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Maximum Likelihood Estimation for Uncertain Moving Average Model Under Imprecise Observations

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Abstract: Time series analysis serves as a practical way to investigate data that changes over time, and has great potential for development in fields such as weather prediction and financial engineering. However, due to the imperfections in technology, it is not always possible to obtain precise observational data. Therefore, modeling with uncertain time series is more suitable. Choosing appropriate parameter estimation methods is a critical aspect of the modeling process for uncertain time series. This paper proposes using maximum likelihood estimation for the parameter estimation of the first-order uncertain moving average (UMA) model, and for predicting future values and calculating the confidence intervals. Subsequently, the effectiveness of this method is demonstrated through numerical examples. Firstly, we employ an iterative method to convert the first-order UMA model into an uncertain autoregressive (UAR) model. Secondly, the maximum likelihood method is used to estimate the unknown parameters of the UMA model. Additionally, for this model where the observations are linear uncertain variables, a specific maximum likelihood estimation approach is provided. Thirdly, following the estimation values obtained, future data predictions and confidence intervals are calculated. Furthermore, when the observations are linear uncertain variables, corresponding confidence intervals are also provided. Finally, two practical cases are presented to demonstrate the practicality of the method. Moreover, contrasted with the least squares method, the results indicate that this method can enhance the accuracy of predictions.

Keywords: Uncertainty Theory, Uncertain Time Series Analysis, Uncertain Moving Average Model, Maximum Likelihood Estimation

1. Introduction

A time series consists of data points ordered by time. The key aim of time series analysis is to forecast future values. Initially, Yule [1] explored how to assess and investigate the periodicity in temporal sequence data that are subject to disturbances in 1927. Since then, various time series models have been discussed by many researchers, such as Walker [2], Box and Jenkins [3] and others. However, some models are predominantly based on the linear case. Traditional linear time series models have limitations in analyzing certain complex phenomena. To make up weakness, Tong and Lim [4] explored a novel category of nonlinear models-threshold autoregressive model. In practical applications, many time series data are multivariate, such as economic indicators and meteorological data. These data not only change over time but also have interrelationships between variables. Therefore, Lewis and Reinsel [5] studied how to predict multivariate time series using autoregressive model. Overall, time series analysis has provided more accurate and effective analytical tools for a variety of fields, enabling better forecasting and understanding of complex dynamic systems.

Traditional time series analysis relies on a probabilistic framework, which requires the observation data to be accurate. However, in many cases, accurate observations cannot be obtained due to time, economy, collection methods and other factors. For example, annual carbon emissions, daily stock trading prices and daily temperature changes are not accurate observation data. At this point, traditional time series analysis struggles to provide a reasonable explanation. Therefore, Liu [6] proposed the uncertainty theory to address issues involving uncertainty. This theory is a mathematical framework that aims to describe and quantify uncertainty through an axiomatic approach. In 2009, this theory was further developed [7]. Furthermore, the imprecise observed values were treated as uncertain variables by Liu [7]. Subsequently, Yao [8] extended traditional statistical inference models to handle inaccurate observational data. Building on this research, Ye and Liu [9] put forward the uncertain hypothesis test. This method demonstrates how to check the validity of regression models and assess the reasonableness of statistical hypotheses in light of the observed data, and offers a fresh way to deal with uncertain data and points to new possibilities for the development of uncertain statistics. Currently, the uncertainty theory has already demonstrated its extensive potential for application. Among its diverse applications, uncertain statistics stands out as a significant subdivision of uncertainty theory, with uncertain time series analysis being a key component of statistical models.

On this basis, some scholars have presented a variety of uncertain time series models to project future values with inaccurate observations. This work has subsequently elicited a substantial response from the research community, prompting numerous scholars to engage in follow-up studies. Up to now, uncertain time series models include the uncertain autoregressive (UAR) model [10], the uncertain moving average (UMA) model [11], and the uncertain autoregressive moving average (UARMA) model [12]. The data points mentioned above exhibit a linear relationship. However, in actual practice, many time series data show characteristics of non-linearity, randomness, and seasonality. To model the observation data, Tang [13] proposed the uncertain threshold autoregressive (UTAR) model. Furthermore, time series data are often multivariate, such as carbon emission indicators and meteorological data, where there are interrelationships among variables. In light of this, Zhang et al. [14] further proposed the uncertain vector autoregressive (UVAR) model . At the same time, due to the complex and diverse nature of the environment, the above static models are no longer sufficient to effectively grasp and control. Therefore, to enhance the precision of future event forecasting, Shi and Wang [15] proposed an uncertain time-varying autoregressive model. Actually, prior to the construction of an uncertain time series model, we had already obtained a set of observational data. Nevertheless, parameter estimation remains an essential part of model building. Therefore, Liu and Yang [16] put forward a cross-validation method to determine the optimal model order that minimizes the loss function by segmenting the dataset into various subsets. Yang and Liu [10] suggested employing least squares approach for parameter estimation of the UAR model. To address the limitations of the least squares method, Xin et al. [17] put forward maximum likelihood estimation, and the experimental results indicated that least squares estimation is more accurate without outliers. With multiple outliers, maximum likelihood estimation yields more accurate predictions than least squares estimation. Additionally, Zhang et al. [18] developed least absolute shrinkage and selection operator (LASSO) approach for parameter estimation. For LASSO, the L1 regularization forces some coefficients to be zero, thus losing the information of the corresponding variables. To make up weakness, Chen and Yang [19] proposed a ridge method, which enhances the generalization performance of the model by introducing an L2 regularization penalty term. Furthermore, the uncertain time series model has been applied in some fields, such as epidemiology to predict the spread of infectious diseases [20, 21], financial market [22], trajectory analysis of moving objects [21], etc.

Yang and Ni [11] put forward the UMA model based on uncertain observations and employed least squares for parameter estimation. Xin et al. [23] later advanced this model by proposing a higher-order UMA model, also relying on the least squares method. As is well known, the least squares method is not well-suited for datasets that contain a significant number of outliers. Given that maximum likelihood estimation is a significant parameter estimation method in uncertain statistics and has a solid theoretical foundation, this paper adopts it for parameter estimation of the UMA model, aiming to improve reliability and robustness of the estimation by addressing the limitations of the least squares method. Firstly, an iterative method is employed to convert the first-order UMA model into an UAR model. Secondly, the maximum likelihood method is utilized for parameter estimation of this model. Additionally, for this model with observations being linear uncertain variables, general and specific approaches to maximum likelihood estimation are provided. Thirdly, using the fitted UMA model, future data predictions cand confidence intervals can be determined. Moreover, when the observations are linear uncertain variables, corresponding confidence intervals are also provided. Finally, two practical cases are presented to demonstrate the practicality of this method. The prediction results are contrasted with those from the least squares method, and the results indicate that this method can enhance the accuracy of predictions.

This paper comprises six sections. Section 2 will show us the process of transforming a first-order UMA model into an UAR model through iterative techniques. Section 3 will use maximum likelihood method for parameter estimation of the first-order UMA model. Additionally, for this model with observations being linear uncertain variables, we will present a specific maximum likelihood method for this model. In section 4, following the estimation values obtained, we will obtain future data predictions and confidence intervals, and present a specific forecast interval when the observations are linear uncertain variables. Section 5 will give numerical examples to demonstrate the viability of the maximum likelihood estimation. This demonstrates that maximum likelihood estimation is more practical than least squares estimation when handling outliers. Section 6 will present a brief conclusion.

2. UMA Model

The UMA model [11] serves as a prevalent instrument in statistical analysis. Supposing that the present value is represented as a linear function of previous uncertain error terms, this model excels in handling time series data with short-term fluctuations and a stable long-term trend, making it particularly suitable for short-term forecasting and scenarios requiring rapid prediction results. To achieve the goal of better forecasting, the UMA model is used to model the observations in uncertain time series and to predict future value. The q order UMA model is defined by

$$X_t = b_0 - b_1\epsilon_{t-1} - \cdots - b_q\epsilon_{t-q} + \epsilon_t \tag{1}$$

where X_t denote imprecise observation data at time t for each $t = 1, 2, \ldots, n$, $b_0, b_1, \ldots, b_q$ are unknown parameters. And $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are independent and uniformly distributed uncertain variables with $E(\epsilon_t) = e$ and $V(\epsilon_t) = \sigma^2$.

When $q = 1$, the UMA model is defined by

$$X_t = b_0 - b_1\epsilon_{t-1} + \epsilon_t. \tag{2}$$

To achieve the transformation, substitute $t = 1, 2, \ldots, n-1$ into the above formula (2), we obtain

$$\begin{aligned}
\epsilon_1 &= (X_1 - b_0) + b_1\epsilon_0,\\
\epsilon_2 &= X_2 - b_0 + b_1\epsilon_1\\
&= (X_2 - b_0) + b_1(X_1 - b_0) + b_1^2\epsilon_0,\\
\epsilon_3 &= X_3 - b_0 + b_1\epsilon_2\\
&= (X_3 - b_0) + b_1(X_2 - b_0) + b_1^2(X_1 - b_0) + b_1^3\epsilon_0,\\
&\vdots\\
\epsilon_{t-1} &= X_{t-1} - b_0 + b_1\epsilon_{t-2}\\
&= \sum_{i=0}^{t-2} b_1^i(X_{t-1-i} - b_0) + b_1^{t-1}\epsilon_0.
\end{aligned}$$

Therefore, the UMA model (2) is iterated to become the following formula,

$$\begin{aligned}
X_t &= b_0 - b_1\epsilon_{t-1} + \epsilon_t\\
&= b_0 - b_1\left(\sum_{i=0}^{t-2} b_1^i(X_{t-1-i} - b_0) + b_1^{t-1}\epsilon_0\right) + \epsilon_t\\
&= b_0 - \sum_{i=0}^{t-2} b_1^{i+1}(X_{t-1-i} - b_0) - b_1^t\epsilon_0 + \epsilon_t\\
&= b_0 - \sum_{i=1}^{t-1} b_1^i(X_{t-i} - b_0) - b_1^t\epsilon_0 + \epsilon_t.
\end{aligned} \tag{3}$$

Suppose that $\epsilon_0 \equiv 0$, the model (3) is simplified as

$$X_t = b_0 - \sum_{i=1}^{t-1} b_1^i(X_{t-i} - b_0) + \epsilon_t. \tag{4}$$

Clearly, the model (4) takes the form of an UAR model.

3. Maximum Likelihood Estimation for UMA Model

The fundamental principle of maximum likelihood estimation lies in selecting the parameter values that maximize the probability of observing the given data . These values are then used as the estimates. This section utilizes maximum likelihood estimation for the first-order UMA model and presents the maximum likelihood estimation under general conditions and specific conditions.

3.1. Maximum Likelihood Estimation

Definition 3.1. (Xin et al. [17]) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent and identically distributed samples and conform to uncertainty distribution $F(y|\boldsymbol{\vartheta})$ where $\boldsymbol{\vartheta}$ denote a collection of parameters. Let $\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n$ be imprecise observations and conform to uncertainty distributions $G_1, G_2, \ldots, G_n$, respectively. Assume that $\xi_1, \xi_2, \ldots, \xi_n$, $\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n$ are independent uncertain variables. Then, the likelihood function is recognized as

$$L(\boldsymbol{\vartheta}|\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n) = \lim_{\Delta y \to 0} \max_{y_1, y_2, \ldots, y_n} \left\{ \frac{\left[F\left(y_i + \frac{\Delta y}{2}\middle|\boldsymbol{\vartheta}\right) - F\left(y_i - \frac{\Delta y}{2}\middle|\boldsymbol{\vartheta}\right)\right]}{\Delta y} \bigwedge \frac{\left[G_i\left(y_i + \frac{\Delta y}{2}\right) - G_i\left(y_i - \frac{\Delta y}{2}\right)\right]}{\Delta y} \right\}.$$

The maximum likelihood estimation of $\boldsymbol{\vartheta}$ is

$$\boldsymbol{\vartheta}^* = \arg\max_{\boldsymbol{\vartheta}} L\left(\boldsymbol{\vartheta}|\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n\right).$$

Theorem 3.1. (Xin et al. [17]) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent and identically distributed samples and conform to uncertainty distribution $F(y|\boldsymbol{\vartheta})$ where $\boldsymbol{\vartheta}$ denote a collection of parameters. Let $\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n$ be imprecise observations and conform to uncertainty distributions $G_1, G_2, \ldots, G_n$, respectively. Assume that $\xi_1, \xi_2, \ldots, \xi_n$, $\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n$ are independent uncertain variables, and $F, G_1, G_2, \ldots, G_n$ are uniformly continuously differentiable, then we have

$$L(\boldsymbol{\vartheta}|\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n) = \max_{y_1, y_2, \ldots, y_n} \bigwedge_{i=1}^{n} F'(y_i|\boldsymbol{\vartheta}) \bigwedge_{i=1}^{n} G_i'(y_i).$$

Theorem 3.2. Assume that $X_1, X_2, ..., X_n$ are a set of observations with uncertainty distributions $\Phi_1, \Phi_2, ..., \Phi_n$, respectively. The uncertain variables $\epsilon_t (t = 1, 2, ..., n)$ in the UMA model (4) are independent and identically distributed and conform to uncertainty distribution $F(z|\boldsymbol{\theta})$

where $\boldsymbol{\theta} = (e, \sigma^2)$ denote a collection of parameters. Assume that $\epsilon_1, \epsilon_2, ..., \epsilon_n, X_1, X_2, ..., X_n$ are independent uncertain variables, and $S_t(z|b_0, b_1)(t = 1, 2, ..., n)$ are the uncertainty distribution of observation of

$$\tilde{z}_t = X_t - b_0 + \sum_{i=1}^{t-1} b_1^i (X_{t-i} - b_0). \tag{5}$$

If $F(z|\boldsymbol{\theta})$, $S_1(z|b_0, b_1)$, $S_2(z|b_0, b_1)$, $\ldots$, $S_n(z|b_0, b_1)$ are uniformly continuously differentiable, then the maximum likelihood estimation of $(\boldsymbol{\theta}, b_0, b_1)$ is

$$(\hat{\boldsymbol{\theta}}, \hat{b}_0, \hat{b}_1) = \arg\max_{\boldsymbol{\theta}, b_0, b_1} L(\boldsymbol{\theta}, b_0, b_1 | \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) \tag{6}$$

where

$$L(\boldsymbol{\theta}, b_0, b_1 \mid \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) = \max_{z_1, z_2, \ldots, z_n} \bigwedge_{t=1}^{n} \left[F'(z_t|\boldsymbol{\theta}) \bigwedge S_t'(z_t|b_0, b_1) \right]. \tag{7}$$

Proof. In order to estimate the unknown parameters $(\boldsymbol{\theta}, b_0, b_1)$ in (4), a relationship is constructed by

$$\tilde{z}_t = X_t - b_0 + \sum_{i=1}^{t-1} b_i^i (X_{t-i} - b_0)$$

for $t = 1, 2, \ldots, n$, which are the differences between the true values and the fitted values. Then $\tilde{z}_t$ $(t = 1,\ 2, \ldots, n)$ can be considered as uncertain observational data from independent and identically distributed samples ϵ_t $(t = 1, 2, \ldots, n)$ that conform to uncertainty distribution $F(z|\boldsymbol{\theta})$. According to Theorem 3.1, the likelihood function is $L(\boldsymbol{\theta}, b_0, b_1 | \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n)$ presented in (7), and the maximum likelihood estimation of $(\boldsymbol{\theta}, b_0, b_1)$ is obtained by solving the optimization problem in (6). The Theorem 3.2 is thus verified.

The fitted UMA model is

$$X_t = \hat{b}_0 + \epsilon_t - \hat{b}_1 \epsilon_{t-1}. \tag{8}$$

The fitted equivalent UAR model is

$$X_t = \hat{b}_0 - \sum_{i=1}^{t-1} (\hat{b}_1)^i (X_{t-i} - \hat{b}_0). \tag{9}$$

3.2. Special Case

Assume that $X_1, X_2, \ldots, X_n$ are independent linear uncertain variables and conform to uncertainty distribution $\mathcal{L}(c_1, k_1), \mathcal{L}(c_2, k_2), \ldots, \mathcal{L}(c_n, k_n)$. Given that Liu [24] proposed operation laws for uncertain variables, we will prove a special maximum likelihood estimation for the first-order UMA model, which is scheduled as outlined below.

Theorem 3.3. (Liu [24]) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables and conform to respective uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$ being regular. If the function $f(\xi_1, \xi_2, \ldots, \xi_n)$ is strictly increasing about $\xi_1, \xi_2, \ldots, \xi_m$ and strictly decreasing about $\xi_{m+1}, \xi_{m+2}, \ldots, \xi_n$, then the inverse distribution of $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has the following form

$$\begin{aligned}\Psi^{-1}(\alpha) =& f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \\ & \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)).\end{aligned}$$

Theorem 3.4. Assume that $X_1, X_2, \ldots, X_n$ are independent linear uncertain variables with uncertainty distributions $\mathcal{L}(c_1, k_1)$, $\mathcal{L}(c_2, k_2)$, $\ldots$, $\mathcal{L}(c_n, k_n)$. The the inverse uncertainty distribution of

$$\begin{aligned}\tilde{z}_t &= X_t - b_0 + \sum_{i=1}^{t-1} b_1^i (X_{t-i} - b_0) \\ &= \sum_{i=0}^{t-1} b_1^i (X_{t-i} - b_0)\end{aligned}$$

is

$$S_t^{-1}(\alpha) = \sum_{i=0}^{t-1} b_1^i (\Upsilon_{t-i}^{-1}(\alpha, b_1) - b_0)$$

in which

$$\Upsilon_{t-i}^{-1}(\alpha, b_1) = \begin{cases} (1-\alpha)c_{t-i} + \alpha k_{t-i}, & \text{if } b_1^i \geq 0 \\ \alpha c_{t-i} + (1-\alpha)k_{t-i}, & \text{if } b_1^i < 0. \end{cases}$$

for $i = 0, 1, \ldots, t-1$ and $t = 1, 2, \ldots, n$.

Proof. Clearly,

$$\tilde{z}_t = \sum_{i=0}^{t-1} b_1^i (X_{t-i} - b_0)$$

is strictly increasing concerning X_{t-i} when $b_1^i \geq 0$, and strictly decreasing concerning X_{t-i} if $b_1^i < 0$ for $t = 1, 2, \ldots, n$ and $i = 0, 1, \ldots, t-1$. By applying Theorem 3.3, the inverse uncertainty distribution of $\tilde{z}_t$ is

$$S_t^{-1}(\alpha) = \sum_{i=0}^{t-1} b_1^i (\Upsilon_{t-i}^{-1}(\alpha, b_1) - b_0)$$

where

$$\Upsilon_{t-i}^{-1}(\alpha, b_1) = \begin{cases} (1-\alpha)c_{t-i} + \alpha k_{t-i}, & \text{if } b_1^i \geq 0 \\ \alpha c_{t-i} + (1-\alpha)k_{t-i}, & \text{if } b_1^i < 0. \end{cases}$$

Theorem 3.5. Assume $X_1, X_2, \ldots, X_n$ are independent linear uncertain variables and conform to uncertainty distribution $\mathcal{L}(c_1, k_1), \mathcal{L}(c_2, k_2), \ldots, \mathcal{L}(c_n, k_n)$, respectively. The uncertain variables $\epsilon_t (t = 1, 2, \ldots, n)$ in the UMA model (4) are independent and identically distributed and conform to uncertainty distribution $F(z|\boldsymbol{\theta})$ where $\boldsymbol{\theta} = (e, \sigma^2)$ denote a collection of parameters. Assume that $\epsilon_1, \epsilon_2, \ldots, \epsilon_n, X_1, X_2, \ldots, X_n$ are independent uncertain variables, and $S_t(z|b_0, b_1)$ $(t = 1, 2, \ldots, n)$ refer the uncertainty distribution of observation

$$\tilde{z}_t = \sum_{i=0}^{t-1} b_1^i (X_{t-i} - b_0)$$

determined by $\mathcal{L}(c_1, k_1), \mathcal{L}(c_2, k_2), \ldots, \mathcal{L}(c_n, k_n)$. If $F(z|\boldsymbol{\theta})$, $S_1(z|b_0, b_1), S_2(z|b_0, b_1), \ldots, S_n(z|b_0, b_1)$ are uniformly continuously differentiable, then the maximum likelihood estimation of $(\boldsymbol{\theta}, b_0, b_1)$ is

$$\left(\hat{\boldsymbol{\theta}}, \hat{b}_0, \hat{b}_1\right) = \underset{\boldsymbol{\theta}, b_0, b_1}{\arg\max} L(\boldsymbol{\theta}, b_0, b_1 | \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n). \tag{10}$$

Proof.

Step 1. Let us review the conception of linear uncertain variable $\mathcal{L}(c, k)$, its inverse uncertainty distribution is

$$\Phi^{-1}(\alpha) = (1 - \alpha)c + \alpha k$$

where c and k are real numbers with $c < k$.

Step 2. Derive the uncertain distribution $S_t'(z|b_0, b_1)$ of the observed value $\tilde{z}_t$. Assume that $X_1, X_2, \ldots, X_n$ are independent linear uncertain variables and conform to uncertainty distribution $\mathcal{L}(c_1, k_1), \mathcal{L}(c_2, k_2), \ldots, \mathcal{L}(c_n, k_n)$. By adopting Theorem 3.4, the inverse uncertainty distribution of

$$\tilde{z}_t = \sum_{i=0}^{t-1} b_1^i (X_{t-i} - b_0)$$

is derived as

$$S_t^{-1}(\alpha) = \sum_{i=0}^{t-1} b_1^i (\Upsilon_{t-i}^{-1}(\alpha; b_1) - b_0) \tag{11}$$

in which

$$\Upsilon_{t-i}^{-1}(\alpha, b_1) = \begin{cases} (1-\alpha)c_{t-i} + \alpha k_{t-i}, & \text{if } b_1^i \geq 0 \\ \alpha c_{t-i} + (1-\alpha)k_{t-i}, & \text{if } b_1^i < 0 \end{cases}$$

for $i = 0, 1, \ldots, t-1$ and $t = 1, 2, \ldots, n$. Obviously, $\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n$ are linear uncertain variables. The uncertainty distribution of $\tilde{z}_t$ are given by $\mathcal{L}(m_t, n_t)$ for $t = 1, 2, \ldots, n$. For (11), when $\alpha = 0$,

$$m_t = \sum_{i=0}^{t-1} b_1^i (\Upsilon_{t-i}^{-1}(0, b_1) - b_0), \tag{12}$$

where

$$\Upsilon_{t-i}^{-1}(0, b_1) = \begin{cases} c_{t-i}, & \text{if } b_1^i \geq 0 \\ k_{t-i}, & \text{if } b_1^i < 0. \end{cases}$$

When $\alpha = 1$,

$$n_t = \sum_{i=0}^{t-1} b_1^i \left(\Upsilon_{t-i}^{-1}(1, b_1) - b_0\right), \tag{13}$$

where

$$\Upsilon_{t-i}^{-1}(1, b_1) = \begin{cases} k_{t-i}, & \text{if } b_1^i \geq 0 \\ c_{t-i}, & \text{if } b_1^i < 0. \end{cases}$$

Therefore, the uncertainty distribution of $\tilde{z}_t$ is

$$S_t(z|b_0, b_1) = \begin{cases} 0, & \text{if } z \leq m_t \\ \frac{z - m_t}{n_t - m_t}, & \text{if } m_t < z \leq n_t \\ 1, & \text{if } n_t < z. \end{cases} \tag{14}$$

The first derivative of $S_t(z|b_0, b_1)$ on the interval (m_t, n_t) are

$$S_t'(z|b_0, b_1) = \frac{1}{n_t - m_t} \tag{15}$$

for $t = 1, 2, \ldots, n$. Since the first derivative of $S_t(z|b_0, b_1)$ are constant in the interval (m_t, n_t), it follows that the first derivative of $S_t(z|b_0, b_1)$ are continuous and bounded throughout in the interval (m_t, n_t) for $t = 1, 2, \ldots, n$. As a consequence, $S_t(z|b_0, b_1)$ are uniformly continuously differentiable in the interval (m_t, n_t) for $t = 1, 2, \ldots, n$.

Step 3. Derive the uncertain distribution $F'(z|\boldsymbol{\theta})$ of sample ϵ_t. Suppose $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are independent and identically distributed samples and conform to normal uncertainty distribution $\mathcal{N}(e, \sigma)$ with parameters e and σ, specifically,

$$F(z|e, \sigma) = \left(1 + \exp\left(\frac{\pi(e - z)}{\sqrt{3}\sigma}\right)\right)^{-1}.$$

Given that $\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n$ are observed, since $F(z|e, \sigma)$ is uniformly continuously differentiable, the first derivative of $F(z|e, \sigma)$ is

$$F'(z|e, \sigma) = \frac{\frac{\pi}{\sqrt{3}\sigma} \exp\left(\frac{\pi(e-z)}{\sqrt{3}\sigma}\right)}{\left(1 + \exp\left(\frac{\pi(e-z)}{\sqrt{3}\sigma}\right)\right)^2}. \tag{16}$$

Step 4. Overall, $\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n$ can be considered as the observation data of $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, and $F(z|\boldsymbol{\theta})$, $S_1(z|b_0, b_1)$, $S_2(z|b_0, b_1)$, $\ldots$, $S_n(z|b_0, b_1)$ are uniformly continuously differentiable. According to Theorem 3.1, for $L(\boldsymbol{\theta}, b_0, b_1 | \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n)$ in (10), we have

$$\begin{aligned} & L(\boldsymbol{\theta}, b_0, b_1 | \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) \\ = & \max_{z_1, z_2, \ldots, z_n} \bigwedge_{t=1}^{n} \left[F'(z_t|\boldsymbol{\theta}) \bigwedge S_t'(z_t|b_0, b_1)\right] \\ = & \max_{z_1, z_2, \ldots, z_n} \bigwedge_{t=1}^{n} \left[\frac{\frac{\pi}{\sqrt{3}\sigma} \exp\left(\frac{\pi(e-z_t)}{\sqrt{3}\sigma}\right)}{\left(1 + \exp\left(\frac{\pi(e-z_t)}{\sqrt{3}\sigma}\right)\right)^2} \bigwedge \frac{1}{n_t - m_t}\right]. \end{aligned} \tag{17}$$

4. Forecast Value and Confidence Interval

This section primarily focuses on forecasting future value X_{n+1} and proposing a forecast interval for X_{n+1}.

4.1. Forecast Value

In practical situations, the forecast value X_{n+1} is considered as an uncertain variable. For the purpose of obtaining the forecast value X_{n+1}, according to the definition of Liu [6], we utilize the expected value of the uncertain forecast variable to determine X_{n+1}. The uncertain predictive variable X_{n+1} is specified as

$$X_{n+1} = \hat{b}_0 - \sum_{i=1}^{n} (\hat{b}_1)^i (X_{n+1-i} - \hat{b}_0) + \hat{\epsilon}_{n+1}, \tag{18}$$

where $\hat{\epsilon}_{n+1}$ conforms to uncertainty distribution $F(z|\hat{\boldsymbol{\theta}})$. Then the expected value of X_{n+1} can be obtained by

$$\begin{aligned} u_{n+1} &= E[X_{n+1}] \\ &= E\left[\hat{b}_0 - \sum_{i=1}^{n} (\hat{b}_1)^i (X_{n+1-i} - \hat{b}_0) + \hat{\epsilon}_{n+1}\right] \\ &= \hat{b}_0 - \sum_{i=1}^{n} (\hat{b}_1)^i E[X_{n+1-i} - \hat{b}_0] + E[\hat{\epsilon}_{n+1}] \\ &= \hat{b}_0 - \sum_{i=1}^{n} (\hat{b}_1)^i E[X_{n+1-i} - \hat{b}_0] + \hat{e}. \end{aligned} \tag{19}$$

4.2. Confidence Interval

To determine the confidence interval of predicted value X_{n+1}, we employ the parameters $\hat{\boldsymbol{\theta}}$ in which $\hat{\boldsymbol{\theta}} = (\hat{e}, \hat{\sigma})$ have been estimated in section 3, and assume that $\epsilon_t (t = 1, \ldots, n)$ conform to an uncertainty distribution $F(z|\hat{\boldsymbol{\theta}})$, If the uncertainty distributions $F, \Phi_1, \Phi_2, ..., \Phi_n$ are independent and regular, the inverse uncertainty distribution of X_{n+1} is derived as

$$\hat{\Phi}_{n+1}^{-1}(\alpha) = \hat{b}_0 - \sum_{i=1}^{n} (\hat{b}_1)^i \left(\Upsilon_{n+1-i}^{-1}(\alpha, \hat{b}_1) - \hat{b}_0\right) + F^{-1}(\alpha) \tag{20}$$

where

$$\Upsilon_{n+1-i}^{-1}(\alpha, \hat{b}_1) = \begin{cases} \Phi_{n+1-i}^{-1}(1-\alpha), & \text{if } (\hat{b}_1)^i \geq 0 \\ \Phi_{n+1-i}^{-1}(\alpha), & \text{if } (\hat{b}_1)^i < 0. \end{cases}$$

Assume that $\hat{\epsilon}_{n+1}$ conforms to a normal uncertainty distribution $\mathcal{N}(\hat{e}, \hat{\sigma})$, and $F^{-1}(\alpha)$ is the inverse uncertainty distribution of $\hat{\epsilon}_{n+1}$, that is

$$F^{-1}(\alpha) = \hat{e} + \frac{\hat{\sigma}\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}. \tag{21}$$

From the inverse distribution $\hat{\Phi}_{n+1}^{-1}(x)$, we are able to get the distribution $\hat{\Phi}_{n+1}(x)$ of X_{n+1}. Finally, based on the research conducted by Yang and Liu [10] in 2019, we take β as a confidence level, and identify the best solution $\hat{h}$ that satisfy

$$\hat{\Phi}_{n+1}(u_{n+1} + \hat{h}) - \hat{\Phi}_{n+1}(u_{n+1} - \hat{h}) \geq \beta. \tag{22}$$

Then the β confidence interval of X_{n+1} is given by

$$u_{n+1} \pm \hat{h}. \tag{23}$$

Remark 1. Assume $X_1, X_2, \ldots, X_n$ are independent linear uncertain variables, and conform to uncertainty distribution $\mathcal{L}(c_1, k_1), \mathcal{L}(c_2, k_2), \ldots, \mathcal{L}(c_n, k_n)$, the inverse distribution of X_{n+1} is

$$\hat{\Phi}_{n+1}^{-1}(\alpha) = \hat{b}_0 - \sum_{i=1}^{n} (\hat{b}_1)^i \left(\Upsilon_{n+1-i}^{-1}(\alpha, \hat{b}_1) - \hat{b}_0\right) + F^{-1}(\alpha) \tag{24}$$

where

$$\Upsilon_{n+1-i}^{-1}(\alpha, \hat{b}_1) = \begin{cases} \alpha c_{n+1-i} + (1-\alpha) k_{n+1-i}, & \text{if } (\hat{b}_1)^i \geq 0 \\ (1-\alpha) c_{n+1-i} + \alpha k_{n+1-i}, & \text{if } (\hat{b}_1)^i < 0, \end{cases}$$

and

$$F^{-1}(\alpha) = \hat{e} + \frac{\hat{\sigma}\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

5. Numerical Examples

In this section, to demonstrate the effectiveness and feasibility of this approach, we will present numerical examples of the temperature in Beijing and the stock price of Tesla to illustrate the validity and robustness of this method.

5.1. Temperature Prediction

Since temperature records feature daily maxima and minima, they can be modeled as linear uncertain variables. Consequently, this study uses the temperature data from October 2024 in Fengtai District, Beijing, as observational values. Assume that $X_1, X_2, \ldots, X_{30}$ represent the temperatures in Beijing from October 1 to 30, 2024. They are independent linear uncertain variables $\mathcal{L}(c_t, k_t)$, with c_t indicating the lowest temperature and k_t indicating the highest temperature on day t. These temperature observation data are shown in Table 1 and available at https://tianqi.2345.com. Let us demonstrate the application of the maximum likelihood estimation for predicting the temperature on October 31, November 1, November 2, and November 3, employing a first-order UMA model.

Firstly, we assume that $\epsilon_t (t = 1, 2, \ldots, 30)$ conform to a normal uncertainty distribution characterized by mean e and standard deviation σ. Employing Theorem 3.5, we calculate the optimal solution for

$$\arg \max_{\boldsymbol{\theta}, b_0, b_1} L(\boldsymbol{\theta}, b_0, b_1 | \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n)$$

where

$$\begin{aligned}&L(\boldsymbol{\theta}, b_0, b_1|\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_n)\\ &= \max_{z_1,z_2,\dots,z_n} \bigwedge_{t=1}^{n} \left[F'(z_t|\boldsymbol{\theta}) \bigwedge S'_t(z_t|b_0,b_1)\right]\\ &= \max_{z_1,z_2,\dots,z_n} \bigwedge_{t=1}^{n} \left[\frac{\frac{\pi}{\sqrt{3}\sigma}\exp\left(\frac{\pi(e-z_t)}{\sqrt{3}\sigma}\right)}{\left(1+\exp\left(\frac{\pi(e-z_t)}{\sqrt{3}\sigma}\right)\right)^2} \bigwedge \frac{1}{n_t - m_t}\right].\end{aligned}$$

To obtain the estimated value of (b_0, b_1, e, σ),

$$(\hat{b}_0, \hat{b}_1, \hat{e}, \hat{\sigma}) = (12.1550, -0.1271, 0.5610, 3.6021).$$

Hence the fitted UMA model is

$$\begin{aligned}X_t =&\hat{b}_0 - \sum_{i=1}^{t-1}(\hat{b}_1)^i(X_{t-i} - \hat{b}_0)\\ =&12.1550 - \sum_{i=1}^{t-1}(-0.1271)^i(X_{t-i} - 12.1550),\end{aligned}$$

and the estimated values of mean and variance of ϵ_t are

$$\hat{e} = 0.5610, \quad \sigma^2 = 12.9751$$

for $t = 1, 2, \dots, 30$.

Secondly, we assume that ϵ_{31} conforms to normal uncertainty distribution $\mathcal{N}(0.5610, 3.6021)$, which is independent of $X_1, X_2, \dots, X_{30}$. Then the expected value of X_{31} is

$$\begin{aligned}u_{31} &= \hat{b}_0 - \sum_{i=1}^{n}(\hat{b}_1)^i E[X_{n+1-i} - \hat{b}_0] + \hat{e}\\ &= 12.1550 - \sum_{i=1}^{30}(-0.1271)^i(E[X_{31-i}] - 12.1550) + 0.5610\\ &= 12.8755.\end{aligned}$$

The inverse uncertainty distribution of X_{31} is

$$\begin{aligned}\hat{\Phi}_{31}^{-1}(\alpha) = 12.1550 - \sum_{i=1}^{30}(-0.1271)^i(\Upsilon_{31-i}^{-1}(\alpha, -0.1271)\\ - 12.1550) + F^{-1}(\alpha)\end{aligned}$$

where

$$\begin{aligned}&\Upsilon_{31-i}^{-1}(\alpha, -0.1271)\\ &= \begin{cases}\alpha c_{31-i} + (1-\alpha)k_{31-i}, & \text{if } (-0.1271)^i \ge 0\\ (1-\alpha)c_{31-i} + \alpha k_{31-i}, & \text{if } (-0.1271)^i < 0,\end{cases}\end{aligned}$$

and

$$F^{-1}(\alpha) = 0.5610 + \frac{3.6021\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}.$$

Finally, assume the confidence level β be 75%, the best solution for

$$\hat{\Phi}_{31}\left(u_{31} + \hat{h}\right) - \hat{\Phi}_{31}\left(u_{31} - \hat{h}\right) \ge 75\%$$

is

$$\hat{h} = 4.1333,$$

where $\hat{\Phi}_{31}$ is the uncertainty distribution of X_{31}, and can be determined by $\hat{\Phi}_{31}^{-1}$. Thus the 75% confidence interval of X_{31} is

$$12.8755 \pm 4.1333.$$

In summary, the predicted minimum temperature for October 31 is 8.7422, and the predicted maximum temperature is 17.0088. The maximum likelihood estimation prediction confidence interval is illustrated in Figure 1. In reality, the actual temperature recorded on October 31 was [9, 17]. According to the prediction results from the least squares estimation, we gain the predicted value of u_{31} as 13.7333 and a 75% confidence interval of 4.6217. Next, Remark 2 is provided to explain the comparison of prediction results for temperature using maximum likelihood estimation as opposed to least squares estimation.

Remark 2. In this case, we define the error as the discrepancy between the actual values and the predicted values derived from maximum likelihood estimation and least squares method.

$$\begin{aligned}Q_1 &= |8.7422 - 9| + |17.0088 - 17| = 0.2666,\\ Q_2 &= |9.1116 - 9| + |18.3550 - 17| = 1.4666.\end{aligned}$$

The comparative results of the two methods are displayed in Table 2, where Q_1 is the error of the prediction result from the maximum likelihood estimation, and Q_2 is the error of the predictive result from the least squares estimation. The findings indicate that the maximum likelihood approach yields lower errors compared to the least squares approach.

From the perspective of prediction interval, the results show that the prediction interval made by the maximum likelihood estimation covers the actual temperature recorded completely on October 31. However, the prediction interval made by least squares estimation does not.

In summary, this implies that the UMA model utilizing maximum likelihood estimation is more practical.

Subsequently, to demonstrate the practicality of the maximum likelihood estimation, we used both the maximum likelihood estimation and least squares estimation methods to forecast the temperature for three consecutive days and compared the predictions with the actual temperature values.

Firstly, leveraging the actual temperature data from October 31, we assume that X_{31} conforms to the uncertain variable $\mathcal{L}(9, 17)$. Then we forecast the temperature for November 1 using maximum likelihood estimation method, and obtain a predicted value of u_{32} as 15.8514 for November 1, with a 75% confidence interval of 5.9242. According to the prediction results from the least squares estimation, we gain the predicted value of u_{32} as 13.7096 and a 75% confidence interval of 4.5687. The actual temperature on November 1 is [10, 20].

Secondly, leveraging the actual temperature data from November 1, we assume that X_{32} conforms to the uncertain variable $\mathcal{L}(10, 20)$. Then we forecast the temperature for November 2 using maximum likelihood estimation method, and obtain a predicted value of u_{33} as 12.8375 for November 2, with a 75% confidence interval of 4.1914. According to the prediction results from the least squares estimation, we gain the predicted value of u_{33} as 13.7499 and a 75% confidence interval of 4.5353. The actual temperature on November 2 is [8, 16].

Finally, leveraging the actual temperature data from November 2, we assume that X_{33} conforms to the uncertain variable $\mathcal{L}(8, 16)$. Then we forecast the temperature for November 3 using maximum likelihood estimation method, and obtain a predicted value of u_{34} as 11.9237 for November 3, with a 95% confidence interval of 8.7113. According to the prediction results from the least squares estimation, we gain the predicted value of u_{34} as 13.6969 and a 95% confidence interval of 8.4669. The actual temperature on November 3 is [4, 21].

Next, we will compare the errors between the predicted results obtained from the two methods and the actual temperature values for November 1, November 2, and November 3, respectively. The comparison results are shown in Table 3.

From Table 3, the findings indicate that the maximum likelihood approach yields lower errors compared to the least squares approach. Thus, it can be concluded that maximum likelihood estimation is more practical for predicting future values.

Table 1. *The temperature of Beijing from October 1 to 30.*

X_1	X_2	X_3	X_4
$\mathcal{L}(7, 17)$	$\mathcal{L}(7, 20)$	$\mathcal{L}(8, 22)$	$\mathcal{L}(9, 21)$
X_5	X_6	X_7	X_8
$\mathcal{L}(13, 23)$	$\mathcal{L}(10, 19)$	$\mathcal{L}(8, 22)$	$\mathcal{L}(10, 23)$
X_9	X_{10}	X_{11}	X_{12}
$\mathcal{L}(10, 20)$	$\mathcal{L}(8, 22)$	$\mathcal{L}(10, 23)$	$\mathcal{L}(13, 24)$
X_{13}	X_{14}	X_{15}	X_{16}
$\mathcal{L}(13, 18)$	$\mathcal{L}(12, 21)$	$\mathcal{L}(10, 21)$	$\mathcal{L}(13, 18)$
X_{17}	X_{18}	X_{19}	X_{20}
$\mathcal{L}(14, 16)$	$\mathcal{L}(5, 19)$	$\mathcal{L}(2, 13)$	$\mathcal{L}(1, 11)$
X_{21}	X_{22}	X_{23}	X_{24}
$\mathcal{L}(6, 13)$	$\mathcal{L}(4, 17)$	$\mathcal{L}(5, 18)$	$\mathcal{L}(7, 22)$
X_{25}	X_{26}	X_{27}	X_{28}
$\mathcal{L}(10, 19)$	$\mathcal{L}(10, 16)$	$\mathcal{L}(4, 18)$	$\mathcal{L}(7, 19)$
X_{29}	X_{30}		
$\mathcal{L}(7, 19)$	$\mathcal{L}(10, 17)$		

Table 2. *Comparison of two approaches to predict weather temperature on October 31.*

	maximum likelihood	**least squares**
75% confidence interval	$[8.7422, 17.0088]$	$[9.1116, 18.3550]$
Error	0.2666	1.4666

Table 3. *Comparison of the prediction results of the two methods.*

	maximum likelihood on November 1	least squares on November 1
75% confidence interval	$[9.9272, 21.7756]$	$[9.1409, 18.2783]$
Error	1.8484	2.5808
	maximum likelihood on November 2	least squares on November 2
75% confidence interval	$[8.6461, 17.0289]$	$[9.2146, 18.2852]$
Error	1.6750	3.4998
	maximum likelihood on November 3	least squares on November 3
95% confidence interval	$[3.2124, 20.6350]$	$[5.2300, 22.1638]$
Error	1.1526	2.3938

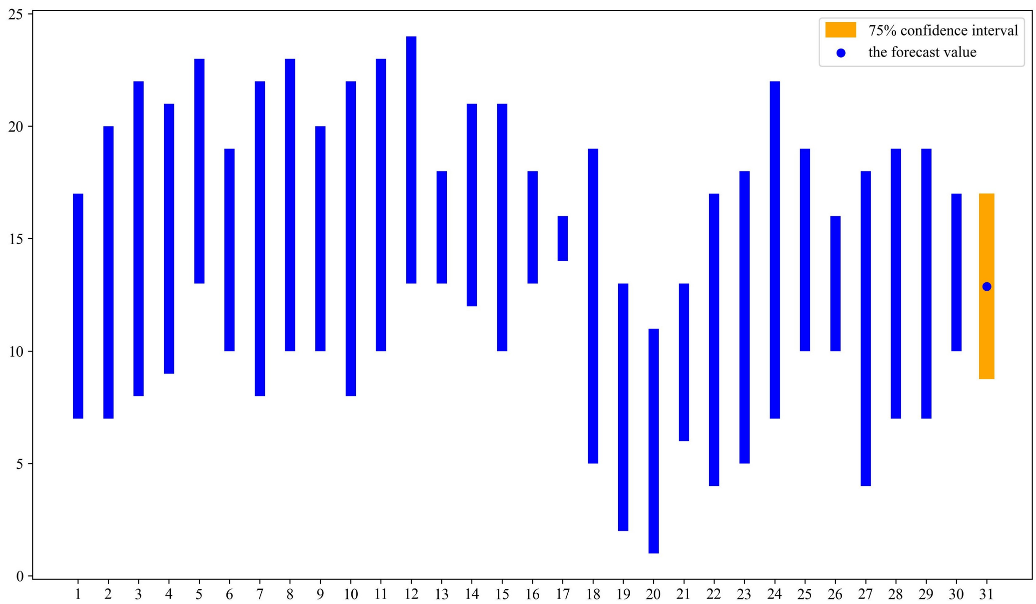

Figure 1. *The 75% confidence interval of weather temperature forecast.*

5.2. Tesla Stock Price Prediction

Assume $X_1, X_2, \ldots, X_{24}$ are the stock price of Tesla from December 24, 2024, through January 30, 2025. They are independent linear uncertain variable $\mathcal{L}(c_t, k_t)$, with c_t indicating the minimum price and k_t indicating the maximum price on day t. These values are presented in Table 4 and available at https://cn.investing.com. Let's now demonstrate the application of the maximum likelihood method for predicting the stock price of Tesla on January 31, February 3, February 4, and February 5, 2025, respectively, employing a first-order UMA model.

Firstly, we assume that $\epsilon_t (t = 1, 2, \ldots, 24)$ conform to a normal uncertainty distribution characterized by mean e and standard deviation σ. Employing Theorem 3.5, we calculate the optimal solution for

$$\arg \max_{\boldsymbol{\theta}, b_0, b_1} L(\boldsymbol{\theta}, b_0, b_1 | \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n)$$

where

$$\begin{aligned} & L(\boldsymbol{\theta}, b_0, b_1 | \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) \\ & = \max_{z_1, z_2, \ldots, z_n} \bigwedge_{t=1}^{n} \left[F'(z_t | \boldsymbol{\theta}) \bigwedge S_t'(z_t | b_0, b_1) \right] \\ & = \max_{z_1, z_2, \ldots, z_n} \bigwedge_{t=1}^{n} \left[\frac{\frac{\pi}{\sqrt{3}\sigma} \exp\left(\frac{\pi(e - z_t)}{\sqrt{3}\sigma}\right)}{\left(1 + \exp\left(\frac{\pi(e - z_t)}{\sqrt{3}\sigma}\right)\right)^2} \bigwedge \frac{1}{n_t - m_t} \right]. \end{aligned}$$

To obtain the estimated value of (b_0, b_1, e, σ),

$$(\hat{b}_0, \hat{b}_1, \hat{e}, \hat{\sigma}) = (415.9221, -0.6394, 0.7070, 9.9439).$$

Hence the fitted UMA model is

$$\begin{aligned} X_t &= \hat{b}_0 - \sum_{i=1}^{t-1} (\hat{b}_1)^i (X_{t-i} - \hat{b}_0) \\ &= 415.9221 - \sum_{i=1}^{t-1} (-0.6394)^i (X_{t-i} - 415.9221), \end{aligned}$$

and the estimated values of mean and variance of ϵ_t are

$$\hat{e} = 0.7070, \quad \sigma^2 = 98.8811$$

for $t = 1, 2, \ldots, 24$.

Secondly, we assume that ϵ_{25} conforms to normal uncertainty distribution $\mathcal{N}(0.7070, 9.9439)$, which is independent of $X_1, X_2, \ldots, X_{24}$. Then the forecast value of X_{25} is

$$\begin{aligned} u_{25} &= \hat{b}_0 - \sum_{i=1}^{n} \left(\hat{b}_1\right)^i E\left[X_{n+1-i} - \hat{b}_0\right] + \hat{e} \\ &= 415.9221 - \sum_{i=1}^{24} (-0.6394)^i \left(E\left[X_{25-i}\right] - 415.9221\right) \\ &+ 0.7070 \\ &= 412.6730. \end{aligned}$$

The inverse uncertainty distribution of X_{25} is

$$\begin{aligned} \hat{\Phi}_{25}^{-1}(\alpha) = 415.9221 - \sum_{i=1}^{24} (-0.6394)^i \\ \left(\Upsilon_{25-i}^{-1}(\alpha, -0.6394) \ -415.9221\right) + F^{-1}(\alpha) \end{aligned}$$

where

$$\Upsilon^{-1}_{25-i}(\alpha,-0.6394) = \begin{cases} \alpha c_{25-i} + (1-\alpha)k_{25-i}, & \text{if } (-0.6394)^i \geq 0 \\ (1-\alpha)c_{25-i} + \alpha k_{25-i}, & \text{if } (-0.6394)^i < 0, \end{cases}$$

and

$$F^{-1}(\alpha) = 0.7070 + \frac{9.9439\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

Finally, let the confidence level β be 75%, the best solution for

$$\hat{\Phi}_{25}\left(u_{25}+\hat{h}\right) - \hat{\Phi}_{25}\left(u_{25}-\hat{h}\right) \geq 75\%$$

is

$$\hat{h} = 15.7724,$$

where $\hat{\Phi}_{25}$ is the uncertainty distribution of X_{25}, and can be determined by $\hat{\Phi}^{-1}_{25}$. Thus the 75% confidence interval of X_{25} is

$$412.6730 \pm 15.7724.$$

According to our predictions, the lowest price of Tesla on January 31 will be 396.9006 and the highest price of Tesla on January 31 will be 428.4454. The maximum likelihood estimation prediction confidence interval is illustrated in Figure 2. In fact, the real stock price of Tesla on January 31 was [401.34, 419.99]. According to the forecast results from the least squares estimation, we gained the forecast value is 408.6758 and the forecast interval is 22.5740, specifically, the lowest price of Tesla on January 31 will be 386.1018 and the highest price of Tesla on January 31 will be 431.2498. Next, Remark 3 provides an explanation comparing the prediction outcomes for Tesla stock price using maximum likelihood estimation as opposed to least squares estimation.

Remark 3. In this case, we define the interval lengths of the predicted results obtained by these two methods. The comparative results of the interval lengths are displayed in Table 5.

From the perspective of the prediction interval, the results show that maximum likelihood method has a smaller interval length than least squares method.

From the perspective of the predicted values, the minimum value predicted from least squares method is lower than one acquired through maximum likelihood estimation, while the maximum value predicted from the least squares method exceeds that of the maximum likelihood estimation.

Hence, the predictions from the maximum likelihood estimation are more accurate.

Subsequently, to demonstrate the practicality of the maximum likelihood estimation, we used both the maximum likelihood estimation and least squares estimation methods to forecast Tesla stock price for three consecutive days and compared the predictions with the actual stock price values.

Firstly, leveraging the actual Tesla stock price data from January 31, we assume that X_{25} conforms to the uncertain variable $\mathcal{L}(401.34, 419.99)$. Then we forecast the Tesla stock price for February 3 using maximum likelihood estimation method, and obtain a predicted value of u_{26} as 381.1011 for February 3, with a 75% confidence interval of 12.0263. According to the prediction results from the least squares estimation, we gain the predicted value of u_{26} as 381.3989 and a 75% confidence interval of 16.8269. The actual Tesla stock price on February 3 is [374.36, 389.17].

Secondly, leveraging the actual Tesla stock price data from February 3, we assume that X_{26} conforms to the uncertain variable $\mathcal{L}(374.36, 389.17)$. Then we forecast Tesla stock price for February 4 using maximum likelihood estimation method, and obtain a predicted value of u_{27} as 385.9122 for February 4, with a 75% confidence interval of 13.5470. According to the prediction results from the least squares estimation, we gain the predicted value of u_{27} as 384.3512 and a 75% confidence interval of 19.1924. The actual Tesla stock price on February 4 is [381.4, 394].

Finally, leveraging the actual Tesla stock price data from February 4, we assume that X_{27} conforms to the uncertain variable $\mathcal{L}(381.4, 394)$. Then we forecast the Tesla stock price for February 5 using maximum likelihood estimation method, and obtain a predicted value of u_{28} as 382.9031 for February 5, with a 75% confidence interval of 13.3628. According to the prediction results from the least squares estimation, we gain the predicted value of u_{28} as 382.5276 and a 75% confidence interval of 24.2558. The actual Tesla stock price on February 5 is [375.53, 388.39].

Next, we will compare the interval lengths of the predicted Tesla stock prices obtained from the two methods for February 3, February 4, and February 5, respectively. The comparison results are shown in Table 6.

From Table 6, the results show that maximum likelihood method has a smaller interval length than least squares method. Thus, it can be concluded that maximum likelihood estimation is more practical for predicting future values.

Table 4. *The stock price of Tesla.*

X_1	X_2	X_3	X_4
$\mathcal{L}(435.14, 462.78)$	$\mathcal{L}(451.02, 465.33)$	$\mathcal{L}(426.5, 450)$	$\mathcal{L}(415.75, 427)$
X_5	X_6	X_7	X_8
$\mathcal{L}(402.54, 427.93)$	$\mathcal{L}(373.04, 392.73)$	$\mathcal{L}(379.45, 411.88)$	$\mathcal{L}(401.7, 426.43)$
X_9	X_{10}	X_{11}	X_{12}
$\mathcal{L}(390, 414.33)$	$\mathcal{L}(387.4, 402.5)$	$\mathcal{L}(377.29, 399.28)$	$\mathcal{L}(380.07, 403.79)$

X_{13}	X_{14}	X_{15}	X_{16}
$\mathcal{L}(394.54, 422.64)$	$\mathcal{L}(405.66, 429.8)$	$\mathcal{L}(409.13, 424)$	$\mathcal{L}(419.75, 439.74)$
X_{17}	X_{18}	X_{19}	X_{20}
$\mathcal{L}(406.31, 433.2)$	$\mathcal{L}(414.59, 428)$	$\mathcal{L}(408.95, 420.73)$	$\mathcal{L}(405.78, 418.88)$
X_{21}	X_{22}	X_{23}	X_{24}
$\mathcal{L}(389, 406.69)$	$\mathcal{L}(386.5, 400.59)$	$\mathcal{L}(384.48, 398.59)$	$\mathcal{L}(384.41, 412.5)$

Table 5. *Comparison of two approaches to predict the stock price of Tesla on January 31.*

	maximum likelihood	least squares
75% confidence interval	$[396.9006, 428.4454]$	$[386.1018, 431.2498]$
Interval length	31.5448	45.1480

Table 6. *Comparison of the prediction results of the two methods.*

	maximum likelihood on February 3	least squares on February 3
75% confidence interval	$[369.0748, 393.1274]$	$[364.5720, 398.2258]$
Interval length	24.0526	33.6538
	maximum likelihood on February 4	least squares on February 4
75% confidence interval	$[372.3652, 399.4592]$	$[365.1588, 403.5436]$
Interval length	27.0940	38.3848
	maximum likelihood on February 5	least squares on February 5
75% confidence interval	$[369.5403, 396.2659]$	$[358.2718, 406.7834]$
Interval length	26.7256	48.5116

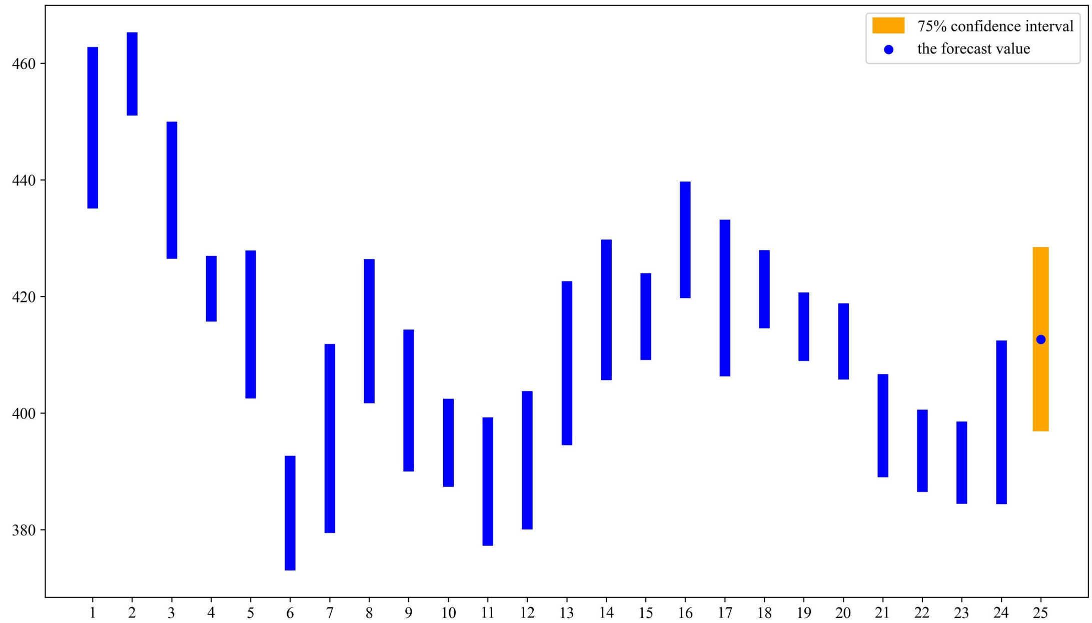

Figure 2. *The 75% confidence interval of Tesla stock price forecast.*

6. Conclusion

This paper employs a first-order UMA model to analyze imprecise observations and predict future values. Firstly, this study utilizes the maximum likelihood method for the first-order UMA model, and provides a maximum likelihood estimation for it when the observations are linear uncertain variables. Secondly, utilizing the obtained estimation values, future data predictions and confidence intervals are calculated. Additionally, confidence intervals are provided for the case where the observations are linear uncertain variables. Finally, numerical experiments demonstrate that maximum likelihood method outperforms least squares estimation in dealing with outliers.

Through numerical examples, it is found that the UMA model is highly effective in addressing imprecise observations. Specifically, it has shown promise in applications such as weather temperature prediction and Tesla stock price forecasting. For different data types, the likelihood function requires separate representation. For example, when the observed values conform to a normal distribution or a linear distribution, their likelihood functions are very different, which increases the complexity of calculations. In this case, confronted with this complex problem, it is essential to have an adequate knowledge of uncertainty theory.

In this paper, we transform the first-order UMA model into an infinite-order UAR model and then truncate it to an n-order UAR model. This process may introduce some bias and has certain limitations. From the perspective of selecting the value of n, choosing an appropriate value of n is crucial for the performance of the model. If the value of n is too small, it may fail to capture the complex features in the data. If the value of n is too large, the model may overfit, that is, it may fit the training data too closely. From the perspective of model approximation, no matter what value of n is chosen, the n-order UAR model is always an approximation of the first-order UMA model. This approximate transformation may lead to a loss of prediction accuracy.

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Abbreviations

UAR	Uncertain Autoregressive
UMA	Uncertain Moving Average
UARMA	Uncertain Autoregressive Moving Average
UTAR	Uncertain Threshold Autoregressive
UVAR	Uncertain Vector Autoregressive
LASSO	Least Absolute Shrinkage and Selection Operator

Author Contributions

Xiaosheng Wang: Conceptualization, Formal Analysis, Investigation, Writing - review & editing

Ben He: Conceptualization, Formal Analysis, Investigation, Writing - review & editing

Data Availability Statement

The data supporting the outcome of this research work has been reported in this manuscript.

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Yule, G. U. On a method of investigating periodicities in disturbed series, with special reference to Wolfer's sunspot numbers. *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character.* 1927, 226, 267-298. https://doi.org/10.1098/rsta.1927.0007

[2] Walker, G. T. On periodicity in series of related terms. *Proceedings of the Royal Society of London. Series A. Containing Papers of a Mathematical and Physical Character.* 1931, 131(818), 518-532. https://doi.org/10.1098/rspa.1931.0069

[3] Box, G. E. P., Jenkins, G. M. *Time series analysis: Forecasting and control.* San Francisco: Holden-Day; 1970.

[4] Tong, H., Lim, K. S. Threshold Autoregression, Limit Cycles and Cyclical Data. *Journal of the Royal Statistical Society Series: B (Statistical Methodology).* 1980, 42(3), 245-268. https://doi.org/10.1111/j.2517-6161.1980.tb01126.x

[5] Lewis, R., Reinsel, G. C. Prediction of multivariate time series by autoregressive model fitting. *Journal of Multivariate Analysis.* 1985, 16(3), 393-411. https://doi.org/10.1016/0047-259X(85)90027-2

[6] Liu, B. *Uncertainty Theory.* 2nd ed. Berlin: Springer-Verlag; 2007. https://doi.org/10.1007/978-3-540-73165-8

[7] Liu, B. Some research problems in uncertainty theory. *Journal of Uncertain Systems.* 2009, 3(1), 3-10.

[8] Yao, K. Uncertain statistical inference models with imprecise observations. *IEEE Transactions on Fuzzy Systems.* 2018, 26(2), 409-415. https://doi.org/10.1109/TFUZZ.2017.2666846

[9] Ye, T., Liu, B. Uncertain significance test for regression coefficients with application to regional economic analysis. *Communications in Statistics - Theory and Methods.* 2022, 52, 7271-7288. https://doi.org/10.1080/03610926.2022.2042562

[10] Yang, X., Liu, B. Uncertain time series analysis with imprecise observations. *Fuzzy Optimization and Decision Making.* 2019, 18, 263-278. https://doi.org/10.1007/s10700-018-9298-z

[11] Yang, X., Ni, Y. Least-squares estimation for uncertain moving average model. *Communications in Statistics - Theory and Methods.* 2021, 50(17), 4134-4143. https://doi.org/10.1080/03610926.2020.1713373

[12] Lu, J., Peng, J., Chen, J., Sugeng, K. A. Prediction method of autoregressive moving average models for uncertain time series. *International Journal of General Systems.* 2020, 49(5), 546-572. https://doi.org/10.1080/03081079.2020.1748616

[13] Tang, H. Uncertain threshold autoregressive model with imprecise observations. *Communications in Statistics - Theory and Methods.* 2021, 51(24), 8776-8785. https://doi.org/10.1080/03610926.2021.1906433

[14] Zhang, G., Shi, Y., Sheng, Y. (2023). Least absolute deviation estimation for uncertain vector autoregressive model with imprecise data. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems.* 2023, 31(3), 353-370. https://doi.org/10.1142/S0218488523500186

[15] Shi, Y., Wang, L. Uncertain time-varying autoregressive model with imprecise observations and its applications. *Journal of Computational and Applied Mathematics.* 2025, 463, 116508. https://doi.org/10.1016/j.cam.2025.116508

[16] Liu, Z., Yang, X. Cross validation for uncertain autoregressive model. *Communications in Statistics - Simulation and Computation.* 2020, 51(8), 4715-4726. https://doi.org/10.1080/03610918.2020.1747077

[17] Xin, Y., Gao, J., Yang, X., Yang, J. Maximum likelihood estimation for uncertain autoregressive moving average model with application in financial market. *Journal of Computational and Applied Mathematics.* 2023, 417, 114604. https://doi.org/10.1016/j.cam.2022.114604

[18] Zhang, Z., Yang, X., Gao, J. Uncertain Autoregressive Model via LASSO Procedure. *International Journal of Uncertainty, Fuzziness and Knowledge-based Systems.* 2020, 28(6), 939-956. https://doi.org/10.1142/S0218488520500415

[19] Chen, D., Yang, X. Ridge Estimation for Uncertain Autoregressive Model with Imprecise Observations. *International Journal of Uncertainty, Fuzziness and Knowledge-based Systems.* 2021, 29(1), 37-55. https://doi.org/10.1142/S0218488521500033

[20] Ye, T., Yang, X. Analysis and prediction of confirmed COVID-19 cases in China with uncertain time series. *Fuzzy Optimization and Decision Making.* 2021, 20(2), 209-228. https://doi.org/10.1007/s10700-020-09339-4

[21] Xie, J., Lio, W. Uncertain nonlinear time series analysis with applications to motion analysis and epidemic spreading. *Fuzzy Optimization and Decision Making.* 2024, 23(2), 279-294. https://doi.org/10.1007/s10700-024-09421-1

[22] Zhang, Y., Gao, J. Nonparametric uncertain time series models: Theory and application in brent crude oil spot price analysis. *Fuzzy Optimization and Decision Making.* 2024, 23(2), 239-252. https://doi.org/10.1007/s10700-024-09419-9

[23] Xin, Y., Yang, X., Gao, J. Least squares estimation for the high-order uncertain moving average model with application to carbon dioxide emissions. *International Journal of General Systems.* 2021, 50(6), 724-740. https://doi.org/10.1080/03081079.2021.1950150

[24] Liu, B. *Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty.* Berlin: Springer-Verlag; 2010. https://doi.org/10.1007/978-3-642-13959-8