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Marginalized Maximum Likelihood Estimation Method for the Three-parameter Lognormal Distribution

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Abstract: This paper proposes an adjustment method to overcome the difficulties encountered by the maximum likelihood method in the case of the three-parameter lognormal distribution. Endeed, when the threshold parameter is close to the smallest order statistic, the standard likelihood function is no longer bounded. In this case, maximum likelihood estimators are no longer accessible. Our strategy is twofold: first, we construct adaptive bounds intended to contain the location and shape parameters with a probability close to one as the sample size increases. Second, we construct a marginal likelihood function, which we maximize using an optimization method available in the R software through the "nlnimb" package. This likelihood function is based on the (n-1) largest order statistics. Finally, Monte Carlo simulation studies are used to analyze the asymptotic behavior of the constructed intervals and to study the asymptotic properties of the proposed estimators through bias and the Root Mean-Squared Error(RMSE).

Keywords: Order statistics, Maximum Likelihood Estimation, Left Censored Samples, Three Parameter Lognormal Distribution

1. Introduction

Maximum likelihood estimation is a widely used method for estimating parameters in statistical models. In this context, we consider the three-parameter lognormal distribution, which is commonly used to model skewed data in various fields such as finance and engineering [10, 13]. However, in some situations, the data may be left-censored, meaning that values below a certain threshold are not observed. In such cases, the maximum likelihood estimation approach needs to be modified to account for the censored data. We will then explore the method of maximum likelihood estimation for left censored samples of the three parameter lognormal distribution, and how it can be applied to real-world problems. A random variable X has the three-parameter lognormal distribution if it cumulative distribution function (cdf) is indexed by a vector of three real parameters $\theta = (\gamma, \mu, \sigma)$and is defined as follows:

$$F(x, \theta) = \varPhi\left(\frac{\ln(x-\gamma)-\mu}{\sigma}\right), \sigma \succ 0, -\infty \prec \mu \prec +\infty, \gamma \prec x$$

where $\varPhi$ is the cdf of the standard normal distribution. The

corresponding probability density function (pdf) is

$$f(x, \theta) = \frac{1}{(x-\gamma)\,\sigma\sqrt{2\pi}} \exp\left\{-\frac{[\ln(x-\gamma)-\mu]^2}{2\sigma^2}\right\}, x \succ \gamma$$

There is a close relationship between normal and lognormal distributions. If $Y = \ln(x-\gamma)$is normally distributed with mean μ and standard deviation σ, then the distribution of X becomes a three-parameter lognormal distribution with parameter $\theta = (\gamma, \mu, \sigma)$. The parameter γ denotes the threshold value and is called the location parameter, while μ and $\sigma$are scale and shape parameters respectively. Samples from distributions of this type have previously been studied by various writers.Cohen[3] has dealt with maximum likelihood estimation in the three parameter lognormal distribution when samples are complete. Hill [4] considered a Bayesian point of view for estimation and explored some natural posterior distributions.Munro and Wixley [1] gave estimators based on order statistics of small samples from a three parameter lognormal distribution.When the threshold parameter is unknown in the three-parameter lognormal distribution, the regularity conditions required for maximum likelihood estimation are not satisfied. This is because the likelihood function is unbounded above and below in the parameter space, which causes the likelihood function to be non-regular. Severall authors have investigate to coutourn this difficulty [2, 5, 6, 11, 12]. Ouédraogo and co. [7] proposed a maximum likelihood method for fitting a three-parameter gamma distribution using an Minorize-Maximization(MM) algorithm on the $n-1$ largest order statistics. Monte Carlo simulations were conducted to assess the bias and root mean square error of the estimator. In this study, a new estimation method based on the likelihood of the $n-1$ largest values of a sample of size n from a three-parameter lognormal distribution is developed. The rest of the paper is organized as follows. The next section presents the method. Results of a Monte Carlo study conducted to evaluate the performance of the proposed bounds and estimators for θ are presented in the third section. The last section is devoted to a concluding remarks.

2. The Method: A Comprehensive Guide

2.1. Marginalized Likelihood Method for the Three-parameter Lognormal Distribution

Precedently, simular method have been considered by Katchekpele and co. [7] for the three-parameter Gamma and the three-parameter Weibull model. Its method is formulate as follow: consider independent and identically distributed (i.i.d) sequence $(X_i)\, i = 1 : n$ and let $f(x, \theta)$ be the underlying marginal density. Notes $(X_{i:n})\, i = 1 : n$ the associated sequence of order statistics. It is well known that the joint density of $(X_{i:n})\, i = 1 : n$ is

$$f_{(X_{i:n})_{i=1:n}}(x, \theta) = n! \prod_{i=1}^{n} f(x_i, \theta)\, 1_{\{\gamma \prec x_{1:n} \prec x_{2:n} \prec \cdots \prec x_{n:n}\}}$$

where $x = (x_1, x_2, \cdots, x_n)$. The joint density of the sequence $(X_{i:n})\, i = 2 : n$ is therefore

$$f_{(X_{i:n})_{i=1:n}}(x, \theta) = n! F(x_{2:n}, \theta) \prod_{i=2}^{n} f(x_i, \theta)\, 1_{\{\gamma \prec x_{1:n} \prec x_{2:n} \prec \cdots \prec x_{n:n}\}}$$

Definition 2.1. Let $(x_i)_{i=1:n}$ denote a sample of size n from a three-parameter Lognormal distribution with unknown parameters vector $\theta = (\gamma, \mu, \sigma)$ and $(x_{i:n})_{i=1:n}$ is the sequence of the non-decreasing sorted values. An marginalized log-likelihood function based on the $n-1$ largest order statistics is given as follows:

$$L(x, \theta) = n! F(x_{2:n}, \theta) \prod_{i=2}^{n} f(x_{i:n}, \theta)$$

Taking natural logarithms of each side leads to the following marginal order statistic based loglikelihood function $l(x_{2:n} \cdots x_{n:n}, \theta)$

$$\begin{aligned} l(x_{2:n} \cdots x_{n:n}, \theta) &= \ln(n!) - (n-1)\ln\left(\sqrt{2\pi}\right) - (n-1)\ln\sigma \\ &- \sum_{i=2}^{n} \ln(x_{i:n} - \gamma) + \ln(F(x_{2:n}, \theta)) - \frac{1}{2}\sum_{i=2}^{n}\left[\frac{\ln(x_{i:n} - \gamma) - \mu}{\sigma}\right]^2 \end{aligned}$$

2.2. Threshold and Shape Parameters Ranges

Proposition 2.1. Let $(X_i)_{i=1:n}$ be independentes random variables distributed according to the three parameter lognormal distribution with vector parameter $\theta = (\gamma, \mu, \sigma)$. Let's denote $X_{1:n} = \min(X_1, X_2, ..., X_n)$and $X_{n:n} = \max(X_1, X_2, ..., X_n)$. The following results for the threshold parameter γ hold.

$$\begin{aligned} P(3X_{1:n} - 2X_{n:n} \leq \gamma \leq 3X_{n:n} - 2X_{1:n}) = \\ n\int_0^{\infty}\left[G\left(\frac{3u}{2}\right) - G(u)\right]^{n-1} g(u)\,du \\ -n\int_0^{\infty}\left[G\left(\frac{2u}{3}\right) - G(u)\right]^{n-1} g(u)\,du, \end{aligned}$$

where $g(.)$ and $G(.)$ denote probability and cumulative density functions of the two parameters lognormal distribution with shape parametre σ and scale parameter μ.

Proof: Probability in above equation can be decomposed as

$$\begin{aligned} P(3X_{1:n} - 2X_{n:n} \leq \gamma \leq 3X_{n:n} - 2X_{1:n}) &= \\ P(\gamma \leq 3X_{n:n} - 2X_{1:n}) - P(\gamma \leq 3X_{1:n} - 2X_{n:n}) \end{aligned}$$

Now, to evaluate probability $P(\gamma \leq 3X_{n:n} - 2X_{1:n})$ one can remark that $3X_{1:n} - 2X_{n:n} \leq \gamma$ is equivalente to $3(X_{1:n} - \gamma) \leq 2(X_{n:n} - \gamma)$. Let's denote $U = X_{1:n} - \gamma$ and $V = X_{n:n} - \gamma$. Then $P(\gamma \leq 3X_{n:n} - 2X_{1:n}) = P(3U \leq 2V)$. Note that the joint distribution of u, v is given as

$$h(u,v)=\frac{n!}{(n-2)!}\left[G(v)-G(u)\right]^{n-2}g(u)\,g(v),$$
$$-\infty\prec u\prec v\prec+\infty$$

where $g(.)$ and $G(.)$ denote probability and cumulative density functions of $X_i-\gamma$ respectively. Note that $X_i-\gamma$ is distributed according to two-parameters lognormal distribution with parameters μ and σ. Then,

$$\begin{aligned}
&P(3U\leq 2V)=P\left(\frac{3U}{2}\leq V\right)\\
&=\int_0^{\infty}\int_{\frac{3u}{2}}^{\infty}n(n-1)\left[G(v)-G(u)\right]^{n-2}g(u)\,g(v)\,dvdu\\
&=n\int_0^{\infty}\left[(G(v)-G(u))^{n-1}\right]_{\frac{3u}{2}}^{\infty}g(u)\,du\\
&=-\left[(1-G(u))^n\right]_0^{\infty}-n\int_0^{\infty}\left(G\left(\frac{3u}{2}\right)-G(u)\right)^{n-1}g(u)\,du\\
&=1-n\int_0^{\infty}\left[G\left(\frac{3u}{2}\right)-G(u)\right]^{n-1}g(u)\,du
\end{aligned}$$

In simular case one can prove that

$$\begin{aligned}
&P(\gamma\leq 3X_{1:n}-2X_{n:n})=P\left(\frac{2U}{3}\leq V\right)\\
&=\int_0^{\infty}\int_{\frac{2u}{3}}^{\infty}n(n-1)\left[G(v)-G(u)\right]^{n-2}g(u)\,g(v)\,dvdu\\
&=n\int_0^{\infty}\left[(G(v)-G(u))^{n-1}\right]_{\frac{2u}{3}}^{\infty}g(u)\,du\\
&=-\left[(1-G(u))^n\right]_0^{\infty}-n\int_0^{\infty}\left(G\left(\frac{2u}{3}\right)-G(u)\right)^{n-1}g(u)\,du\\
&=1-n\int_0^{\infty}\left[G\left(\frac{2u}{3}\right)-G(u)\right]^{n-1}g(u)\,du
\end{aligned}$$

Proposition 2.2. Let $(X_i)_{i=1:n}$ be sequence of n independentes random variables from the three parameter lognormal distribution with parameter vector $\theta=(\gamma,\mu,\sigma)$. Using the same notations as those of the proposition2.1 then

$$P(0\prec\sigma\leq 3X_{n:n}-3X_{1:n})$$
$$=1-n\int_0^{\infty}\left[G\left(x+\frac{1}{3}\right)-G(x)\right]^{n-1}g(x)\,dx$$

where $G(.)$ and $g(.)$ denote the cummulative and probability density function respectively of the two-parameter lognormal with shape parameter 1 and scale parameter μ.

Proof: Since σ denote the scale parameter of lognormal distribution it's stricly positivity is always true. $0\prec\sigma\leq 3X_{n:n}-3X_{1:n}$ is then equivalent to $\frac{1}{3}\prec U_{n:n}-U_{1:n}$ where $U_i=\frac{X_i-\gamma}{\sigma}$ and the subscript $n:n$ and $1:n$ denote the largest and smallest order statistic associated to the random sequence $(U_i)_{i=1:n}$. It follows that

$$P(0\prec\sigma\leq 3X_{n:n}-3X_{1:n})=P\left(\frac{1}{3}\prec U_{n:n}-U_{1:n}\right)$$

Let's denote by $W_n=U_{n:n}-U_{1:n}$ the sample range associated to $(U_i)_{i=1:n}$. Then,

$$P\left(\frac{1}{3}\prec U_{n:n}-U_{1:n}\right)=1-P\left(W_n\leq\frac{1}{3}\right)\qquad(1)$$

Since $(X_i)_{i=1:n}$ is derived from three-parameter lognormal distribution with vector parameter θ, it follows that

$$P\left(W_n\leq\frac{1}{3}\right)=n\int_0^{\infty}\left[G\left(x+\frac{1}{3}\right)-G(x)\right]^{n-1}g(x)\,dx$$

where $G(.)$ and $g(.)$ denote the cummulative and probability density function respectively of the two-parameter lognormal with shape parameter 1 and scale parameter μ.

3. Simulation Studies

3.1. Probability of the Proposed Ranges

In this subsection, we investigate the behavior of the proposed parameters range through a Monte Carlo simulation study. For doing this let's consider the following theoretical values of the threshold parameter γ: $0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0$ while the scale and shape parameters are fixed as $\mu=0$ and $\sigma=1$ respectively. The sample size is taken to be $10, 25, 50, 75$ and 100. Foreach configuration, the probability of each range were compute as given in Proposition 2.1 and Proposition 2.2. Note that integral in Propositions above can be obtained through Monte Carlo integration as detailled in [9]. So, generating sample of size B, say $x_1, x_2, ..., x_B$ from the underlying distribution with density function $g(.)$, the integral can be approximate as follow:

$$\int_0^{\infty}f(x)\,g(x)\,dx=\frac{1}{B}\sum_{i=1}^{B}f(x_i)$$

In the sequel, setting B equal 5000 ensures convergence. All computations were running with R computing environment (R Core Team[8]).

Table 1. *Probability table of Lognormal threshold parameter range.*

γ	n				
	10	**25**	**50**	**75**	**100**
0.5	1.000	1.000	1.000	1.000	1.000
1.0	0.9999	1.000	1.000	1.000	1.000
1.5	0.9999	1.000	1.000	1.000	1.000
2.0	0.9999	1.000	1.000	1.000	1.000
2.5	0.9997	0.9978	1.000	1.000	1.000
3.0	0.9988	0.9988	1.000	1.000	1.000
3.5	0.9964	0.9999	1.000	1.000	1.000
4.0	0.9907	0.9999	0.9999	1.000	1.000
4.5	0.9795	0.9999	0.9999	1.000	1.000
5.0	0.9605	0.9999	1.000	0.9999	1.000

Table 2. *Probability table of Lognormal shape parameter range.*

γ	n				
	10	**25**	**50**	**75**	**100**
0.5	0.9997	1.000	1.000	1.000	1.000
1.0	0.9999	1.000	1.000	1.000	1.000
1.5	0.9999	1.000	1.000	1.000	1.000
2.0	0.9999	1.000	1.000	1.000	1.000
2.5	0.9998	1.000	1.000	1.000	1.000
3.0	0.9995	1.000	1.000	1.000	1.000
3.5	0.9985	0.9999	1.000	1.000	1.000
4.0	0.9964	0.9999	0.9999	1.000	1.000
4.5	0.9920	0.9999	0.9999	1.000	1.000
5.0	0.9837	0.9999	0.9999	1.000	1.000

The above tables presente probabilities that the theoretical value of the threshold parameter (Table 1) and that of the shape parameter (Table 2) fall in the proposed interval when the threshold parameter takes different values. For small samples and for both intervals, this probability deteriorates as the theoretical value of the scale parameter increases, while remaining close to 1. However, it improves very quickly for medium and large samples reaching unity. These intervals are therefore highly recommended for the estimation of the parameters of the three-parameter lognormal distribution.

3.2. Marginalized Likelihood Method Estimates

To evaluate the performance of the estimators of the proposed method, simulation study has been carried out. These simulations was run by selecting the following values of the threshold parameter γ: 0.5, 1.0, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5 and 5.0 when the shape and the scale parameters are taken fixed as $\sigma = 1$ and $\mu = 1$. The behavior of the estimators performance is evaluated through the bias and root-mean-squared error (RMSE). The simulations were performed using the R environment via its package "nlminb". To do this, we use scale and shape parameters proposed bounds and take the threshold one into its domain.

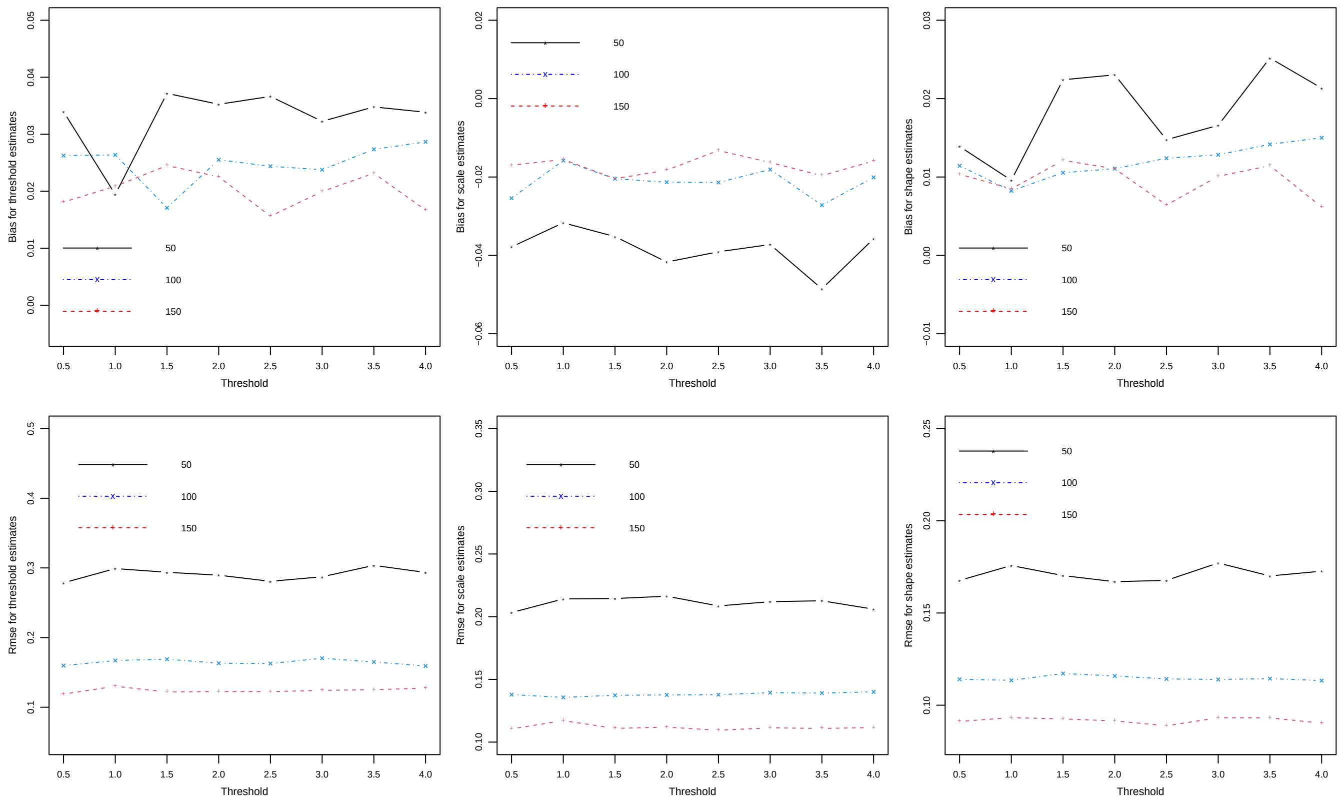

Figure 1. *Bias and Rmse for parameters' estimates bosed on* 1000 *samples.*

The graph above suggests that the proposed estimators are competitive in terms of bias and root mean squared error. This performance improves rapidly as the sample size increases. Indeed, the bias and the RMSE of the estimates of the three parameters decrease when sample size increase. Only the bias associated with the scale parameter is negative. Consequently, all other proposed estimators tend to overestimate the theoretical value of the parameters.

4. Concluding Remarks

We proposed a method called the ”marginalized maximum likelihood method” for estimating the parameters of the three-parameter lognormal distribution. This method is based on a marginal likelihood function using the $n-1$ largest order statistics. Furthermore, adaptive bounds for locating the threshold and shape parameters were also proposed. These bounds were used for parameter estimation via a constrained optimization method available in the R software. Through a Monte Carlo simulation study, we showed that:

1. the proposed parameters ranges contain the true value of the parameters with probability close to unity.
2. the proposed estimators exhibit competitive performance in terms of bias and RMSE.

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Abbreviations

MM	Minorize-Maximization
RMSE	Root-Mean-Squared Error

Conflicts of Interest

The authors declare no conflicts of interest.

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