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On Fixed Point Results of Suzuki-Type Contractions on Controlled Metric Spaces with Applications

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Abstract: In this paper, we introduce the concept of Suzuki-type contractions on controlled metric spaces and prove a fixed point theory. This extends and generalises the already existing results of Suzuki-type contractions on $b-$metric spaces and extended $b-$metric spaces to controlled metric spaces. Some illustrative examples are presented in order to amplify our findings. It is shown that Suzuki-type contractions in the setting of controlled metric spaces provide greater generality and flexibility compared to the setting of metric spaces. We do this by constructing an example where a Suzuki-type contraction does not guarantee a fixed point in a standard metric space but does in a controlled metric space. In this setting, the control function in the controlled metric helps to stabilise iterative sequences in proving the fixed point theory and indeed in the application. Finally, our main result is applied to show the existence of a solution for the fredholm type integral equation. The results obtained in this paper contribute to the broader study of fixed point theory and its applications in mathematical analysis and applied sciences.

Keywords: Controlled Metric, $b-$Metric Space, Extended $b-$Metric Space, Fixed Point, Fredholm Integral Equation, Suzuki-type Contractions

1. Introduction

Fixed point theory has a wide range of applications in a number of disciplines of mathematics such as functional analysis, obtaining numerical solutions of integral and differential equations, optimisation, approximation theory, variational and linear inequalities and many more. For instance, the Banach contraction principle provides a method for solving functional equations. Various researchers have generalised the Banach contraction principle in many directions(see [4–8, 20, 21]).

Mlaiki et al. [13] introduced the concept of controlled metric spaces which is also a generalisation of b-metric spaces. This concept of controlled metric spaces is very useful in the proving of the existence and uniqueness theorems for different types of integral and differential equations. See also [3, 19] for some recent works on controlled metric spaces.

Suzuki [20] generalised the Banach contraction principle by introducing Suzuki-type contractions of metric spaces. See also Kikkawa and Suzuki in [10]. In 2017, Leyew and Abbas [12] discussed the concept of a generalised $\alpha-$Suzuki - Geraghty type contraction mapping and obtained fixed point results in the framework of an $f-$orbitally complete $b-$metric space. In 2015, Roshan et. al [15] introduced some Suzuki-type fixed point results in $b-$metric spaces as a way of generalisation. In this framework, the $b-$metric was not necessarily continuous. Various Suzuki-type theorems in $b-$metric spaces were discussed by Latif et. al in [11] (see also [1, 16, 17]). For some recent work on Suzuki-type contractions, see also [2, 22].

In this paper, the main results are in section 3 where we introduce the concept of Suzuki-type contractions on controlled metric spaces and prove a fixed point theorem. This extends and generalises the already existing results of Suzuki-type contractions on $b-$metric spaces and extended $b-$metric spaces to controlled metric spaces. Some examples are presented in order to amplify our findings. Finally, we apply the obtained results to show the existence of a solution for the Fredholm type integral equation in the setting of controlled metric spaces, and provide an example for further illustration. The results obtained in this paper generalise, amplify and unifiy the existing results in the literature.

2. Preliminaries

In this section, we present some important definitions and concepts from the
literature which will be needed in the sequel.

Definition 2.1. [6]

Let X be a non empty set and $s \geq 1$ a given real number. A function $d : X \times X \to [0, \infty)$ is called a $b-$metric on X if, for all $x, y, z \in X$, the following conditions hold:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$; and
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$

The pair (X, d) is called a $b-$metric space and the real number $s \geq 1$ is called the coefficient of (X, d). Every metric space is a $b-$metric space with coefficient $s = 1$. The converse is generally not true.

Kamran et al. [9] initiated the concept of extended $b-$metric spaces as follows.

Definition 2.2.

Let X be a non empty set and $p : X \times X \to [1, \infty)$ be a function. A function
$d_p : X \times X \to [0, \infty)$ is called an extended $b-$metric if for all $x, y, z \in X$, the following conditions hold:

1. $d_p(x, y) = 0$ if and only if $x = y$;
2. $d_p(x, y) = d_p(y, x)$; and
3. $d_p(x, z) \leq p(x, z)[d(x, y) + d(y, z)]$

The pair (X, d_p) is called an extended $b-$metric space. Clearly, every $b-$metric space is an extended $b-$metric space with coefficient $p(x, y) = s \geq 1$. The converse is generally not true.

Mlaiki et al. [13] introduced a new kind of generalised $b-$metric space called controlled metric space.

Definition 2.3.

Let X be a non-empty set and $\alpha : X \times X \to [1, \infty)$. Then a mapping
$d_\alpha : X \times X \to [0, \infty)$ is called a controlled metric, if for all $x, y, z \in X$, it satisfies the following axioms;

1. $d_\alpha(x, y) = 0$ if and only if $x = y$;
2. $d_\alpha(x, y) = d_\alpha(y, x)$
3. $d_\alpha(x, z) \leq \alpha(x, y)d_\alpha(x, y) + \alpha(y, z)d_\alpha(y, z)$.

The pair (X, d_α) is called a controlled metric space.

We define convergence, Cauchy sequences and completeness in controlled metric spaces as follows:

Definition 2.4. [18]

Let (X, d_α) be a controlled metric space and $\{x_n\}_{n\geq 0}$ be a sequence in X. Then

1. the sequence $\{x_n\}$ converges to some x in X if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_\alpha(x_n, x) < \epsilon$ for all $n \geq N$. In this case we write $\lim_{n\to\infty} x_n = x$,
2. the sequence x_n is Cauchy if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_\alpha(x_m, x_n) < \epsilon$ for all $m, n \geq N$,
3. the controlled metric space (X, d_α) is called complete if every Cauchy sequence in X is convergent in X.

In 1922, Banach first stated the much celebrated fixed point theory as follows:

Definition 2.5.

Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying

$$d(Tx, Ty) \leq kd(x, y),$$

for all $x, y \in X$ where $0 < k \leq 1$. Then T has a unique fixed point in X.

In 2008, Suzuki [21] introduced a generalised Banach's contraction in compact metric spaces as follows:

Definition 2.6.

Define a non-increasing function θ from $[0, 1)$ onto $(\frac{1}{2}, 1]$ by

$$\theta(r) = \begin{cases} 1, & if \quad 0 \leq r \leq \frac{1}{2}(\sqrt{5} - 1) \\ \frac{1-r}{r^2}, & if \quad \frac{1}{2}(\sqrt{5} - 1) \leq r \leq \frac{1}{2} \\ \frac{1}{1+r}, & if \quad \frac{1}{2} \leq r < 1 \end{cases}$$

Let (X, d) be a complete metric space and let T be a mapping from X into itself. Assume that there exists $r \in [0, 1)$ such that for every $x, y \in X$,

$$\theta(r)d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq rd(x, y).$$

Then there exists a unique fixed point z of T. Moreover, $\lim_{n\to\infty} T^n(x) = z$, for all $x \in X$.

The other version of Nemytckii fixed point theorem which was proved by Suzuki [21] is as follows:

Theorem 2.7.

Let (X, d) be a compact metric space. Let $T : X \to X$ be a self-map, satisfying for all $x, y \in X$, $x \neq y$ the condition

$$\tfrac{1}{2}d(x, Tx) \leq d(x, y) \quad \text{implies} \quad d(Tx, Ty) < d(x, y).$$

Then T has a unique fixed point in X.

We follow [15] for the theorem on Suzuki type contractions on $b-$metric spaces as follows:

Theorem 2.8.

Let (X, d) be a complete $b-$metric space. Let $T, S : X \to X$ be two self-maps and $\theta : [0, 1) \to (\frac{1}{2}, 1]$ be defined by

$$\theta(r) = \begin{cases} 1, & if \quad 0 \leq r \leq \frac{\sqrt{5}-1}{2} \\ \frac{1-r}{r^2}, & if \quad \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{2} \\ \frac{1}{1+r}, & if \quad \frac{1}{2} \leq r < 1 \end{cases}$$

Suppose there exists $r \in [0, 1)$ such that for each $x, y \in X$, the following condition is satisfied:

$$\tfrac{1}{b}(r)min\{d(x, Tx)\ ,\ d(x, Sx)\} \leq d(x, y)$$

implies

$$max\{d(Sx, Sy)\ ,\ d(Tx, Ty)\ ,\ d(Sx, Ty)\ ,\ d(Sy, Tx)\} \leq \tfrac{r}{b^2}d(x, y).$$

Then T and S have a unique fixed point $z \in X$.

3. Main Results

In this section, we prove a new fixed point theorem for Suzuki type contractions in the setting of controlled metric spaces. We begin with the following definition:

Example 3.1.

Let (X, d_α) be a complete controlled metric space with $\alpha : X \times X \to [1, \infty)$. Then the mapping $T : X \to X$ is said to be a Suzuki type contraction on X if for all $x, y \in X$ the following condition holds:

$$\tfrac{1}{2} d_\alpha(x, Tx) \leq d_\alpha(x, y) + \alpha(x, y) \text{ implies}$$
$$d_\alpha(Tx, Ty) \leq \alpha(x, y) d_\alpha(x, y)$$

Example 3.2.

Let $X = \{1, 2, 3\}$ and $d_\alpha(x, y) = max\{x, y\}$, for all $x, y \in X$. Define the mapping $T : X \to X$ by $T(x) = \frac{1}{x^2}$ and a function $\alpha : X \times X \to [1, \infty)$ by $\alpha(x, y) = x + 2y - 1$. Then T is a Suzuki type contraction on X.

Proof.

We show that the condition of Definition 3.1 is satisfied. Consider the following cases:

Case I

If $x = 1$ and $y = 2$, then we have that
$\frac{1}{2} d_\alpha(1, T(1)) = \frac{1}{2} d_\alpha(1, 1) = \frac{1}{2} < d_\alpha(1, 2) + \alpha(1, 2) = 6$ implies
$d_\alpha(T(1), T(2)) = d_\alpha(1, \frac{1}{4}) = 1 < \alpha(1, 2) d_\alpha(1, 2) = 8$

Case II

If $x = 1$ and $y = 3$, then we have that
$\frac{1}{2} d_\alpha(1, T(1)) = \frac{1}{2} d_\alpha(1, 1) = \frac{1}{2} < d_\alpha(1, 3) + \alpha(1, 3) = 9$ implies
$d_\alpha(T(1), T(3)) = d_\alpha(1, \frac{1}{9}) = 1 < \alpha(1, 3) d_\alpha(1, 3) = 18$

Case III

If $x = 2$ and $y = 1$, then we have that
$\frac{1}{2} d_\alpha(2, T(2)) = \frac{1}{2} d_\alpha(2, \frac{1}{4}) = 2 \leq d_\alpha(2, 1) + \alpha(2, 1) = 5$ implies
$d_\alpha(T(2), T(1)) = d_\alpha(\frac{1}{4}, 1) = 1 < \alpha(2, 1) d_\alpha(2, 1) = 6$

Case IV

If $x = 2$ and $y = 3$, then we have that
$\frac{1}{2} d_\alpha(2, T(2)) = \frac{1}{2} d_\alpha(2, \frac{1}{4}) = 1 < d_\alpha(2, 3) + \alpha(2, 3) = 10$ implies
$d_\alpha(T(2), T(3)) = d_\alpha(\frac{1}{4}, \frac{1}{9}) = \frac{1}{4} < \alpha(2, 3) d_\alpha(2, 3) = 21.$

Case V

If $x = 3$ and $y = 1$, then we have that
$\frac{1}{2} d_\alpha(3, T(3)) = \frac{1}{2} d_\alpha(3, \frac{1}{9}) = \frac{3}{2} < d_\alpha(3, 1) + \alpha(3, 1) = 7$ implies
$d_\alpha(T(3), T(1)) = d_\alpha(\frac{1}{9}, 1) = 1 < \alpha(3, 1) d_\alpha(1, 3) = 12.$

Case VI

If $x = 3$ and $y = 2$, then we have that
$\frac{1}{2} d_\alpha(3, T(3)) = \frac{1}{2} d_\alpha(3, \frac{1}{9}) = \frac{3}{2} < d_\alpha(3, 2) + \alpha(3, 2) = 9$ implies
$d_\alpha(T(3), T(2)) = d_\alpha(\frac{1}{9}, \frac{1}{4}) = \frac{1}{4} < \alpha(3, 2) d_\alpha(3, 2) = 18.$

Continuing in this way, one can check the other cases, that is, for $x = y = 1$, $x = y = 2$ and $x = y = 3$, that they hold. Hence, Definition 3.1 is satisfied so that T is a Suzuki type contraction on (X, d_α).

Example 3.3.

Let (X, d_α) be a complete controlled metric space with $\alpha : X \times X \to [1, \infty)$. Let
$T : X \to X$ be a Suzuki type contraction on X. Suppose that

$$\frac{1}{2} d_\alpha(x, Tx) \leq d_\alpha(x, y) + \alpha(x, y) \ \ implies \ \ d_\alpha(Tx, Ty) \leq \lambda d_\alpha(x, y) \tag{1}$$

for all $x, y \in X$, where $\lambda \in (0, 1)$. For $x_0 \in X$, take $x_n = T^n x_0$. Assume that

$$\underset{m \geq 1}{sup} \lim_{k \to \infty} \frac{\alpha(x_{k+1}, x_{k+2}) \alpha(x_{k+1}, x_m)}{\alpha(x_k, x_{k+1})} < \frac{1}{\lambda}. \tag{2}$$

Furthermore, suppose that for all $x \in X$, we have that

$$\lim_{n \to \infty} \alpha(x_n, x) \ \ and \ \ \lim_{n \to \infty} \alpha(x, x_n) \ \ exist \ and \ are \ finite. \tag{3}$$

Then T has a unique fixed point $z \in X$.

Proof.

Consider the sequence $x_n = T^n x_0$. From (1), we obtain

$$\frac{1}{2} d_\alpha(x_0, T^2 x_0) = \frac{1}{2} d_\alpha(x_0, T(Tx_0)) < d_\alpha(x_0, Tx_0) + \alpha(x_0, Tx_0) = d_\alpha(x_0, x_1) + \alpha(x_0, x_1)$$

implies

$$d_\alpha(T^n x_0, T^n x_1) = d_\alpha(x_n, x_{n+1}) \leq \lambda^n d_\alpha(x_0, x_1) \quad \forall n \geq 0.$$

For all natural numbers $n < m$ and from the triangle inequality in definition 2.3, we have

$$\begin{aligned} d_\alpha(x_n, x_m) &\leq \alpha(x_n, x_{n+1}) d_\alpha(x_n, x_{n+1}) + \alpha(x_{n+1}, x_m) d_\alpha(x_{n+1}, x_m) \\ &\leq \alpha(x_n, x_{n+1}) d_\alpha(x_n, x_{n+1}) + \alpha(x_{n+1}, x_m) \alpha(x_{n+1}, x_{n+2}) d_\alpha(x_{n+1}, x_{n+2}) + \\ &\quad \alpha(x_{n+1}, x_m) \alpha(x_{n+2}, x_m) d_\alpha(x_{n+2}, x_m) \\ &\leq \alpha(x_n, x_{n+1}) d_\alpha(x_n, x_{n+1}) + \alpha(x_{n+1}, x_m) \alpha(x_{n+1}, x_{n+2}) d_\alpha(x_{n+1}, x_{n+2}) + \end{aligned}$$

$$\alpha(x_{n+1},x_m)\alpha(x_{n+2},x_m)\alpha(x_{n+2},x_{n+3})d_\alpha(x_{n+2},x_{n+3})+$$
$$\alpha(x_{n+1},x_m)\alpha(x_{n+2},x_m)\alpha(x_{n+3},x_m)d_\alpha(x_{n+3},x_m)$$
$$\leq \cdots$$
$$\leq \alpha(x_n,x_{n+1})d_\alpha(x_n,x_{n+1})+\sum_{k=n+1}^{m-2}\left(\prod_{j=n+1}^{k}\alpha(x_j,x_m)\right)\alpha(x_k,x_{k+1})d_\alpha(x_k,x_{k+1})+\prod_{i=n+1}^{m-1}\alpha(x_i,x_m)d_\alpha(x_{m-1},x_m)$$
$$\leq \alpha(x_n,x_{n+1})\lambda^n d_\alpha(x_0,x_1)+\sum_{k=n+1}^{m-2}\left(\prod_{j=n+1}^{k}\alpha(x_j,x_m)\right)\alpha(x_k,x_{k+1})\lambda^k d_\alpha(x_0,x_1)+\prod_{i=n+1}^{m-1}\alpha(x_i,x_m)\lambda^{m-1}d_\alpha(x_0,x_1)$$

Since $\alpha(x,y)\geq 1$ for all $x,y\in X$ and letting $W_r=\sum_{k=0}^{r}\left(\prod_{j=0}^{k}\alpha(x_j,x_m)\right)\alpha(x_k,x_{k+1})\lambda^k$ we have that

$$d_\alpha(x_n,x_m)\leq d_\alpha(x_0,x_1)[\lambda^n\alpha(x_n,x_{n+1})+(W_{m-1}-W_n)] \tag{4}$$

Now we show that $\{W_n\}$ is a Cauchy sequence in X. From (2) and by using the ratio test, we are sure that $\underset{n\to\infty}{lim} W_n$ exists so that $\{W_n\}$ is Cauchy. Taking the limit in (4) as $n,m\to\infty$, we have

$$\underset{n\to\infty}{lim} d_\alpha(x_n,x_m)=0, \tag{5}$$

so that $\{x_n\}$ is Cauchy in (X,d_α). Thus, by the completeness of (X,d_α), $\{x_n\}$ converges to some $z\in X$. We now remain to show that z is a fixed point of T. From the triangle inequality in Definition 2.3, we have that
$d_\alpha(z,z_{n+1})\leq\alpha(z,x_n)d_\alpha(z,x_n)+\alpha(x_n,x_{n+1})d_\alpha(x_n,x_{n+1})$, so that from (2) and (5) this yields;

$$\underset{n\to\infty}{lim} d_\alpha(z,x_{n+1})=0 \tag{6}$$

By applying the triangle inequality in Definition 2.3 again, we obtain

$$d_\alpha(z,Tz)\leq\alpha(z,x_{n+1})d_\alpha(z,x_{n+1})+\alpha(x_{n+1},Tz)d_\alpha(x_{n+1},Tz)\leq\alpha(z,x_{n+1})d_\alpha(z,x_{n+1})+\lambda\alpha(x_{n+1},Tz)d_\alpha(x_n,z).$$

From (3) and (6) and taking limit as $n\to\infty$ it follows that $d_\alpha(z,Tz)=0$, so that $Tz=z$, implying that $z\in X$ is the fixed point.

To prove uniqueness, we assume to the contrary that T has two fixed points, say, z_1 and z_2. Then $d_\alpha(z_1,z_2)=d_\alpha(Tz_1,Tz_2)\leq\lambda d_\alpha(z_1,z_2)$
which can only hold if $d_\alpha(z_1,z_2)=0$, so that $z_1=z_2$. Thus, T has a unique fixed point.

Example 3.4.

Consider $X=\{0,1,2\}$ and let

$$d_\alpha(x,y)=\begin{cases}0, & if\quad x=y\\ x, & if\ \ x>y\\ y, & if\ \ y>x\end{cases}$$

Define $\alpha: X\times X\to[1,\infty)$ by $\alpha(x,y)=|x+y|+1$. Then (X,d_α) is a controlled metric space. To see this, we show that Definition 2.3 is satisfied.

1. By the definition of d_α, the property that $d_\alpha(x,y)=0$ if and only if $x=y$ holds.
2. By the definition of d_α, the property that $d_\alpha(x,y)=d_\alpha(y,x)$ for all $x,y\in X$ holds.
3. $d_\alpha(0,1)=1\leq\alpha(0,2)d_\alpha(0,2)+\alpha(1,2)d_\alpha(1,2)=10$
$d_\alpha(0,2)=2\leq\alpha(0,1)d_\alpha(0,1)+\alpha(2,1)d_\alpha(2,1)=6$
$d_\alpha(1,2)=2\leq\alpha(1,0)d_\alpha(1,0)+\alpha(2,0)d_\alpha(2,0)=6.$

Hence, $d_\alpha(x,z)\leq\alpha(x,y)d_\alpha(x,y)+\alpha(y,z)d_\alpha(y,z)$ holds.

Consider a self-map $T: X\to T$ defined by $T(x)=\frac{x}{x+1}$, where $x\in X$. We show that this mapping satisfies the conditions of a Suzuki contraction.

Taking $x=0$ and $y=1$, we have that $\frac{1}{2}d_\alpha(0,T(0))=0\leq d_\alpha(0,1)+\alpha(0,1)=3\implies d_\alpha(T(0),T(1))=d_\alpha(0,\frac{1}{2})=\frac{1}{2}<\alpha(0,1)d_\alpha(0,1)=2$

Taking $x=0$ and $y=2$, we obtain $\frac{1}{2}d_\alpha(0,T(0))=0\leq d_\alpha(0,2)+\alpha(0,2)=6\implies d_\alpha(T(0),T(2))=d_\alpha(0,\frac{2}{3})=\frac{2}{3}<\alpha(0,2)d_\alpha(0,2)=6$

Continuing with the other cases, we can check that Definition 3.1 holds so that T is a Suzuki type contraction on a controlled metric space, (X,d_α).

Taking $\lambda=\frac{1}{3}$, we have that conditions (1) and (2) in Theorem 3.3 are satisfied. By definition of $\alpha(x,y)$, we have that $\underset{n\to\infty}{lim}\alpha(x_n,x)$ and $\underset{n\to\infty}{lim}\alpha(x,x_n)$ exist and are finite. Hence, all the assumptions of Theorem 3.3 are satisfied and so T has a unique fixed point in X, which is $z=0$.

4. Application

4.1. Fredholm Integral Equation

Fixed point theorems have widely been used to guarantee the existence and uniqueness of results of the solution for non linear integral equations under various conditions. In this section, we investigate the solution of Fredholm-type integral equation,

$$f(t)=\lambda(t)+\int_a^b K(t,s)f(s)ds)<K(t,s,\zeta(t)) \tag{7}$$

where $K : [a,b] \to \mathbb{R}$ is the integral kernel and $\lambda : [a,b] \to \mathbb{R}$ is a function.

Theorem 4.1.

Let (X, d_α) be a complete controlled metric space and let $T : X \to X$ be a Suzuki type contraction on (X, d_α). Define $\alpha : X \times X \to [1,\infty)$ by the metric $\alpha(f,\mu) = \underset{a\leq t\leq b}{Sup}\, [\frac{2}{5}(|f(t)| + |\mu(t)|)$. Assume that for all $t, s \in [a,b]$ and $\beta : X \to [0,1)$, the following properties hold;

1. $|K(t,s,f(t))| + |K(t,s,\mu(t)| \leq \beta(\underset{a\leq t\leq b}{Sup}|f(t)| + |\mu(t)|)(|f(t)| + |\mu(t)|)$,
2. $K(t,s,\int_a^b K(t,s,f(t)ds) < K(t,s,f(t))$

Then equation (7) has a unique solution. *Proof.*

Let $X = (0,\infty)$ and $T : X \to X$ be defined by

$$Tf(t) = \lambda(t) + \textstyle\int_a^b K(t,s,f(t))ds,$$

Then

$$\alpha(Tf, T\mu) = \underset{a\leq t\leq b}{Sup}\left(\frac{2|Tf(t)| + 2|T\mu(t)f(t)|}{5}\right).$$

Implying that

$$\alpha(Tf(t), T\mu(t)) = \frac{2|Tf(t)| + 2|T\mu(t)f(t)|}{5}, = \frac{2|\int_a^b K(t,s,f(t))ds| + 2|\int_a^b K(t,s,\mu(t))ds|}{5},$$
$$\leq \frac{2\int_a^b |K(t,s,f(t))|ds + 2\int_a^b |K(t,s,\mu(t))|ds}{5}, = \frac{2\int_a^b \left[|K(t,s,f(t))| + |K(t,s,\mu(t))|\right] ds}{5},$$
$$\leq \frac{2\int_a^b \beta\left(\underset{a\leq t\leq b}{\mathrm{Sup}}|f(t)| + |\mu(t)|\right)(|f(t)| + |\mu(t)|)\, ds}{5}, \leq \beta\left(\underset{a\leq t\leq b}{\mathrm{Sup}}|f(t)| + |\mu(t)|\right)\alpha\left(f(t), \mu(t)\right),$$

which satisfies conditions 1 and 2 above. Hence, equation (7) has a unique solution.

Theorem 4.2.

Solve the Fredholm type Integral Equation

$$f(t) = 3cost + \textstyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sint f(y)dy.$$

Solution

We first take note that

$$\underset{a\leq t\leq b}{\mathrm{Sup}}\int_a^b |K(t,y)|dy = \underset{0\leq t\leq \pi}{\mathrm{Sup}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin t|dy = \underset{0\leq t\leq \pi}{\mathrm{Sup}}|\sin t|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dy = \pi$$

Let $f_0(t) = \pi$.

$$f_1(t) = 3\cos t + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin t \cdot \pi dy = 3\cos t + \pi \sin t\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dy = 3\cos t + \pi^2 \sin t.$$
$$f_2(t) = 3\cos t + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin t\left(3\cos y + \pi^2 \sin y\right) dy = 3\cos t + 6\sin t.$$
$$f_3(t) = 3\cos t + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin t\left(3\cos y + 6\sin y\right) dy = 3\cos t + 6\sin t.$$

Hence, our fixed point is $f(t) = 3cost + 6sint$. This clearly satisfies the given Fredholm type Integral Equation and so it is a solution.

5. Conclusion

In this paper, we have presented a Suzuki type contraction on controlled metric space as a new result. We have also proved a fixed point theorem for this new result. Additionally, illustrative examples are provided to ensure a detailed understaning of our results. We have utilised the developed fixed point theory to show the existence of a solution for a Fredholm type Integral Equation as a matter of application. Our results may be extended to more types of Suzuki contractions as further research.

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Conflicts of Interest

The authors declare no conflict of interest.

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