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Research Article

A Posteriori Error Estimates and Convergence of Error Indicator by FEM for a Semi-linear Elliptic Source-boundary Control Problem

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Abstract

In this paper, we obtain convergence of a posteriori error indicator to 0 when the mesh size h goes to 0 for the finite element approximation of source-boundary control problems governed by a system of semi-linear elliptic equations. We give the upper and lower bound of a posteriori error, and convergency of a posteriori error indicator.

Keywords

Semi-linear Elliptic Equations, Source Control, A Posteriori Error Estimate

1. Introduction

Differential equations and their control problems appear a lot in the thermal engineering, metallurgical engineering, chemical process and in the field of a system of optiamality. In many previous works, a posteriori error estimates of a differential equation have been done by using finite element method and spectral method [1-3, 7-9, 14, 16-18], and these methods are being extended to optimal control problems [4-6, 10-12, 13, 15].

If the solution of a differential equation is smooth, a posterior error estimates are obtained by spectral method [4, 5] but if not, by finite element method [6, 10, 12, 13, 15]. One of the difficulties in estimating a posteriori error by finite element method(FEM) is to verify the convergence of a posteriori error indicator to 0 when the mesh size h goes to 0, for an error indicator η is denoted as the form of sum with respect to elements of an indicator η_K, i.e., as the form of infinite series when h goes to 0 unlike spectral method.

$$\|y-y_h\|^2 \le \eta^2 = Ch^2\sum_K \|\eta_K\|_{0,K}^2 = \eta_\Omega^2 h^2,\quad \eta_\Omega^2 = \sum_K \|\eta_K\|_{0,K}^2$$

In [12], for control problem of a linear differential equation, the convergence of an error indicator was studied in the meaning of a subsequence. In [5], for a semi-linear parabolic source control problem, the upper bound estimates of a posteriori error by spectral method were obtained but not the convergence of an indicator.

The purpose of this work is to obtain a finite element posteriori error estimates for a semi-linear elliptic source control problem and study the convergence of an error indicator.

We are not aware of any existing work on a posteriori error estimates and the convergenc of error indicator by

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FEM for a semi-linear elliptic control problem.

For convergence of an error indicator, we've obtained a finite element posterio-ri lower estimates, estimated a priori error and osc (osc: oscilliation) of the residual, introduced a proper finite dementional norm and applied proof method using Lis theorem.in the Hilvert space.

Our paper is constructed as follows. In section 2, we give establishiment of problem and optimality condition, in section 3, 4 upper, lower bound of a posteriori error estimates of finite element, in section 5, priori error estimates and in section 6, convergency of error indicator of FEM.

Let $\|\cdot\|_{k,}$ $(\cdot,\cdot)_k$ denote the $H^k(\Omega)$ norm and inner product. We set $H := H^0(\Omega) = L^2(\Omega)$, $U := L^2(\Omega)$ with $k = 0$. .We denote by $\|\cdot\|_\Gamma$, $(\cdot,\cdot)_\Gamma$ or $\|\cdot\|_{0,\Gamma}$, $(\cdot,\cdot)_{0,\Gamma}$ the $L^2(\Gamma)$ norm and inner product, and by $\|\cdot\|_{0,s}$, $<\cdot,\cdot>_{0,s}$ $(s > 2)$ the $L^s(\Omega)$-norm and duality product between $L^s(\Omega)$ and $L^{s'}(\Omega)(1/s + 1/s' = 1)$.

2. Setting of Problems and Optimality Condition

Let Ω be a bounded convex polygonal domain in R^2 with boundary Γ.

Let us consider the boundary problem of differential equation as follows.

$$\begin{cases} -div(\lambda\nabla y) + \phi(y) = f + u, & \Omega \\ \dfrac{\partial y}{\partial n} + \varphi(y) = g, & \Gamma \end{cases} \tag{1}$$

$u \in U_1$, $g \in U_2$ are to be controlled and the feasible sets are convex closed subsets of $L^2(\Omega)$, $L^2(\Gamma)$ as follows.

$$U_1 := \left\{ u \in L^2(\Omega) \middle| \ u(x) \ge 0,\ x \in \Omega \right\}, \quad f \in L^2(\Omega)$$
$$U_2 := \left\{ g \in L^2(\Gamma) \middle| \ g_0 \le g(x),\ x \in \Gamma \right\}, \quad g_0 < 0$$

[Assumption 1] We assume the following properties for ϕ, φ.

$\phi(t), \varphi(t) \in W^{2,\infty}(-R,R)$, $0 < R < +\infty$, and they are strictly monotonic in R^1.

$\phi(0) = 0$, $\varphi(0) = 0$, $\phi'(t) \ge 0$, $\varphi'(t) \ge 0$, $t \in R$

The weak form of model (1) is (2) as follow

$$\int_\Omega \lambda\nabla y \nabla \upsilon dx + \int_\Omega \phi(y)\upsilon dx = \\ = \int_\Omega (f+u)\upsilon dx + \int_\Gamma g\upsilon dl, \forall \upsilon \in V \tag{2}$$

We assume that there exist unique weak solutions y of (2) in $V = H^1(\Omega)$.

The cost functional is defined as follows:

where $z_d \in L^2(\Omega)$ is given, $U := U_1 \times U_2$.

[Problem 1] $\inf_{(u,g)\in U} J(y;u,g)$

where $y = y(u,g)$ is a solution of (2).

Assume that there exists at least one solution of Problem 1.

[Theorem 1] (Optimality conditions)

Assume that $\bar{u}$ is a solution of Problem 1. There exists $\bar{p} \in V$ that satisfies:

$$\begin{aligned} &(\nabla\bar{p},\nabla\upsilon)_0 + (\phi'(\bar{y})\bar{p},\upsilon)_0 + (\varphi'(\bar{y})\bar{p},\upsilon)_{0,\Gamma} = (\bar{y} - z_d,\upsilon)_0,\ \forall\upsilon \in V, \\ &(\nabla\bar{y},\nabla\upsilon)_0 + (\phi(\bar{y}),\upsilon)_0 + (\varphi(\bar{y}),\upsilon)_{0,\Gamma} = (f + \bar{u},\upsilon)_0 + (\bar{g},\upsilon)_{0,\Gamma},\ \forall\upsilon \in V, \end{aligned} \tag{3}$$

where $\bar{y} = y(\bar{u},\bar{g})$ are unique solutions of (2) when $u = \bar{u}$, $g = \bar{g}$.

Proof. Let $\dot{y}_u$ denote Gateaux differential of the solution of (1) $y = y(u,g)$ at $(\bar{u},\bar{g})$ in the direction of $u - \bar{u}$,

$\dot{y}_g$ in the $\dot{y}_g$ in the direction of $g - \bar{g}$.

$\dot{y}_u$, $\dot{y}_g$ satisfy the following equations, respectively.

$$\begin{aligned} &(\nabla\dot{y}_u,\nabla\upsilon)_0 + (\phi'(\bar{y})\dot{y}_u,\upsilon)_0 + (\varphi'(\bar{y})\dot{y}_u,\upsilon)_{0,\Gamma} = (u - \bar{u},\upsilon)_0, \quad \forall\upsilon \in V \\ &(\nabla\dot{y}_g,\nabla\upsilon)_0 + (\phi'(\bar{y})\dot{y}_g,\upsilon)_0 + (\varphi'(\bar{y})\dot{y}_g,\upsilon)_{0,\Gamma} = (g - \bar{g},\upsilon)_{0,\Gamma}, \quad \forall\upsilon \in V \end{aligned} \tag{4}$$

The Gateaux differentials of the cost functional at $(\bar{u},\bar{g})$ in the direction of $u - \bar{u}$, $g - \bar{g}$ are as follows:

$$(J'_u(\bar{y};\bar{u},\bar{g}), u - \bar{u})_0 = (\bar{y} - z_d, \dot{y}_u)_0 + (\nu\bar{u}, u - \bar{u})_0 \ge 0, \\ \forall u \in U_1$$

$$(J_g'(\bar{y};\bar{u},\bar{g}),g-\bar{g})_{0,\Gamma}=(\bar{y}-z_d,\dot{y}_g)_0+(\nu\bar{g},g-\bar{g})_0\geq 0,\quad \forall g\in U_2$$

Let us consider the unique solution $\bar{p}$ of the following adjoint system.

$$(\nabla\bar{p},\nabla\upsilon)_0+(\phi'(\bar{y})\bar{p},\upsilon)_0+(\varphi'(\bar{y})\bar{p},\upsilon)_{0,\Gamma}= =(\bar{y}-z_d,\upsilon)_0,\quad \forall\upsilon\in V$$

By setting $\upsilon=p$ in the first equation of (4) and $\upsilon=\dot{y}_u$ in the adjoint system, and comparing them, we get

$$(\bar{y}-z_d,\dot{y}_u)_0=(\bar{p},u-\bar{u})_0.$$

Similarly, by setting $\upsilon=\bar{p}$ in the second equation of (4) and $\upsilon=\dot{y}_g$ in the adjoint system, and comparing them, we get

$$(\bar{y}-z_d,\dot{y}_g)_{0,\Gamma}=(\bar{p},g-\bar{g})_{0,\Gamma}.$$

Thus, we have

$$(J_u'(\bar{y};\bar{u},\bar{g}),u-\bar{u})=(\bar{p}+\nu\bar{u},\,u-\bar{u})_0\geq 0,\quad \forall u\in U_1$$
$$(J_g'(\bar{y};\bar{u},\bar{g}),g-\bar{g})=(\bar{p}+\nu\bar{g},\,g-\bar{g})_{0,\Gamma}\geq 0,\ \forall g\in U_2$$

The proof is completed.

3. Upper Bound of a Posteriori Error

Now, we partition Ω into regular triangles $K_j\,(j=1,2,\cdots,M)$ and denote the diameter of a triangle K by h_K, h_E, respectively. We set $h:=\max_{1\leq k\leq M}h_k$, let T_h be the family of triangles, E_h the set of all the sides of the triangles and denote finite element function space of $H^1(\Omega)$ as:

$$V_h:=\left\{\upsilon_h\in H^1(\Omega)\cap C(\bar{\Omega})\middle|\upsilon_h|_K\in P_1(K),\ \forall K\in T_h\right\}$$

where $P_1(K)$ denotes a polynomial space whose order is less or equal than one on element K.

$$\begin{aligned}&U_h:=U\cap V_h,\ U_h=U_{1h}\times U_{2h},U_{1h}=U_1\times V_h,\ U_{2h}=U_2\times V_h,\\&(\nabla y_h,\nabla\upsilon_h)_0+(\phi(y_h),\upsilon_h)_0+(\phi(y_h),\upsilon_h)_{0,\Gamma}=(f+u_h,\upsilon_h)_0+(g_h,\upsilon_h)_{0,\Gamma},\\&\qquad\forall\upsilon_h\in V_h,\ u_h\in U_{1h},\ g_h\in U_{2h}\end{aligned}\tag{5}$$

where $y_h\in V_h$ is the unique solution of (5).

The finite element approximation of (2) is as follows:
Problem 2.

$$\min_{(u_h,g_h)\in U_h}\left\{\frac{1}{2}\int_\Omega(y_h-z_d)^2dx+\frac{\nu}{2}\int_\Omega u_h^2dx+\frac{\nu}{2}\int_\Gamma g_h^2dl\right\}$$

Lemma 1. Problem 2 has the unique solution $\bar{u}_h\in U_h$, which satisfies the following system (optimality system)

$$\int_\Omega(\nu\bar{u}_h+p_h)(\upsilon_h-\bar{u}_h)dx\geq 0,\quad \bar{u}_h\in U_{1h},\ \forall\upsilon_h\in U_{1h},$$
$$\int_\Gamma(\nu\bar{g}_h+p_h)(w_h-\bar{g}_h)dl\geq 0,\quad \bar{g}_h\in U_{2h},\ \forall w_h\in U_{2h},$$

$$\begin{aligned}&(\nabla\bar{y}_h,\nabla\upsilon_h)_0+(\phi(\bar{y}_h),\upsilon_h)_0+(\varphi(\bar{y}_h),\upsilon_h)_{0,\Gamma}=(f+\bar{u}_h,\upsilon_h)_0+(g_h,\upsilon_h)_{0,\Gamma}\\&\quad\forall\upsilon_h\in V_h\subset V=H^1(\Omega),\end{aligned}\tag{6}$$

$$\begin{aligned}&(\nabla p_h,\nabla\upsilon_h)_0+(\phi'(\bar{y}_h)p_h,\upsilon_h)_0+(\phi'(\bar{y}_h)p_h,\upsilon_h)_{0,\Gamma}=(\bar{y}_h-z_d,\upsilon_h)_0,\\&\quad\forall\upsilon_h\in V_h\subset V=H^1(\Omega),\end{aligned}\tag{7}$$

where $p_h\in V_h$ is the unique solution of (7).

Proof. It's similar to the proof of Theorem 1.

Lemma 2 ([2, 3]) Let $I_h:V\to V_h\subset H^1(\Omega)$ denote an orthogonal projection operator. For $\forall\upsilon\in H^1(K),\ \forall K\in T_h,\ \forall E\in E_h$, we have:

1) $$\|\upsilon-I_h\upsilon\|_{0,K}\leq Ch_k\|\upsilon\|_{1,\omega_K},$$
$$\|\upsilon-I_h\upsilon\|_{0,E}\leq Ch_E^{1/2}\|\upsilon\|_{1,\omega_E}.$$

$$\omega_K := \bigcup_{K\cap K'\neq\varnothing} K',\quad \omega_E := \bigcup_{K\cap E\neq\varnothing} K$$

$$\forall \upsilon \in H^1(K),\ \forall K \in T_h,\ \forall E \in E_h$$

2) $(\|\upsilon\|_{0,E} \le C(h_K^{-1/2}\|\upsilon\|_{0,K} + h_K^{1/2}\|\nabla\upsilon\|_{0,K})$

3) $\|\upsilon\|_{0,\infty,K} \le Ch_K^{-1}\|\upsilon\|_{0,K}$, $\|\upsilon\|_{1,K} \le Ch_K^{-1}\|\upsilon\|_{0,K}$

Lemma 3

1) If $(\bar u, \bar g)$ is the solution of problem 1, then $\bar u \in C^0(\Omega)_+$, $\bar g \in C^0(\Gamma)_+$.

2) The solutions y_h, p_h of (6), (7) are bounded independent of h.

(proof) 1) From theorem 1, we have

$$(\bar p + \nu\bar u, u - \bar u)_0 \ge 0,\quad \forall u \in U_1,$$
$$(\bar p + \nu\bar g, g - \bar g)_{0,\Gamma} \ge 0,\quad \forall g \in U_2,$$

and this is equivalent to the followings.

$$(\nu\bar u - (-\bar p),\ u - \bar u)_0 \ge 0,\ \forall u \in U_1,$$
$$(\nu\bar g - (-\bar p),\ g - \bar g)_{0,\Gamma} \ge 0,\ \forall g \in U_2,$$

$$\Leftrightarrow \begin{array}{l} \bar u = \dfrac{1}{\nu}\max\{-\bar p,\ 0\},\quad \Omega \\ \bar g = \dfrac{1}{\nu}\max\{-\bar p,\ 0\},\quad \Gamma \end{array}$$

But $\bar y \in V$ and $\bar y \in L^r(\Omega)(1 < r < +\infty)$

Since $\phi(\bar y) \in L^3(\Omega)$, $\varphi(\bar y) \in L^3(\Gamma)$ by assumption 1, $\bar y \in H^2(\Omega)$ from the smoothness of the solution of an elliptic equation and $\bar p$ has a unique solution in. $V = H^2(\Omega)$

From $\bar p \in H^2(\Omega)$, the continuity of $H^2(\Omega \subset C(\Omega)$ and $H^{3/2}(\Gamma) \subset C(\Gamma)$ and the properties of operator $\max$, the assertion of the theorem holds.

2) From the fact that the functional $J(\cdot)$ is coercive with respect to $\bar u_h$, $\bar g_h$, $\bar u_h$, $\bar g_h$ are bounded independent of h. In the below equation satisfied by $\bar y_h$

$$(\nabla\bar y_h, \nabla\upsilon_h)_0 + (\phi(\bar y_h), \upsilon_h)_0 + (\varphi(\bar y_h), \upsilon_h)_{0,\Gamma} = (f + \bar u_h, \nabla\upsilon_h)_0 + \\ + (\bar g_h, \upsilon_h)_{0,\Gamma},\ \forall \upsilon_h \in V_h$$

when considering the asumption 1, we get

$$\|\bar y_h\|_1 \le \bar C\|f + \bar u_h\|_0 \le C$$

when $\upsilon_h = \bar y_h$.

Here, $\bar C := 1/C_1$, C_1 are norm equivalent constants of spaces $H^1(\Omega)$, $H_0^1(\Omega)$ and C is a constant independent of h.

In the adjoint equation

$$(\nabla\bar p_h, \nabla\upsilon_h)_0 + (\phi'(\bar y_h)\bar p_h, \upsilon_h)_0 + (\varphi'(\bar y_h)\bar p_h, \upsilon_h)_{0,\Gamma} = \\ = (\bar y_h - z_d, \upsilon_h)_0,\quad \forall \upsilon_h \in V_h^0 \subset V = H_0^1(\Omega)$$

considering assumption 1, we get

$$\|\bar p_h\|_1 \le \bar C\|\bar y_h - z_d\|_0 \le C,$$

when $\upsilon_h = \bar p_h$.

Let us estimate the upper bound of a posteriori error.

The following expression holds. ([5])

$$(J'(u), w) = (\nu u + p, w)_0$$
$$(J'(u_h), w_h) = (\nu u_h + p_h, w_h)_0$$
$$(J'(u_h), w) = (\nu u_h + p^h, w)_0$$

Here, $p^h := p(u_h, g_h) \in V$ satisfies the following system

$$\begin{aligned} &(\int_\Omega \nabla y^h \nabla\upsilon dx + \int_\Omega \phi(y^h)\upsilon dx + \int_\Gamma \varphi(y^h)\upsilon dl = \int_\Omega (f + u_h)\upsilon dx + \int_\Gamma g_h\upsilon dl,\quad \forall\upsilon \in V, \\ &\int_\Omega \nabla p^h\nabla\upsilon dx + \int_\Omega \phi'(y^h)p^h\upsilon dx + \int_\Gamma \varphi'(y^h)p^h\upsilon dl = \int_\Omega (y^h - z_d)\upsilon dx,\quad \forall\upsilon \in V \end{aligned} \tag{8}$$

Where $y^h := y(u_h, g_h)$ is a unique solution of the first expression of (5).

[Assumption 2] Functional is strictly convex in the neighborhood of a solution and there exists a proper constant so that

$$(\tilde J'(\tilde\upsilon) - \tilde J'(\tilde w), \tilde\upsilon - \tilde w) \ge C(\|u - w_1\|^2_{L^2(\Omega)} + \|g - w_2\|^2_{L^2(\Gamma)}. \tag{9}$$

$w:=(w_1,w_2)$ is an arbitrary element in the neighborhood of $\tilde{\upsilon}:=(u,g)$.

In the future, assumptions 1, 2 are assumed to be satisfied.

Lemma 4 When are the solutions of (3) and (6), (7) respectively, it holds that

$$\|u-u_h\|_0^2 \le C\left\|p^h - p_h\right\|_1^2 + C(\eta_5^2+\eta_6^2)$$

where $p^h := p(u_h, g_h)$ is a function defined in (8), $C>0$ is a constant independent of h and $\eta_5^2 := \|u_h + p_h\|_0^2$, $\eta_6^2 := \|g_h + p_h\|_0^2$.

(proof) From assumption 2 (expression (9)), we have

$$\begin{aligned}
&C(\|u-u_h\|_0^2 + \|g-g_h\|_{0,\Gamma}^2) \le \\
&\le (\tilde{J}'(u,g) - \tilde{J}'(u_h,g_h), (u-u_h, g-g_h)) = \\
&= \int_\Omega (u+p)(u-u_h)dx + \int_\Gamma (g+p)(g-g_h)dl + \\
&+ \int_\Omega (u_h+p^h)(u_h-u)dx + \int_\Gamma (g_h+p^h)(g_h-g)dl \le
\end{aligned}$$

Here, we consider that

$$\int_\Omega (u+p)(u-u_h)dx + \int_\Gamma (g+p)(g-g_h)dl \le 0, \quad \forall (u_h,g_h)\in U_h \subset U$$

from the optimality condition.

From the above inequality,

$$\begin{aligned}
&C(\|u-u_h\|_0^2 + \|g-g_h\|_{0,\Gamma}^2) \le \int_\Omega (u_h+p_h)(u_h-u)dx + \\
&+ \int_\Omega (p^h - p_h)(u_h-u)dx + \int_\Omega (g_h+p_h)(g_h-g)dx + \\
&+ \int_\Gamma (p^h-p_h)(g_h-g)dl \le C(\delta)(\left\|p^h-p_h\right\|_0^2 + \left\|p^h-p_h\right\|_{0,\Gamma}^2) + \\
&+ \delta(\|u_h-u\|_0^2 + \|g_h-g\|_{0,\Gamma}^2) + C(\eta_5^2+\eta_5^2) \le \\
&\le C(\delta)\left\|p^h-p_h\right\|_1^2 + \delta(\|u-u_h\|_0^2 + \|g_h-g\|_{0,\Gamma}^2) + C(\eta_5^2+\eta_5^2) \\
&\le C(\delta)\left\|p^h-p_h\right\|_1^2 + \delta(\|u-u_h\|_0^2 + \|g_h-g\|_{0,\Gamma}^2) + C(\eta_5^2+\eta_5^2)
\end{aligned}$$

Thus, we come to the following conclusion

$$\|u-u_h\|_0^2 + \|g-g_h\|_{0,\Gamma}^2 \le C\left\|p^h-p_h\right\|_1^2 + C(\eta_5^2+\eta_5^2).$$

Lemma 5 Let be solutions of (6), (7) and (8), respectively. Then it holds that

$$\left\|y_h - y^h\right\|_1^2 + \left\|p_h - p^h\right\|_1^2 \le C\sum_{i=1}^4 \eta_i^2$$

where η_i^2 are as follows.

$$\eta_1^2 = \sum_{K\in T_h} h_K^2 \|\gamma_K\|_{0,K}^2, \quad \gamma_K := y_h - z_d + \Delta p_h - \phi'(y_h)p_h$$

$$\eta_2^2 = \sum_{E\in E_h} h_E \|\gamma_E\|_{0,E}^2,$$

$$\gamma_E := \begin{cases} \left[\dfrac{\partial p_h}{\partial n}\right]_E = (\nabla p_h - \nabla\overline{p}_h)\cdot n_K, & E\cap\Gamma = \varnothing \\ \dfrac{\partial p_h}{\partial n}\Big|_E + \varphi'(y_h), & E\cap\Gamma \ne \varnothing \end{cases}$$

And $\left[\dfrac{\partial p_h}{\partial n}\right]_E$ is the jump value of the directional derivative in the common boundary E of two elements $K, \bar{K}$.

$$\eta_3^2 = \sum_{K\in T_h} h_K^2 \|r_K\|_{0,K}^2, \quad r_K := \Delta y_h - \phi(y_h) + f + u_h$$

$$\eta_4^2 = \sum_{E\in E_h} h_E \|r_E\|_{0,E}^2,$$

Similarly, $\left[\dfrac{\partial y_h}{\partial n_K}\right]_E$ is the jump value of the directional derivative in the common boundary E of two elements $K, \bar{K}$.

(proof) Step 1: We set,. Considering the assumption 1 about, we get

$$
\begin{aligned}
C\|\xi\|_1^2 &\le (\nabla\xi,\nabla\xi)_0+(\phi'(y^h)\xi,\xi)_0+(\varphi'(y^h)\xi,\xi)_{0,\Gamma}\le \\
&\le (\nabla p^h,\nabla\xi)_0+(\phi'(y^h)p^h,\xi)_0+(\varphi'(y^h)p^h,\xi)_{0,\Gamma}- \\
&-(\nabla p_h,\nabla\xi)_0-(\phi'(y^h)p_h,\xi)_0-(\varphi'(y^h)p_h,\xi)_{0,\Gamma}= \\
&=(y^h-z_d,\xi)_0-(\nabla p_h,\nabla\xi)_0-(\phi'(y_h)p_h,\xi)_0-(\varphi'(y_h)p_h,\xi)_{0,\Gamma}+ \\
&+((\phi'(y_h)-\phi'(y^h))p_h,\xi)_0+((\varphi'(y_h)-\varphi'(y^h))p_h,\xi)_{0,\Gamma}=
\end{aligned}
$$

Here, $C:=\min(1,\alpha)$, α are positive, seen in theorem 1. Let us make a use of lemma 2 adding

$$
\begin{aligned}
(\nabla p_h,\nabla\xi_I)_0+(\phi'(\bar{y}_h)p_h,\xi_I)_0+(\phi'(\bar{y}_h)p_h,\xi_I)_{0,\Gamma}- \\
-(\bar{y}_h-z_d,\xi_I)_0=0,\quad \xi_I\in V_h
\end{aligned}
$$

to the above expression.

$$
\begin{aligned}
C\|\xi\|_1^2 &\le (y_h-z_d,\xi-\xi_I)_0-(\nabla p_h,\nabla\xi-\xi_I)_0- \\
&-(\phi'(y_h)p_h,\xi-\xi_I)_0+(\varphi'(y_h)p_h,\xi-\xi_I)_{0,\Gamma}+ \\
&+((\phi'(y_h)-\phi'(y^h))p_h,\xi)_0+ \\
&+((\varphi'(y_h)-\varphi'(y^h))p_h,\xi)_{0,\Gamma}+(y^h-y_h,\xi)_0=
\end{aligned}
$$

$$
\begin{aligned}
&=\sum_{K\in T_h}(\gamma_K,\xi-\xi_I)_{0,K}+\sum_{E\in E_h}(\gamma_E,\xi-\xi_I)_{0,E}+ \\
&+((\phi'(y_h)-\phi'(y^h))p_h,\xi)_0+ \\
&+((\varphi'(y_h)-\varphi'(y^h))p_h,\xi)_{0,\Gamma}+(y^h-y_h,\xi)_0
\end{aligned}
$$

Considering $\xi-\xi_I\in V=H^1(\Omega)$ and 1) of lemma 2, we get

$$
\begin{aligned}
C\|\xi\|_V^2 &\le \sum_{K\in T_h}\|\gamma_K\|_{0,K}h_K\|\xi\|_{1,K}+\sum_{E\in E_h}\|\gamma_E\|_{0,E}h_E^{1/2}\|\xi\|_{1,K}+ \\
&+C\|y_h-y^h\|_{0,4}\|p_h\|_0\|\xi\|_{0,4}+C\|y_h-y^h\|_{0,\Gamma,4}\cdot \\
&\cdot\|p_h\|_0\|\xi\|_{0,\Gamma,4}+\|y_h-y^h\|_1\|\xi\|_1\le
\end{aligned}
$$

$$
\le C(\delta)\Big(\sum_{K\in T_h}\|\gamma_K\|_{0,K}^2h_K^2+\sum_{E\in E_h}\|\gamma_E\|_{0,E}^2h_E\Big)+C\|y_h-y^h\|_1^2+\delta\|\xi\|_1^2
$$

.

Thus, we get the following inequality.

$$
\|p^h-p_h\|_1\le C\|y_h-y^h\|_1^2+\eta_1^2+\eta_2^2.
$$

Step 2: We set $\xi:=y^h-y_h$, $\xi_I:=I_h\xi\in V_h\subset V$, $\xi\in V$.

Taking into account the assumption 1, we get the following estimation.

$$
\begin{aligned}
C\|\xi\|_1^2 &\le (\nabla y^h,\nabla\xi)_0+(\phi(y^h),\xi)_0+(\varphi(y^h),\xi)_{0,\Gamma}- \\
&-(\nabla y_h,\nabla\xi)_0-(\phi(y_h),\xi)_0-(\varphi(y_h),\xi)_{0,\Gamma}= \\
&=(f+u_h,\xi)_0+(g_h,\xi)_{0,\Gamma}-(\nabla y_h,\nabla\xi)_0- \\
&-(\phi(y_h),\xi)_0-(\varphi(y_h),\xi)_{0,\Gamma}
\end{aligned}
$$

where $C:=\min(1,l)$, l is a strictly monotonic constant of in assumption 1.

Adding

$$
\begin{aligned}
(\nabla y_h,\nabla\xi_I)_0+(\phi(y_h),\xi_I)_0+(\varphi(y_h),\xi_I)_{0,\Gamma}- \\
-(f+u_h,\xi_I)_0-(g_h,\xi_I)_{0,\Gamma}=0
\end{aligned}
$$

to the above expressing and using lemma 2, then we get

$$
\begin{aligned}
&=(f+u_h,\xi-\xi_I)_0+(g_h,\xi-\xi_I)_{0,\Gamma}-(\nabla y_h,\nabla\xi-\xi_I)_0- \\
&-(\varphi(y_h),\xi-\xi_I)_{0,\Gamma}=\sum_{K\in T_h}(r_K,\xi-\xi_I)_{0,K}- \\
&-\sum_{K\in T_h}(r_E,\xi-\xi_I)_{0,\partial K}\le \\
&\le\sum_{K\in T_h}(h_K\|r_K\|_{0,K})(h_K^{-1}\|\xi-\xi_I\|_{0,K})+\sum_{E\in E_h}(h_E^{1/2}\|r_E\|_{0,E})(h_E^{-1/2}\|\xi-\xi_I\|_{0,E})
\end{aligned}
$$

Therefore, we get

$$
C\|\xi\|_1^2\le C(\delta)\Big(\sum_{K\in T_h}(h_K^2\|r_K\|_K^2)+\sum_{E\in E_h}(h_E\|r_E\|_{0,E}^2)+\delta\|\xi\|_1^2
$$

and the following holds.

$$
\|y^h-y_h\|_1^2\le C(\eta_3^2+\eta_4^2).
$$

In the following theorem, we give the upper bound estimate of a posteriori errorwhich is one of the main results of this paper.

[Theorem 2] If $((u,g);\ y,\ p),\ ((u_h,g_h);\ y_h,\ p_h)$ are the solutions of (3) and (6), (7) respectively, then the following relation holds.

$$\|u-u_h\|_0^2+\|g-g_h\|_{0,\Gamma}^2+\|y-y_h\|_1^2+\|p-p_h\|_1^2\le C\sum_{i=1}^{6}\eta_i^2.$$

(proof) From lemma 4 and 5, it holds that

$$\|u-u_h\|_0^2+\|g-g_h\|_{0,\Gamma}^2\le C\left\|p_h-p^h\right\|_1^2\le C\sum_{i=1}^{6}\eta_i^2. \quad (10)$$

On the other hand, it holds that

$$\|y-y_h\|_1^2\le\left\|y-y^h\right\|_1^2+\left\|y^h-y_h\right\|_1^2, \quad (11)$$

$$\|p-p_h\|_1^2\le\left\|p-p^h\right\|_1^2+\left\|p^h-p_h\right\|_1^2 \quad (12)$$

Subtracting the equation satisfied by $y,\ y^h$, setting a test function as $\upsilon=y-y^h$ and using the strong monotonicity of $\phi(\cdot)$ and the monotonicity of $\varphi(\cdot)$, we get

$$\left\|y-y^h\right\|_1^2\le C\|u-u_h\|_0\left\|y-y^h\right\|_0+\|g-g_h\|_{0,\Gamma}\left\|y-y^h\right\|_{0,\Gamma}\le$$
$$\le C(\|u-u_h\|_0+\|g-g_h\|_{0,\Gamma})\left\|y-y^h\right\|_1$$

From above, we get the following inequality

$$\left\|y-y^h\right\|_1^2\le C(\|u-u_h\|_0^2+\|g-g_h\|_{0,\Gamma}^2)\,. \quad (13)$$

Similarly, subtracting the equation satisfied by $p,\ p^h$, setting a test function as $\upsilon=p-p^h$ and considering assumption 1, we get

$$\left\|p-p^h\right\|_1^2\le C\left\|y-y^h\right\|_1^2\le C(\|u-u_h\|_0^2+\|g-g_h\|_{0,\Gamma}^2). \quad (14)$$

From (10) - (14), we get the result.

4. Lower Bound of a Posteriori Error

We introduce the following function by barycentric coordinates $\lambda_{K_1},\ \lambda_{K_2},\ \lambda_{K_3}$ of element $K\in T_h$. [2]

$$b_K:=\begin{cases}27\cdot\lambda_{K,1}\cdot\ \lambda_{K,2}\cdot\lambda_{K,3}, & K\\ 0, & \Omega\setminus K\end{cases}$$

We introduce the following function when $E\in E_h$ is a common edge of elements $K,\ K'$.

$$b_E:=\begin{cases}4\lambda_{K,i}\cdot\lambda_{K,j}, & K\\ 4\lambda_{K,i}\cdot\lambda_{K,m}, & K'\\ 0, & \Omega\setminus\omega_E\end{cases}$$

$\omega_E:=K\cup K'$

Lemma 6 ([2]) For each element $K\in T_h$ and edge $E\in E_h$, b_K, b_E have the following properties.

$$\sup b_K\subset K,\ b_K\in[0,1],\ \max_{x\in K} b_K=1,$$
$$\int_K b_K dx=\frac{9}{20}|K|\le Ch_K^2,\quad \|\nabla b_K\|_{0,K}\le Ch_K^{-1}\cdot\|b_K\|_{0,K}$$

$$\sup b_E\subset\omega_E,\ b_E\in[0,1],\quad \max_{x\in\omega_E} b_E=1,$$

$$\int_{\omega_E} b_E dx=\frac{1}{3}|\omega_E|\le Ch_E^2,\quad \|\nabla b_E\|_{0,\omega_E}\le Ch_E^{-1}\cdot\|b_E\|_{0,\omega_E}$$

[Theorem 3] For the lower bound of a finite element posteriori error, the following estimation holds.

$$\sum_{i=1}^{6}\eta_i^2\le C(\|y-y_h\|_1^2+\|p-p_h\|_1^2+\|u-u_h\|_0^2+\|g-g_h\|_{0,\Gamma}^2)+osc+Ch.$$
$$osc(T_h):=osc(\gamma_K)+osc(\gamma_E)+osc(r_K)+osc(r_E)+osc(z_h)+osc(z_{1h}),$$
$$osc(\gamma_K):=\sum_{K\in T_h}h_K^2\|\gamma_K-\bar{\gamma}_K\|_{0,K}^2,\quad osc(\gamma_E):=\sum_{E\in E_h}h_K\|\gamma_E-\bar{\gamma}_E\|_{0,E}^2,$$

$$osc(r_K):=\sum_{K\in T_h}h_K^2\left\|r_K-\overline{r}_K\right\|_{0,K}^2,\quad osc(r_E):=\sum_{E\in E_h}h_K\left\|r_E-\overline{r}_E\right\|_{0,E}^2$$
$$osc(z_h):=\sum_{K\in T_h}\left\|z_h-\overline{z}_h\right\|_{0,K}^2,\quad osc(z_{1h}):=\sum_{E\in E_h}\left\|z_{1h}-\overline{z}_{1h}\right\|_{0,E}^2$$

where is independent of and denotes the different constants in the proof below, and is oscilliation

(proof) Let us estimate η_1^2.

It holds that

$$\left\|\gamma_K\right\|_{0,K}^2\le\left\|\overline{\gamma}_K\right\|_{0,K}^2+\left\|\gamma_K-\overline{\gamma}_K\right\|_{0,K}^2,\ \overline{\gamma}_K:=\frac{1}{|K|}\int_K\gamma_K dx,$$
$$I_K:=(\overline{\gamma}_K-\gamma_K,b_K\overline{\gamma}_K)_{0,K}$$

$$\begin{aligned}\left\|\overline{\gamma}_K\right\|_{0,K}^2&\sim(\overline{\gamma}_K,b_K\overline{\gamma}_K)_{0,K}=(\gamma_K,b_K\overline{\gamma}_K)_{0,K}+(\overline{\gamma}_K-\gamma_K,b_K\overline{\gamma}_K)_{0,K}=\\&=(y_h-z_d-\phi'(y_h)p_h+\Delta p_h,b_K\overline{\gamma}_K)_{0,K}+I_K=\\&=(y_h-z_d-\phi'(y_h)p_h+\Delta p_h,b_K\overline{\gamma}_K)_{0,K}+\\&+(-(y-z_d)+\phi'(y)p-\Delta p,b_K\overline{\gamma}_K)_{0,K}+I_K=\\&=(y_h-y,b_K\overline{\gamma}_K)_{0,K}+(\phi'(y)p-\phi'(y_h)p_h,b_K\overline{\gamma}_K)_{0,K}+\\&+(\Delta(p_h-p),b_K\overline{\gamma}_K)_{0,K}=I_1+I_2+I_3+I_K.\end{aligned}$$

Using 2) and 3) of lemma 2, we estimate the above integrals.

$$\begin{aligned}I_1&\le\left\|y_h-y\right\|_{0,K}\left\|b_K\overline{\gamma}_K\right\|_{0,K}\le\left\|y_h-y\right\|_{1,K}\left\|b_K\right\|_0\overline{\gamma}_K\le\\&\le C\left\|y_h-y\right\|_{1,K}h_K^{-1/2}\left\|\overline{\gamma}_K\right\|_{0,K}\le C\left\|y_h-y\right\|_{1,K}h_K^{-1}\left\|\overline{\gamma}_K\right\|_{0,K}\end{aligned}$$

$$\begin{aligned}I_2&\le C\left\|y_h-y\right\|_{0,4,K}\left\|p_h\right\|_{0,4,K}\left\|b_K\overline{\gamma}_K\right\|_{0,K}\le\\&\le Ch_K^{-1}\left\|y_h-y\right\|_{0,K}\left\|p_h\right\|_{0,K}\left\|\overline{\gamma}_K\right\|_{0,K}\le\\&\le Ch_K^{-1}\left\|y_h-y\right\|_{0,K}\left\|\overline{\gamma}_K\right\|_{0,K}\end{aligned}$$

$$I_3\le\left\|\Delta(p_h-p)\right\|_{0,K}\left\|b_K\overline{\gamma}_K\right\|_{0,K}\le Ch_K^{-1}\left\|p_h-p\right\|_{1,K}\left\|\overline{\gamma}_K\right\|_{0,K},$$

$$I_K:=(\overline{\gamma}_K-\gamma_K,b_K\overline{\gamma}_K)_{0,K}\le\left\|\overline{\gamma}_K-\gamma_K\right\|_{0,K}\left\|\overline{\gamma}_K\right\|$$

$$\left\|\overline{\gamma}_K\right\|_{0,K}\le Ch_K^{-1}(\left\|y_h-y\right\|_{1,K}+\left\|p_h-p\right\|_{1,K})+\left\|\gamma_K-\overline{\gamma}_K\right\|_{0,K},$$
$$h_K^2\left\|\gamma_K\right\|_{0,K}^2\le C(\left\|y_h-y\right\|_{0,K}^2+\left\|p_h-p\right\|_{0,K}^2)+h_K^2\left\|\gamma_K-\overline{\gamma}_K\right\|_{0,K}^2,$$

From the above relations, we get the following result.

$$\eta_1^2\le\sum_{K\in T_h}h_K^2\left\|\gamma_K\right\|_{0,K}^2\le\ C(\left\|y_h-y\right\|_1^2+\left\|p_h-p\right\|_1^2)+osc(\gamma_K),$$

Let us estimate η_2^2.

$$\|\gamma_E\|_{0,E}^2 \le \|\bar{\gamma}_E\|_{0,E}^2 + \|\gamma_E - \bar{\gamma}_E\|_{0,E}^2, \ \bar{\gamma}_E := \frac{1}{|E|}\int_E \gamma_E dx,$$
$$I_E := (\bar{\gamma}_E - \bar{\gamma}_E, b_E\bar{\gamma}_E)_{0,E}$$

1) In case of $E \cap \Gamma = \varnothing$

$$\|\bar{\gamma}_E\|_{0,E}^2 \sim (\bar{\gamma}_E, b_E\bar{\gamma}_E)_{0,E} = (\gamma_E, b_E\bar{\gamma}_E)_{0,E} + I_E =$$
$$= (\nabla p_h, \nabla(b_E\bar{\gamma}_E)_{0,\omega_E} + (\Delta p_h, b_E\bar{\gamma}_E)_{0,\omega_E} = (\nabla(p_h - p), \nabla(b_E\bar{\gamma}_E))_{0,\omega_E} +$$

$$= (\nabla p_h, \nabla(b_E\bar{\gamma}_E)_{0,\omega_E} + (\Delta p_h, b_E\bar{\gamma}_E)_{0,\omega_E} = (\nabla(p_h - p), \nabla(b_E\bar{\gamma}_E))_{0,\omega_E} +$$

$$+(\phi'(y_h)p_h - (y_h - z_d), b_E\bar{\gamma}_E)_{0,\omega_E} + (\Delta p_h - \phi'(y_h)p_h + y_h - z_d, b_E\bar{\gamma}_E)_{0,\omega_E} =$$

$$= (\nabla(p_h - p), \nabla(b_E\bar{\gamma}_E))_{0,\omega_E} + I_E +$$
$$+ (\phi'(y_h)p_h - \phi'(y)p + y - y_h, b_E\bar{\gamma}_E)_{0,\omega_E} +$$
$$+ (\gamma_K, b_E\bar{\gamma}_E)_{0,\omega_E} = I_1 + I_2 + I_3 + I_E.$$

Using inverse inequality, we estimate the above integrals

$$I_1 \le Ch_K^{-1}\|p - p_h\|_{1,\omega_E}\|b_E\bar{\gamma}_E\|_{0,\omega_E} \le Ch_K^{-1}\|p - p_h\|_{1,\omega_E}\|b_E\|_{0,\omega_E}|\bar{\gamma}_E| \le$$

$$\le C\|p - p_h\|_{1,\omega_E}|\bar{\gamma}_E| \le Ch_K^{-1/2}\|p - p_h\|_{1,\omega_E}\|\bar{\gamma}_E\|_{0,E},$$

Similarly, we get the following expression.

$$I_2 \le Ch_K^{-1/2}(\|p - p_h\|_{1,\omega_E} + \|y - y_h\|_{1,\omega_E})\|\bar{\gamma}_E\|_{0,E},$$

$$I_3 \le \|\gamma_K\|_{0,\omega_E}\|b_E\bar{\gamma}_E\|_{0,\omega_E} \le \|\gamma_K\|_{0,\omega_E}\|b_E\|_{0,\omega_E}|\bar{\gamma}_E| \le$$
$$\le Ch_K^{1/2}\|\gamma_K\|_{0,\omega_E}\|\bar{\gamma}_E\|_{0,E},$$

$$I_E = (\bar{\gamma}_E - \bar{\gamma}_E, b_E\bar{\gamma}_E)_{0,E} \le \|\bar{\gamma}_E - \bar{\gamma}_E\|_{0,E}\|\bar{\gamma}_E\|_{0,E},$$

$$h_K^{1/2}\|\bar{\gamma}_E\|_{0,E} \le C(\|p - p_h\|_{1,\omega_E} + \|y - y_h\|_{1,\omega_E}) + Ch_K\|\gamma_K\|_{0,\omega_E} + h_K^{1/2}\|\bar{\gamma}_E - \bar{\gamma}_E\|_{0,E}.$$

Taking into account the above expressions, we get

$$\eta_2^2 = \sum_{E \in E_h} h_E\|\gamma_E\|_{0,E}^2 \le C(\|p - p_h\|_1^2 + \|y - y_h\|_1^2) + C\sum_{K \in T_h} h_K^2\|\gamma_K\|_{0,K}^2 + osc(\gamma_E) =$$
$$= C(\|p - p_h\|_1^2 + \|y - y_h\|_1^2) + \eta_1^2 + osc(\gamma_E),$$

2) In case of $E \cap \Gamma \ne \varnothing$
Like in 1), we get

$$\|\bar{\gamma}_E\|_{0,E}^2 \sim (\bar{\gamma}_E, b_E\bar{\gamma}_E)_{0,E} = (\gamma_E, b_E\bar{\gamma}_E)_{0,E} + (\bar{\gamma}_E - \gamma_E, b_E\bar{\gamma}_E)_{0,E} =$$
$$= (\Delta p_h, b_E\bar{\gamma}_E)_{0,K} + (\nabla p_h, \nabla(b_E\bar{\gamma}_E))_{0,K} + (\varphi'(y_h)p_h, b_E\bar{\gamma}_E)_{0,E} + I_E =$$
$$= (\Delta p_h, b_E\bar{\gamma}_E)_{0,K} + (\nabla p_h, \nabla(b_E\bar{\gamma}_E))_{0,K} +$$

$$+ (\varphi'(y_h)p_h, b_E\bar{\gamma}_E)_{0,E} - (\nabla p, \nabla(b_E\bar{\gamma}_E))_{0,K} - (\phi'(y)p, b_E\bar{\gamma}_E)_{0,K} -$$
$$-(\varphi'(y)p, b_E\bar{\gamma}_E)_{0,E} + (y - z_d, b_E\bar{\gamma}_E)_{0,K} + I_E =$$

$$= (\nabla(p - p_h), \nabla(b_E\bar{\gamma}_E))_{0,K} + (\phi'(y_h)p_h - \phi'(y)p, b_E\bar{\gamma}_E)_{0,K} -$$

$$+ (\varphi'(y_h)p_h - \varphi'(y)p, b_E\bar{\gamma}_E)_{0,E} + (\Delta p_h - \phi'(y_h)p_h + y_h - z_d, b_E\bar{\gamma}_E)_{0,K} +$$
$$+ (y - z_d, b_E\bar{\gamma}_E)_{0,K} - (y_h - z_d, b_E\bar{\gamma}_E)_{0,K} + I_E =$$

$$= (\nabla(p - p_h), \nabla(b_E\bar{\gamma}_E))_{0,K} + (\phi'(y_h)p_h - \phi'(y)p + y - y_h, b_E\bar{\gamma}_E)_{0,K} +$$
$$+ (\varphi'(y_h)p_h - \varphi'(y)p, b_E\bar{\gamma}_E)_{0,E} + (\gamma_K, b_E\bar{\gamma}_E)_{0,K} + I_E = I_1 + I_2 + I_3 + I_4 + I_E.$$

We estimate the above integral parts.

$$I_1 \le \|p - p_h\|_1 \|b_E\bar{\gamma}_E\|_1 \le Ch_K^{-1}\|p - p_h\|_1 \|b_E\bar{\gamma}_E\|_{0,K} \le$$
$$\le Ch_K^{-1}\|p - p_h\|_1 \|b_E\|_{0,K}\,\bar{\gamma}_E \le C\|p - p_h\|_1\,\bar{\gamma}_E \le$$
$$\le Ch_K^{-1/2}\|p - p_h\|_1 \|\bar{\gamma}_E\|_0,$$

Here, we consider that

$$\bar{\gamma}_E = Ch_K^{-1/2}\|\bar{\gamma}_E\|_{0,K}.$$

$$I_2 \le Ch_K^{-1/2}(\|p - p_h\|_1 + \|y - y_h\|_1)\|\bar{\gamma}_E\|_{0,E},$$
$$I_3 \le Ch_K^{1/2}\|\gamma_K\|_{0,K}\|\bar{\gamma}_E\|_{0,E},$$

$$I_4 = (\varphi'(y_h)(p - p_h), b_E\bar{\gamma}_E)_{0,E} +$$
$$+ ((\varphi'(y_h) - \varphi'(y))p, b_E\bar{\gamma}_E)_{0,E}$$

$$\le \|\varphi'(y_h)\|_{0,E}\|p - p_h\|_{0,4,E}\|b_E\bar{\gamma}_E\|_{0,4,E} +$$
$$+ C\|y - y_h\|_{0,4,E}\|p\|_{0,4,E}\|b_E\bar{\gamma}_E\|_{0,E} \le$$
$$\le Ch_K^{-1/2}(\|p - p_h\|_{1,K} + \|y - y_h\|_{1,K})\|\bar{\gamma}_E\|_{0,E},$$

$$I_E = (\bar{\gamma}_E - \bar{\gamma}_E, b_E\bar{\gamma}_E)_{0,E} \le \|\bar{\gamma}_E - \bar{\gamma}_E\|_{0,E}\|\bar{\gamma}_E\|_{0,E},$$

Thus, we get the following result.

$$Ch_K^{1/2}\|\bar{\gamma}_E\|_{0,E} \le C(\|p - p_h\|_{1,K} + \|y - y_h\|_{1,K}) + Ch_K\|\gamma_K\|_{0,K} +$$
$$+ h_K^{1/2}\|\bar{\gamma}_E - \bar{\gamma}_E\|_{0,E},$$
$$\eta_2^2 = \sum_{E\in E_h} h_K\|\gamma_E\|_{0,K}^2 \le C(\|p - p_h\|_1^2 + \|y - y_h\|_1^2) + \eta_1^2 +$$
$$+ osc(\gamma_E),$$

Let us estimate η_3^2.

$$\|r_K\|_{0,K}^2 \le \|\bar{r}_K\|_{0,K}^2 + \|r_K - \bar{r}_K\|_{0,K}^2, \quad \bar{r}_K := \frac{1}{|K|}\int_K r_K\,dx,$$
$$I_K := (\bar{r}_K - r_K, b_K\bar{r}_K)_{0,K},$$

$$\|\bar{r}_K\|_{0,K}^2 \sim (\bar{r}_K, b_K\bar{r}_K)_{0,K} = (r_K, b_K\bar{r}_K)_{0,K} +$$

$$+ (\bar{r}_K - r_K, b_K\bar{r}_K)_{0,K} = -(\Delta(y - y_h), b_K\bar{r}_K)_{0,K} +$$
$$+ (\phi(y) - \phi(y_h), b_K\bar{r}_K)_{0,K} - (u - u_h, b_K\bar{r}_K)_0 + I_K \le$$

$$\le Ch_K^{-1}\|y - y_h\|_{1,K}\|b_K\bar{r}_K\|_{0,K} + C\|y - y_h\|_{0,K}\|b_K\bar{r}_K\|_{0,K} +$$

$$+\|u-u_h\|_{0,K}\|b_K\bar{r}_K\|_{0,K}+I_K\le$$
$$\le Ch_K^{-1}\|y-y_h\|_{1,K}\|\bar{r}_K\|_{0,K}+Ch_K^{-1}\|y-y_h\|_{1,K}\|\bar{r}_K\|_{0,K}+$$
$$+Ch_K^{-1}\|u-u_h\|_{0,K}\|\bar{r}_K\|_{0,K}+\|\bar{r}_K-r_K\|_{0,K}.$$

Thus, we get

$$h_K\|\bar{r}_K\|_{0,K}\le C\|y-y_h\|_{1,K}+\|u-u_h\|_{0,K}+h_K\|\bar{r}_K-r_K\|_{0,K},$$
$$\eta_3^2=\sum_{K\in T_h}h_K^2\|r_K\|_{0,K}^2\le C(\|y-y_h\|_1^2+\|u-u_h\|_0^2)+osc(r_K),$$

Let us estimate η_4^2.

1) In case of $E\cap\Gamma=\varnothing$

For convenience, we set $I_E:=(r_E-\bar{r}_E,b_E\bar{r}_E)_{0,E}$.

Considering the calculating process of, we can see that the following expressions hold.

$$\|\bar{r}_E\|_{0,E}^2\sim(\bar{r}_E,b_E\bar{r}_E)_{0,E}=(r_E,b_E\bar{r}_E)_{0,E}+I_E=$$
$$=(\nabla(y_h-y),\nabla(b_E\bar{r}_E))_{0,\omega_E}+(\phi(y_h)-\phi(y),b_E\bar{r}_E)_{0,\omega_E}+$$
$$+(r_K,b_E\bar{r}_E)_{0,\omega_E}+(u-u_h,b_E\bar{r}_E)_{0,\omega_E}+I_E\le$$

$$\le(Ch_K^{-1/2}\|y_h-y\|_{1,\omega_E}+Ch_K^{-1/2}\|u-u_h\|_{0,\omega_E}+Ch_K^{1/2}\|r_K\|_{0,K}+$$
$$+\|\bar{r}_K-r_K\|_{0,K})\|\bar{r}_E\|_{0,E}\le$$

$$h_K^{1/2}\|r_E\|_{0,E}\le C(\|y_h-y\|_{1,\omega_E}+\|u-u_h\|_{0,\omega_E})+Ch_K\|r_K\|_{0,\omega_E}+$$
$$+h_K^{1/2}\|\bar{r}_E-r_E\|_{0,E}.$$

$$\eta_4^2=\sum_{E\in E_h}h_K\|r_E\|_{0,E}^2\le C(\|y_h-y\|_1^2+\|u-u_h\|_0^2)+$$
$$+C\sum_{K\in T_h}h_K^2\|r_K\|_{0,K}^2+osc(r_E)\le$$
$$\le C\|y_h-y\|_1^2+\|u-u_h\|_0^2)+C\eta_3^2+osc(r_E),$$
$$\eta_4^2\le C(\|y_h-y\|_1^2+\|u_h-u\|_0^2)+\eta_3^2+osc(r_E).$$

2) In case of $E\cap\Gamma\ne\varnothing$

In this case, we set $r_E:=\dfrac{\partial y_h}{\partial n}+\varphi(y_h)-g_h$.

$$\|\bar{r}_E\|_{0,E}^2\sim(\bar{r}_E,b_E\bar{r}_E)_{0,E}=(r_E,b_E\bar{r}_E)_{0,E}+(\bar{r}_E-r_E,b_E\bar{r}_E)_{0,E}=$$
$$=(\Delta y_h,b_E\bar{r}_E)_{0,K}+(\nabla y_h,\nabla(b_E\bar{r}_E))_{0,K}+(\varphi(y_h),b_E\bar{r}_E)_{0,E}-$$
$$-(g_h,b_E\bar{r}_E)_{0,E}-(\nabla y,\nabla(b_E\bar{r}_E))_{0,K}-(\phi(y),b_E\bar{r}_E)_{0,K}-$$
$$-(\varphi(y),b_E\bar{r}_E)_{0,E}+(f+u,b_E\bar{r}_E)_{0,K}+(g,b_E\bar{r}_E)_{0,E}+I_E=$$

$$=(\nabla(y_h-y),\nabla(b_E\bar{r}_E))_{0,K}+(r_K,b_E\bar{r}_E)_{0,K}+$$
$$+(\phi(y_h)-\phi(y),b_E\bar{r}_E)_{0,K}+(\varphi(y_h)-\varphi(y),b_E\bar{r}_E)_{0,E}+$$
$$+(u-u_h,b_E\bar{r}_E)_{0,K}+(g-g_h,b_E\bar{r}_E)_{0,E}+I_E=$$
$$=I_1+I_2+I_3+I_4+I_5+I_6+I_E.$$

Let us estimate the above integral parts.

$$I_1\le Ch_K^{-1}\|y-y_h\|_1\|b_E\bar{r}_E\|_{0,K}\le Ch_K^{-1}\|y-y_h\|_1\|\bar{r}_E\|_{0,E},\qquad I_2\le Ch_K^{1/2}\|r_K\|_{0,K}\|\bar{r}_E\|_{0,E},$$
$$I_3\le Ch_K^{-1/2}\|y-y_h\|_1\|\bar{r}_E\|_{0,E},$$

$$I_4=(\varphi(y_h)-\varphi(y),b_E\bar{r}_E)_{0,E}\le C\|y-y_h\|_{0,E}\|b_E\bar{r}_E\|_{0,E}\le$$
$$\le Ch_K^{-1/2}\|y-y_h\|_{0,K}\|\bar{r}_E\|_{0,E}\le Ch_K^{-1/2}\|y-y_h\|_{1,K}\|\bar{r}_E\|_{0,E},$$

$$I_5=(u-u_h,b_E\bar{r}_E)_{0,K}\le C\|u-u_h\|_{0,K}\bar{r}_E\le$$
$$\le Ch_K^{-1/2}\|u-u_h\|_{0,K}\|\bar{r}_E\|_{0,E},$$

$$I_6=(g-g_h,b_E\bar{r}_E)_{0,E}\le\|g-g_h\|_{0,E}\|b_E\bar{r}_E\|_{0,E}\le\|g-g_h\|_{0,E}\|b_E\|_{0,E}|\bar{r}_E|$$
$$\le Ch_K^{-1/2}\|g-g_h\|_{0,E}\|\bar{r}_E\|_{0,E},$$

$$Ch_K^{1/2}\|\overline{r}_E\|_{0,E} \le C(\|y-y_h\|_{1,K}+\|u-u\|_{0,K}+\|g-g\|_{0,K})+Ch_K\|r_K\|_{0,K}+$$
$$+h_K^{1/2}\|\overline{r}_E-r_E\|_{0,E},$$
$$\eta_4^2=\sum_{E\in E_h}h_K\|\gamma_E\|_{0,K}^2 \le C(\|y-y_h\|_1^2+\|u-u_h\|_{0,K}^2+\|g-g_h\|_{0,E}^2)+$$
$$+\eta_3^2+osc(r_E),$$

We estimate η_5^2. For convenience, we accept the following notation

$$z_h := u_h+p_h,,\ \ \overline{z}_h:=\frac{1}{|K|}\int_K z_h dx,\ \ I_K:=(\overline{z}_h-\overline{z}_h, b_K\overline{z}_h)_{0,K}.$$

It is obvious that the inequality

$$\|z_h\|_{0,K}^2 \le \|\overline{z}_h\|_{0,K}^2+\|z_h-\overline{z}_h\|_{0,K}^2$$

holds. We give a similar consideration like before.

$$\|\overline{z}_h\|_{0,K}^2 \sim (\overline{z}_h, b_K\overline{z}_h)_{0,K}=(z_h, b_K\overline{z}_h)_{0,K}+(\overline{z}_h-\overline{z}_h, b_K\overline{z}_h)_{0,K}=$$
$$=(u_h-u, b_K\overline{z}_h)_{0,K}+(u, b_K\overline{z}_h)_{0,K}+(p_{2h}-p, b_K\overline{z}_h)_{0,K}+$$
$$+(p, b_K\overline{z}_h)_{0,K}+I_K=\sum_{i=1}^4 I_i+I_K,$$
$$I_1=(u_h-u, b_K\overline{z}_h)_{0,K} \le \|u_h-u\|_{0,K}\|b_K\overline{z}_h\| \le$$
$$\le \|u_h-u\|_{0,K}\|\overline{z}_h\|_{0,K}$$

$$I_2=(u, b_K\overline{z}_h)_{0,K} \le \|u\|_{0,K}\|b_K\|_{0,K}\overline{z}_h \le$$
$$\le Ch_K^{1/2}\|u\|_{0,K}\|\overline{z}_h\|_{0,K}$$

Here, we used the inequality $\|b_K\|_{0,K} \le Ch_K$.

$$I_3=(p_h-p, b_K\overline{z}_h)_{0,K} \le \|p_h-p\|_{0,K}\|b_K\overline{z}_h\|_{0,K} \le$$
$$\le \|p_h-p\|_{1,K}\|\overline{z}_h\|_{0,K}$$

$$I_4=(p, b_K\overline{z}_h)_{0,K} \le \|p\|_{0,K}\|b_K\|_{0,K}\overline{z}_h \le$$
$$\le Ch_K^{1/2}\|p\|_{0,K}\|\overline{z}_h\|_{0,K}$$
$$\|\overline{z}_h\|_{0,K} \le \|u_h-u\|_{0,K}+Ch_K^{1/2}\|u\|_{0,K}+\|p-p_h\|_{1,K}+$$
$$+Ch_K^{1/2}\|p\|_{0,K}+\|\overline{z}_h-z_h\|_{0,K}$$

$$\eta_5^2 \le \|z_h\|_0^2 \le C(\|u-u_h\|_0^2+\|p-p_h\|_1^2)+$$
$$+Ch(\|u\|_0^2+\|p\|_0^2)+\sum_{K\in T_h}\|z_h-\overline{z}_h\|_{0,K}^2=$$
$$=C(\|u-u_h\|_0^2+\|p-p_h\|_1^2)+Ch+osc(z_h),$$

We estimate η_6^2.

$$z_{1h}:=g_h+p_h,\ \ \overline{z}_{1h}:=\frac{1}{|E|}\int_E z_{1h}dl,\ I_E:=(\overline{z}_{1h}-z_{1h}, b_E\overline{z}_{1h})_{0,E}$$

$$\|z_{1h}\|_{0,E}^2 \le \|\overline{z}_{1h}\|_{0,E}^2+\|z_{1h}-\overline{z}_{1h}\|_{0,E}^2$$

$$\|\overline{z}_{1h}\|_{0,E}^2 \sim (\overline{z}_{1h}, b_E\overline{z}_{1h})_{0,E}=(z_{1h}, b_E\overline{z}_{1h})_{0,E}+(\overline{z}_{1h}-z_{1h}, b_E\overline{z}_{1h})_{0,E}=$$
$$=(g_h-g, b_E\overline{z}_{1h})_{0,E}+(g, b_E\overline{z}_{1h})_{0,E}+(p_h-p, b_E\overline{z}_{1h})_{0,E}$$
$$+(p, b_E\overline{z}_{1h})_{0,E}+I_E=\sum_{i=1}^4 I_i+I_E,$$

By estimating the above integrals, we get

$$I_1=(u_h-u, b_E\overline{z}_{1h})_{0,E} \le \|u_h-u\|_{0,E}\|b_E\overline{z}_{1h}\|_{0,E} \le$$

$$\le Ch_K^{-1/2}\|u_h-u\|_{0,K}\|b_E\|_{0,E}|\overline{z}_{1h}| \le Ch_K^{-1/2}\|u_h-u\|_{0,K}Ch_K h_K^{-1/2}\|\overline{z}_{1h}\|_{0,E}=$$
$$=\|u_h-u\|_{0,K}\|\overline{z}_{1h}\|_{0,E}$$

$$I_2=(u, b_K\overline{z}_{1h})_{0,E} \le \|u\|_{0,E}\|\overline{z}_{1h}b_K\|_{0,E} \le Ch_K^{1/2}\|u\|_{1,K}\|\overline{z}_{1h}\|_{0,E}$$

$$I_3=(p_h-p, b_E\overline{z}_h)_{0,E} \le \|p_h-p\|_{0,E}\|b_E\overline{z}_{1h}\|_{0,E} \le$$
$$\le Ch_K^{1/2}\|p_h-p\|_{1,K}\|b_E\|_{0,E}|\overline{z}_{1h}| \le C\|p_h-p\|_{1,K}\|\overline{z}_{1h}\|_{0,E}$$

$$I_4 = (p, b_K \bar{z}_{1h})_{0,E} \le \|p\|_{0,E} \|\bar{z}_{1h} b_E\|_{0,E} \le C h_K^{1/2} \|p\|_{1,K} \|\bar{z}_{1h}\|_{0,K},$$

$$\|\bar{z}_{1h}\|_{0,K} \le \|u_h - u\|_{0,K} + C h_K^{1/2} \|u\|_{1,K} + \|p - p_h\|_{1,K} + C h_K^{1/2} \|p\|_{1,K} + \\ + \|\bar{z}_{1h} - z_{1h}\|_{0,E}$$

$$\eta_6^2 \le \|z_{1h}\|_0^2 \le C(\|u - u_h\|_0^2 + \|p - p_h\|_1^2) + \\ + Ch(\|u\|_1^2 + \|p\|_1^2) + \sum_{K \in T_h} \|z_{1h} - \bar{z}_{1h}\|_{0,K}^2 = \\ = C(\|u - u_h\|_0^2 + \|p - p_h\|_1^2) + Ch + osc(z_{1h}),$$

By summing $\eta_1^2 - \eta_6^2$, we get the result of the theorem.

5. A Priori Error Estimates

For convenience, we set the cost functional as follows

$$\tilde{J}(u,g) := \frac{1}{2}\|y(u,g) - z_d\|_0^2 + \frac{1}{2}\|u\|_0^2 + \frac{1}{2}\|g\|_{0,\Gamma}^2,$$

$$\tilde{J}_h(u_h, g_h) := \frac{1}{2}\|y_h(u_h, g_h) - z_d\|_0^2 + \frac{1}{2}\|u_h\|_0^2 + \frac{1}{2}\|g_h\|_{0,\Gamma}^2,$$

Then we get the following. ([5])

$$(\tilde{J}'(u,g), v) = (u + p, v_1)_0 + (g + p, v_2)_{0,\Gamma},$$
$$(\tilde{J}_h'(u_h, g_h), v_h) = (u_h + p_h, v_{1h})_0 + (g_h + p_h, v_{2h})_{0,\Gamma},$$
$$(\tilde{J}'(u,g), v) = (u + p^h, v_1)_0 + (g + p^h, v_2)_{0,\Gamma}$$

where $p^h := p^h(u,g) \in V_h$ is the solution of the following equation.

$$(\nabla p^h, \nabla \upsilon_h)_0 + (\phi'(y^h) p^h, \upsilon_h)_0 + (\varphi'(y^h) p^h, \upsilon_h)_{0,\Gamma} = \\ = (y^h - z_d, \upsilon_h)_0, \quad \forall \upsilon_h \in V_h^0 \subset V,$$

$$(\nabla y^h, \nabla \upsilon_h)_0 + (\phi(y^h), \upsilon_h)_0 + (\varphi(y^h), \upsilon_h)_{0,\Gamma} = \\ = (f + u, \upsilon_h)_0 + (g, \upsilon_h)_{0,\Gamma} \quad \forall \upsilon_h \in V_h^0 \subset V,$$

Lemma 7 Let $(y,\ p; u, g),\ (y_h,\ p_h; u_h, g_h)$ be the solutions of (3) and (6), (7) respectively. Then it holds that

$$\frac{\nu}{2}(\|u - u_h\|_0 + \|g - g_h\|_{0,\Gamma}) \le \|p - p_h\|_1 + Ch.$$

where C is a constant independent of h.

(proof) From (3) and (6), (7) the relation which satisfy is as follows.

$$\begin{aligned} &(p + \nu u, \upsilon - u)_0 \ge 0, && \forall \upsilon \in U_1,\ u \in U_1 \\ &(p + \nu g, w - g)_{0,\Gamma} \ge 0, && \forall w \in U_2,\ g \in U_2 \end{aligned} \tag{15}$$

$$\begin{aligned} &(p_h + \nu u_h, \upsilon_h - u_h)_0 \ge 0, && \forall \upsilon_h \in U_{1h},\ u_h \in U_{1h} \\ &(p_h + \nu g_h, w_h - g_h)_{0,\Gamma} \ge 0, && \forall w_h \in U_{2h},\ g_h \in U_{2h} \end{aligned} \tag{16}$$

We choose σ as $0 \le \sigma \le 1$ and fix $\tilde{u}_1 \in U_1 \cap C^0(\Omega),\ \ \tilde{u}_2 \in U_2 \cap C^0(\Gamma)$.

Let $\Pi_h : V \to V_h$ be a standard interpolation operator.

By the smoothness of a control (lemma 3), $u \in C^0(\Omega)_+,\ g \in C^0(\Gamma)_+$ and we define $u^\sigma, u_h^\sigma;\ g^\sigma, g_h^\sigma$ as

$$\begin{aligned} &u^\sigma := (1-\sigma) u_h + \sigma \tilde{u}_1,\ \ u_h^\sigma := (1-\sigma)\Pi_h u + \sigma \Pi_h \tilde{u}, \\ &g^\sigma := (1-\sigma) g_h^n + \sigma \tilde{u}_2,\ \ g_h^\sigma := (1-\sigma)\Pi_h g^n + \sigma \Pi_h \tilde{u}_2 \end{aligned} \tag{17}$$

Then we can see that

$$u^\sigma \in C^0(\Omega)_+,\ \ u_h^\sigma \in U_{1h},\ \ g^\sigma \in C^0(\Gamma)_+,\ \ g_h^\sigma \in U_{2h}.$$

In (15) and (16), we set $\upsilon = u^{\sigma}$, $\upsilon_h = u_h^{\sigma}$, $w = g^{\sigma}$, $w_h = g_h^{\sigma}$ add together and put in order, then we get

$$\begin{aligned} &\nu(\|u-u_h\|_0^2+\|g-g_h\|_0^2) \le (\nu u, u^{\sigma}-u_h)_0 + (\nu u_h, u_h^{\sigma}-u)_0 + (p, u^{\sigma}-u_h)_0 + \\ &+(p_h, u_h^{\sigma}-u)_0 + (p-p_h, u_h-u)_0 + (\nu g, g^{\sigma}-g_h)_0 + \\ &+(\nu g_h, g_h^{\sigma}-g)_0 + (p, g^{\sigma}-g_h)_0 + (p_h, g_h^{\sigma}-g)_0 + (p-p_h, g_h-g)_0 \end{aligned} \tag{18}$$

From (18), we get inequality

$$\begin{aligned} (p-p_h, u_h-u)_0 &\le \|p-p_h\|_1^2 + \frac{\nu}{2}\|u_h-u\|_0^2, \\ (p-p_h, g_h-g)_0 &\le \|p-p_h\|_1^2 + \frac{\nu}{2}\|g_h-g\|_0^2, \end{aligned} \tag{19}$$

Setting $\exists h_0>0,\ \forall h<h_0,\ 0<h<1.\ \ \sigma = h$ and considering the boundedness (lemma 2) of finite element solutions independent of h, we get the estimation.

$$\begin{aligned} &(\nu u_h, u_h^{\sigma}-u)_0 = (\nu u_h, \Pi_h u - u)_0 + \sigma(\nu u_h, \Pi_h(\tilde{u}_1-u))_0 \le Ch, \\ &(p, u^{\sigma}-u_h)_0 = \sigma(p, \tilde{u}_1-u_h)_0 \le Ch, \\ &(p_h, u_h^{\sigma}-u)_0 = (\nu p_h, \Pi_h u-u)_0 + \sigma(\nu p_h, \Pi_h(\tilde{u}_1-u))_0 \le Ch, \end{aligned}$$

$$\begin{aligned} &(\nu g, g^{\sigma}-g_h)_0 = \sigma\nu(g, \tilde{u}_2-g_h)_0 \le \\ &\le \sigma\nu\|g\|_0(\|\tilde{u}_2\|_0+\|g_h\|_0) \le Ch \end{aligned}$$

$$(\nu g_h, g_h^{\sigma}-g)_0 = (\nu g_h, \Pi_h g-g)_0 + \sigma(\nu g_h, \Pi_h(\tilde{u}_2-g))_0 \le Ch,$$

$$\begin{aligned} &(p, g^{\sigma}-g_h^n)_0 = \sigma(p, \tilde{u}_2-g_h^n)_0 \le Ch, \\ &(p_h, g_h^{\sigma}-g)_0 = (\nu p_h, \Pi_h g-g)_0 + \\ &+\sigma(\nu p_h, \Pi_h(\tilde{u}_2-g))_0 \le Ch, \end{aligned} \tag{20}$$

From (18), (19) and (20), we get the result.

The following theorem about a priori error estimates shows the convergence of an approximation solution.

[Theorem 4] Let be the solutions of (3) and (6), (7) respectively. Then it holds that

$$\|u-u_h\|_0^2 + \|g-g_h\|_{0,\Gamma}^2 + \|p-p_h\|_1^2 + \|y-y_h\|_1^2 \le Ch.$$

We denote by C the different constants independent of h.

(proof)

We set $\xi := I_h p - p^h,\ \varsigma := p - I_h p,\ \xi+\varsigma = p-p^h$.

We first estimate $\|p-p^h\|_1$. Taking into account the assumption 1, we get

$$\begin{aligned} &C\|\xi\|_1^2 \le (\nabla\xi,\nabla\xi)_0 + (\phi'(y)\xi,\xi)_0 + (\varphi'(y)\xi,\xi)_{0,\Gamma} = \\ &= (\nabla(\pi_h p),\nabla\xi)_0 + (\phi'(y)\pi_h p,\xi)_0 + (\varphi'(y)\pi_h p,\xi)_{0,\Gamma} - \\ &-(\nabla p^h),\nabla\xi)_0 - (\phi'(y)p^h,\xi)_0 - (\varphi'(y)p^h,\xi)_{0,\Gamma} - (\nabla p,\nabla\xi)_0 + \\ &+(\nabla p,\nabla\xi)_0 - (\phi'(y)p,\xi)_0 + (\phi'(y)p,\xi)_0 - (\varphi'(y)p,\xi)_{0,\Gamma} + \\ &+(\varphi'(y)p,\xi)_{0,\Gamma} = \\ &= -(\nabla\varsigma,\nabla\xi)_0 - (\phi'(y)\varsigma,\xi)_0 - (\varphi'(y)\varsigma,\xi)_{0,\Gamma} + \\ &+(y-z_d,\xi)_0 - (y^h-z_d,\xi)_0 = -(\nabla\varsigma,\nabla\xi)_0 - \\ &-(\phi'(y)\varsigma,\xi)_0 - (\varphi'(y)\varsigma,\xi)_{0,\Gamma} + (y-y^h,\xi)_0 \end{aligned}$$

where $C := \min(1, \alpha)$, α is positive seen in assumption 1.

Considering the Lipschitz continuity of $\phi'(\cdot) \in L^{\infty}(\Omega)$, $\phi'(\cdot)$ and 1) of lemma 2, we get

$$\|p-p^h\|_1 \le \|\xi\|_1 + \|\varsigma\|_1 \le \tilde{C}h + \|y-y^h\|_1. \tag{21}$$

We estimate $\|y-y^h\|_1$.

This time we set

$$\xi := I_h y - y^h,\ \varsigma := y - I_h y,\ \xi+\varsigma = y-y^h.$$

Considering the assumption 1, strong monotonicity of, Lipschitz continuity and 1) of lemma 2, we get

$$\begin{aligned} &C\|\xi\|_1^2 \le (\nabla\xi,\nabla\xi)_0 + (\phi(\pi_h y)-\phi(y^h),\xi)_0 + (\varphi(\pi_h y) - \\ &-\varphi(y^h),\xi)_{0,\Gamma} \le (\nabla(\pi_h y),\nabla\xi)_0 + (\phi(\pi_h y),\xi)_0 + (\phi(\pi_h y),\xi)_0 - \\ &-(\nabla y^h,\nabla\xi)_0 - (\phi(y^h),\xi)_0 - (\phi(y^h),\xi)_{0,\Gamma} - (\nabla y,\nabla\xi)_0 + \\ &+(\nabla y,\nabla\xi)_0 - (\phi(y),\xi)_0 + (\phi(y),\xi)_0 - (\varphi(y),\xi)_{0,\Gamma} + \\ &+(\varphi(y),\xi)_{0,\Gamma} = \end{aligned}$$

$$\begin{aligned} &= -(\nabla\varsigma,\nabla\xi)_0 + (\phi(\pi_h y)-\phi(y),\xi)_0 + (f+u,\xi)_0 + (g,\xi)_{0,\Gamma} \\ &-(f+u,\xi)_0 + (g,\xi)_{0,\Gamma} \le \end{aligned}$$

$$\le \|\xi\|_1\|\varsigma\|_1 + C\|\xi\|_0\|\varsigma\|_0 \le \tilde{C}\|\xi\|_1\|\varsigma\|_1$$

$$\|\xi\|_1 \le C\|\varsigma\|_1 \le Ch\|y\|_2 \le \tilde{C}h,$$

$$\left\|y-y^h\right\|_1 \le \|\xi\|_1 + \|\varsigma\|_1 \le \tilde{C}h. \tag{22}$$

On the other hand, subtracting the equations which y^h, y_h satisfy, it holds that

$$(\nabla(y^h-y_h),\nabla\upsilon_h)_0+(\phi(y^h)-\phi(y_h),\upsilon_h)_0+\varphi(y^h)- \\ -\varphi(y_h),\upsilon_h)_{0,\Gamma}=(u-u_h,\upsilon_h)_0+(g-g_h,\upsilon_h)_{0,\Gamma}, \\ \forall\upsilon_h\in V_h$$

and when we set $\upsilon_h := y^h - y_h \in V_h$ and consider the strong monotonicity of $\phi(\cdot)$ and monotonicity of $\varphi(\cdot)$, then we get

$$\begin{aligned} &C\left\|y^h-y_h\right\|_1^2 \le \left\|\nabla(y^h-y_h)\right\|_0^2 \le \|u-u_h\|_0\left\|y^h-y_h\right\|_0+\|g-g_h\|_{0,\Gamma}\left\|y^h-y_h\right\|_{0,\Gamma} \\ &\left\|y^h-y_h\right\|_1 \le C(\|u-u_h\|_0+\|g-g_h\|_{0,\Gamma}). \end{aligned} \tag{23}$$

Similarly, subtracting the equations which p^h, p_h satisfy, we get

$$\begin{aligned} &(\nabla(p^h-p_h),\nabla\upsilon_h)_0+(\phi'(y^h)p^h-\phi'(y_h)p_h,\upsilon_h)_0+(\varphi'(y^h)p^h-\varphi'(y_h)p_h,\upsilon_h)_{0,\Gamma} \\ &\quad=(y^h-y_h,\upsilon_h)_0,\ \forall\upsilon_h\in V_h^0 \end{aligned}$$

Now when we consider

$$\begin{aligned} \phi'(y^h)p^h-\phi'(y_h)p_h &= \phi'(y^h)(p^h-p_h)+(\phi'(y^h)-\phi'(y_h))p_h \\ \varphi'(y^h)p^h-\varphi'(y_h)p_h &= \varphi'(y^h)(p^h-p_h)+(\varphi'(y^h)-\varphi'(y_h))p_h \end{aligned}$$

and Lipschitz continuity of $\phi'(\cdot)\ge\alpha>0$, $\varphi'(\cdot)\ge 0$, $\phi'(\cdot)$, $\varphi'(\cdot)$ and set a test function as $\upsilon_h := p^h - p_h \in V_h$, we get the following expression

$$\begin{aligned} &C\left\|p^h-p_h\right\|_1^2 \le (\phi'(y_h)-\phi'(y^h))p_h,p^h-p_h)_0+(\varphi'(y_h)-\varphi'(y^h))p_h,p^h-p_h)_{0,\Gamma}+ \\ &\le(y^h-y_h,p^h-p_h)_0 \end{aligned}$$

where $C := \min(1, \alpha)$, α is positive seen in assumption 1.

$$\begin{aligned} &C\left\|p^h-p_h\right\|_1^2 \le C\left\|y^h-y_h\right\|_0\|p_h\|_\infty\left\|p^h-p_h\right\|_0+C\left\|y^h-y_h\right\|_{0,\Gamma}\|p_h\|_{\infty,\Gamma}\left\|p^h-p_h\right\|_{0,\Gamma}+ \\ &+\left\|y^h-y_h\right\|_0\left\|p^h-p_h\right\|_0 \le C\left\|y^h-y_h\right\|_1\left\|p^h-p_h\right\|_1 \end{aligned}$$

$$\left\|p^h-p_h\right\|_1 \le C\left\|y^h-y_h\right\|_1 \le C(\|u-u_h\|_0+\|g-g_h\|_{0,\Gamma}), \tag{24}$$

On the other hand, it holds that

$$\begin{aligned} \|y-y_h\|_1 &\le \left\|y-y^h\right\|_1+\left\|y^h-y_h\right\|_1, \\ \|p-p_h\|_1 &\le \left\|p-p^h\right\|_1+\left\|p^h-p_h\right\|_1, \end{aligned} \tag{25}$$

From lemma 7 and (21)-(25), using the above results, it holds that

$$\frac{\nu}{2}(\|u-u_h\|_0+\|g-g_h\|_{0,\Gamma}) \le \|p-p_h\|_1^2 + Ch \le$$

$$\le \tilde{C}(\|u-u_h\|_0 + \|g-g_h\|_{0,\Gamma}) + Ch$$

$$(\frac{\nu}{2}-\tilde{C})(\|u-u_h\|_0 + \|g-g_h\|_{0,\Gamma}) \le Ch$$

We choose ν to satisfy $\nu > 2C$. Then it holds that

$$\|u-u_h\|_0 + \|g-g_h\|_{0,\Gamma} \le C(\nu)h \tag{26}$$

and considering (21)-(26), we get the result of the theorem. Here, $C(\nu)$ is a constant independent of h.

6. Convergence of a Posteriori Error Indicator

We see the following lemma before the convergence of a posteriori error indicator $\eta_i^2 (i=1,\cdots,6)$.

Lemma 8 For $osc(T_h)$, it holds that

$$osc(T_h) \le Ch$$

(In the proof, we denote by C all the constants independent of h.)

(proof) Step *1*: Let us see

$\sum_{K\in T_h} \|r_K\|_{0,K}^2 + \sum_{E\in E_h} \|r_E\|_{0,E}^2 \le C$ (C is independent of h).

Here, r_K, r_E are defined in lemma 5 to be

$$r_K := \Delta y_h - \phi(y_h) + f + u_h,$$

$$r_E := \begin{cases} \left[\frac{\partial y_h}{\partial n}\right]_E = (\nabla y_h - \nabla \overline{y}_h)\cdot n_K, & E\cap\Gamma = \varnothing \\ \frac{\partial y_h}{\partial n}|_E + \varphi(y_h) - g_h, & E\cap\Gamma \ne \varnothing \end{cases}$$

From lemma 3 and theorem 4, we get the following result about the boundedness, i.e.,

$$\begin{aligned} &\|u_h\|_0 = \|u_h\|_{L^2(\Omega)} \le C,\ \|g_h\|_0 = \|g_h\|_{L^2(\Gamma)} \le C \\ &\|y_h\|_1 \le C,\ \|p_h\|_1 \le C \end{aligned} \tag{27}$$

where C is a constant independent of h.

Thus, $y_h \in L^p(\Omega)\,(1<p<\infty)$ and by assumption 1, $\|\phi(y_h)\|_0 \le C,\ \|\varphi(y_h)\|_{0,\Gamma} \le C.$

From the variational equation satisfied by $y_h \in V_h$

$$\begin{aligned} &(\nabla y_h, \nabla \upsilon_h)_0 + (\phi(y_h), \upsilon_h)_0 + (\varphi(y_h), \upsilon_h =)_{0,\Gamma} \\ &= (f+u_h, \nabla\upsilon_h)_0 + (g_h, \upsilon_h)_{0,\Gamma},\ \forall \upsilon_h \in V_h \end{aligned}$$

we get

$$\begin{aligned} &\sum_{K\in T_h} (\Delta y_h, \nabla \upsilon_h)_{0,K} + \sum_{E\in E_h} (-r_E, \upsilon_h)_{0,E} \le \|\phi(y_h)\|_0 \|\upsilon_h\|_0 + \\ &+ \|f+u_h\|_0 \|\upsilon_h\|_0 \le \\ &\le (\|\phi(y_h)\|_0 + \|f+u_h\|_0)\|\upsilon_h\|_0 \le \tilde{C}\|\upsilon_h\|_0. \end{aligned}$$

We introduce the norm in V_h as

$$\|\upsilon_h\|_h := (\sum_{K\in T_h} \|\upsilon_h\|_{0,K}^2 + \sum_{E\in E_h} \|\upsilon_h\|_{0,E}^2)^{1/2}$$

Since finite dimensional norms are equivalent, we get. $C_1\|\upsilon_h\|_h \le \|\upsilon_h\|_0 \le C_2\|\upsilon_h\|_h$ In V_h, we think of a functional

$$F(y_h)(\upsilon_h) := \sum_{K\in T_h} (\Delta y_h, \upsilon_h)_{0,K} + \sum_{E\in E_h} (-r_E, \upsilon_h)_{0,E}.$$

Since $|F(y_h)(\upsilon_h)| \le C\|\upsilon_h\|_0 \le \tilde{C}\|\upsilon_h\|_h$, $F(y_h)$ is a bounded linear functional in V_h and by Riesz' representation theorem, there exists a unique element $a_h \in V_h$, $F(y_h)(\upsilon_h) = (a_h, \upsilon_h)_h$ such that $\|F(y_h)\| = \|a_h\|_h$.

And it holds that

$$\|a_h\|_h = (\sum_{K\in T_h} \|\Delta y_h\|_{0,K}^2 + \sum_{E\in E_h} \|r_E\|_{0,E}^2)^{1/2} \le \tilde{C} \tag{28}$$

Since $\|\phi(y_h)\|_0 \le C$ (C is independent of h), we get

$$\begin{aligned} &\sum_{K\in T_h} \|r_K\|_{0,K}^2 = \sum_{K\in T_h} \|\Delta y_h - \phi(y_h) + f + u_h\|_{0,K}^2 \le 2\sum_{K\in T_h} \|\Delta y_h\|_{0,K}^2 + \\ &+ 2\|\phi(y_h) + f + u_h\|_0^2 \le C \end{aligned}$$

$$osc(r_K) := \sum_{K\in T_h} h_K^2 \|r_K - \bar r_K\|_{0,K}^2 \le 2(\sum_{K\in T_h} h_K^2 \|r_K\|_{0,K}^2 + $$
$$+ \sum_{K\in T_h} h_K^2 \|\bar r_K\|_{0,K}^2) \le 2h^2(\sum_{K\in T_h} \|r_K\|_{0,K}^2 + \sum_{K\in T_h} \|\bar r_K\|_{0,K}^2) \le$$
$$\le Ch^2 + \sum_{K\in T_h} \|\bar r_K\|_{0,K}^2$$

Since

$$\|\bar r_K\|_{0,K}^2 = \int_K \bar r_K{}^2 dx = \bar r_K{}^2 \cdot |K| =\le \frac{1}{|K|}(\int_K r_K dx)^2$$
$$\le \frac{1}{|K|}(\int_K 1^2 dx)(\int_K r_K^2 dx) \le \|r_K\|_{0,K}^2$$

it holds that

$$osc(r_K) := \sum_{K\in T_h} h_K^2 \|r_K - \bar r_K\|_{0,K}^2 \le 3h^2(\sum_{K\in T_h} \|r_K\|_{0,K}^2) \le Ch^2$$

Considering (28), we get

$$osc(r_E) := \sum_{E\in E_h} h_K \|r_E - \bar r_E\|_{0,E}^2 \le Ch$$

in a similar way and in turn, $osc(r_K) + osc(r_E) \le Ch$.

Step 2: Let us see $\sum_{K\in T_h} \|\gamma_K\|_{0,K}^2 + \sum_{E\in E_h} \|\gamma_E\|_{0,E}^2 \le C$ (C is a constant independent of h). Here, γ_K, γ_E are defined in lemma 5 to be as follows.

$$\gamma_K := y_h - z_d + \Delta p_h - \phi'(y_h) p_h ,$$

$$\gamma_E := \begin{cases} \left[\dfrac{\partial p_h}{\partial n}\right]_E = (\nabla p_h - \nabla \bar p_h)\cdot n_K , & E\cap\Gamma = \varnothing \\ \dfrac{\partial p_h}{\partial n}\Big|_E + \varphi'(y_h) p_h , & E\cap\Gamma \ne \varnothing \end{cases}$$

From (27), it holds that

$$\phi'(y_h) \in L^\infty(\Omega),\ \ \phi'(y_h) p_h \in L^2(\Omega)\ \ \|\phi'(y_h) p_h\|_0 \le C$$

From an adjoint equation

$$(\nabla p_h, \nabla \upsilon_h)_0 + (\phi'(y_h) p_h, \upsilon_h)_0 + (\varphi'(y_h) p_h, \upsilon_h)_{0,\Gamma} =$$
$$= (y_h - z_d, \upsilon_h)_0, \qquad \forall \upsilon_h \in V_h \subset V$$

it holds that

$$(\sum_{K\in T_h} (\Delta p_h, \nabla\upsilon_h)_{0,K} + \sum_{E\in E_h} (-\gamma_E, \nabla\upsilon_h)_{0,E} \le$$
$$\le (\|\phi'(y_h) p_h\|_0 + \|y_h - z_d\|_0)\|\upsilon_h\|_0 \le$$

$\le C\|\upsilon_h\|_0 \le \tilde C\|\upsilon_h\|_h$, (C, $\tilde C$ are independent of h)

In V_h^0 , introducing a functional

$$F(p_h)(\upsilon_h) := \sum_{K\in T_h} (\Delta p_h, \upsilon_h)_{0,K} + \sum_{E\in E_h} (-\gamma_E, \upsilon_h)_{0,E}$$

Then it holds that

$$|F(p_h)(\upsilon_h)| \le (\|\phi'(y_h) p_h\|_0 + \|y_h - z_d\|_0)\|\upsilon_h\|_0$$
$$\le C \le \|\upsilon_h\|_0 \le \tilde C\|\upsilon_h\|_h$$

And, similar to step 1, there is a unique element $\tilde a_h \in V_h^0$ such that $\|F(p_h)\| = (\tilde a_h, \upsilon_h)_h$ and $\|F(p_h)\| = \|\tilde a_h\|_h$. Therefore, it holds that

$$\|\tilde a_h\|_h = (\sum_{K\in T_h} \|\Delta p_h\|_{0,K}^2 + \sum_{E\in E_h} \|\gamma_E\|_{0,E}^2)^{1/2} \le \tilde C ,$$

$$\sum_{K\in T_h} \|\gamma_K\|_{0,K}^2 = \sum_{K\in T_h} \|\Delta p_h - \phi'(y_h) p_h\|_{0,K}^2 \le 2\sum_{K\in T_h} \|\Delta p_h\|_{0,K}^2 +$$
$$+ 2\|\phi'(y_h) p_h\|_0^2 \le C$$

and the following result follows.

$$osc(\gamma_K) + osc(\gamma_E) \le C(h^2 + h\ \ ) \le \tilde C h\ ,$$

Step 3: Let us see

$$osc(z_h) := \sum_{K\in T_h} \|z_h - \bar z_h\|_{0,K}^2 \le Ch^2 ,$$
$$osc(z_{1h}) := \sum_{K\in T_h} \|z_{1h} - \bar z_{1h}\|_{0,K}^2 \le Ch^2$$

Using lemma 3.4 in [9] (or theorem 3.1 in [8]), it holds that

$$\sum_{K\in T_h} \|z_h - \bar z_h\|_{0,K}^2 \le \sum_{K\in T_h} Ch_K^2 \|\nabla z_h\|_{0,K}^2 \le$$
$$\le Ch^2 \sum_{K\in T_h} \|\nabla(u_h + p_h)\|_{0,K}^2 \le Ch^2(\|u_h\|_1^2 + \|p_h\|_1^2) \tag{29}$$

By lemma 1 and 3, we get

$$u_h = \frac{1}{\nu}\max\{-p_h, 0\},\ \ \Omega$$

and since $p_h \in H^1(\Omega),\ \max(p_h, 0) \in H^1(\Omega)$, we get.

$$\frac{\partial u_h}{\partial x_i} = \begin{cases} -\frac{1}{\nu}\frac{\partial p_h}{\partial x_i}, & p_h < 0 \\ 0, & p_h \ge 0 \end{cases} \quad (i = 1,\ 2)$$

where a partial derivative is a weak derivative, i.e.,

$$\forall \varsigma \in C_0^\infty(\Omega),\ \int_\Omega \frac{1}{\nu}\max(-p_h, 0)\cdot\frac{\partial \varsigma}{\partial x_i}dx = \int_{\Omega\cap\{p_h<0\}} \frac{1}{\nu}\frac{\partial p_h}{\partial x_i}\cdot\varsigma dx,\ (i = 1,\ 2)$$

Considering the above and (27), we get

$$\|u_h\|_1^2 = \left\|u_h^2 + |\nabla u_h|^2\right\|_0^2 = \frac{1}{\nu}\int_{\Omega\cap\{p_h<0\}} (p_h^2 + |\nabla p_h|^2)dx \le \frac{1}{\nu}\|p_h\|_1^2 \le C\ .$$

Considering (27) and (29), we get

$$osc(z_h) = \sum_{K\in T_h} \|z_h - \bar{z}_h\|_{0,K}^2 \le Ch^2(\|u_h\|_1^2 + \|p_h\|_1^2) \le \tilde{C}h^2\|p_h\|_1^2 \le \bar{C}h^2\ .$$

In a similar way, we get

$$osc(z_{1h}) = \sum_{E\in E_h\cap\Gamma\ne\varnothing} \|z_{1h} - \bar{z}_{1h}\|_{0,E}^2 \le Ch^2(\|g_h\|_{1,\Gamma}^2 + \|p_h\|_{1,\Gamma}^2) \le \tilde{C}h^2\|p_h\|_{1,\Gamma}^2 \le \bar{C}h^2$$

and thus we get the following result.

$$\begin{aligned} &osc(T_h) := osc(\gamma_K) + osc(\gamma_E) + osc(r_K) + osc(r_E) + \\ &+ osc(z_h) + osc(z_{1h}) \le Ch \end{aligned}$$

Now we give the theorem on the convergence of a posteriori error indicator which is one of the main results in the paper.

[Theorem 5] Let $(y, p; u, g)$, $(y_h, p_h; u_h, g_h)$ be the solutions of (5) and (6), (7) respectively. When $h \to 0$, the following convergence result about a posteriori error indicator holds.

$$\sum_{i=1}^{6} \eta_i^2 \to 0.$$

(proof) From lemma 3, all the constants in a posteriori error and a priori error estimates are bounded independent of h. Using theorem 4 and lemma 8, the result follows from theorem 3.

7. Conclusion

We obtained the optimality condition about a mixed control problem (two variable control) of a semi-linear elliptic equation with a control in the right side of the boundary condition and in the source term, and based on this, got a posteriori error estimates of finite element approximation solution in an adaptive finite element function space and verified the convergence of an error indicator to 0 when the meshsize goes to 0.

Abbreviations

FEM Finite Element Method

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Conflicts of Interest

The authors declare no conflicts of interest.

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