (Master's Thesis) Concordia University, Montreal, QC.
- [8] Black, F. and Scholes, M. (1973). The Pricing of Options and Corporate Liabilities. *The Journal of Political Economy*, 81(3): 637-654.
- [9] Bernard, C. and Boyle, P. (2011). Natural Hedge of Volatility Risk in Equity Indexed Annuities. *Annals of Actuarial Science*, 5. <https://doi.org/10.1017/S1748499511000275>
- [10] Gaillardetz, P. and Lakhmiri, J. (2011). A New Premium Principle for Equity Indexed Annuities. *The Journal of Risk and Insurance*, 78: 245-265.
<https://doi.org/10.1111/j.1539-6975.2011.01420.x>
- [11] Alev M. (2020). Comparison of various Risk measures for an Optimal Portfolio. *Acta Universitatis Apulensis*, 64: 83-115.
<https://doi.org/10.17114/j.aua.2020.63.03>