International Journal of Systems Science and Applied Mathematics
2026; 11(1): 1-5
http://www.sciencepublishinggroup.com/j/ijssam
doi: 10.11648/j.ijssam.20261101.11

Mathematical Models for Vegetation Fire PropagationExistence and Uniqueness Analysis

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To cite this article:

Tchiekre Michel Henri Daugny, Aguemon Wiwegnon Uriel-Longin, Kambire Diopina Gnowille, Akahoua David Vincent Brou, Adou Kablan Jerome. (2026). Mathematical Models for Vegetation Fire PropagationExistence and Uniqueness Analysis. *International Journal of Systems Science and Applied Mathematics, 11*(1), 1-5. https://doi.org/10.11648/j.ijssam.20261101.11

Received: 7 September 2025 ; **Accepted:** 9 October 2025 ; **Published:** 4 February 2026

Abstract: This work presents a rigorous mathematical study of a vegetation fire propagation model specifically adapted to the Ivorian environmental context, where climatic conditions and vegetation types significantly influence fire dynamics. The model aims to describe the thermal behavior of vegetation during combustion and the spatial spread of fire fronts. We first analyze a nonlinear ordinary differential equation (ODE) governing the temporal evolution of temperature, which captures the essential mechanisms of heat production and dissipation during combustion. Using classical results from the theory of differential equations, we establish the existence and uniqueness of solutions under suitable assumptions on the model parameters and initial conditions. The study is then extended to a transport-reaction partial differential equation (PDE) that incorporates spatial effects and allows the description of fire propagation in a heterogeneous domain. This PDE model accounts for both the advective transport of heat and the local reaction terms associated with combustion processes. The analysis relies on tools from functional analysis, including appropriate function spaces and a priori estimates, combined with the method of characteristics to handle the transport component of the equation. Under physically relevant assumptions, we prove the existence and uniqueness of weak solutions to the PDE model. The proposed mathematical framework provides a solid theoretical foundation for vegetation fire modeling in West African environments. Beyond its theoretical interest, this work contributes to a better understanding of fire dynamics and offers a basis for future numerical simulations and risk assessment tools aimed at improving fire prevention and management strategies in Côte d'Ivoire and similar regions.

Keywords: Fire, Mathematical, Model, Ordinary Differential Equation, Partial Differential Equation

1. Introduction

In this work, we study a mathematical model describing the propagation of vegetation fires. Our approach is structured around a research plan organized into several key steps.

Our analysis builds upon the ordinary differential equation (ODE) proposed by Adou et al.[1, 2], which captures the thermal dynamics of vegetation combustion. To account for the spatial spread of fire, this framework was initially extended through its coupling with Porterie?s small-world network approach [3]. While this ODE model exhibited remarkable agreement with experimental data, a crucial theoretical limitation persisted: the existence and uniqueness of its solution had never been rigorously proven. The absence of such a mathematical guarantee posed a fundamental barrier to the comprehensive validation and numerical reliability of the model.

We therefore set out to address this gap by formally establishing the existence and uniqueness of the solution to this ODE. Our proof, presented in Section 3, relies on the Cauchy? Lipschitz theorem and a careful analysis of the Lipschitz continuity of the source term representing thermal transfer.

During this analysis, we identified a fundamental limitation of the original model: the spatial positions of the cells and the advection term (transport flux) were not directly represented in the ODE, being captured only indirectly through the coupling with the small-world network. To explicitly incorporate these critical aspects for a realistic modeling of fire propagation, we undertook the transformation of the ODE into a partial differential equation (PDE).

Building on foundational work in reactive transport [5, 6], we introduced a particle derivative to describe heat transport while retaining the source term for thermal balance. This enabled the formulation of a transport?reaction PDE with physically realistic boundary conditions. We then extended the existence and uniqueness proof to this PDE using the method of characteristics, providing a rigorous mathematical foundation that combines the original model's thermal dynamics with explicit spatial transport.

2. ODE-based Propagation Model by Adou et al

2.1. Brief Presentation of the Model

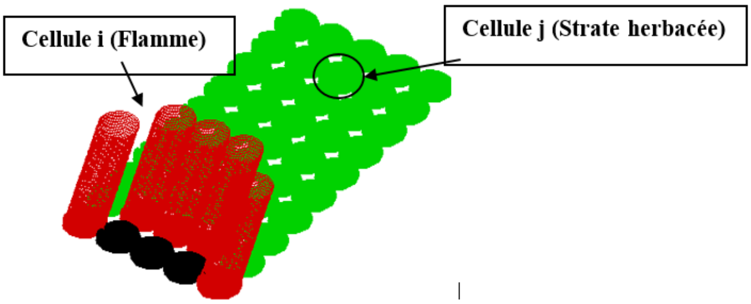

Figure 1. *Schematic representation of the solid flame model, showing a burning cell i and an adjacent healthy cell j within the network.*

The present model [1], is constructed from a two-dimensional regular network of equally sized cells (Figure. 1). It is assumed that each combustible cell j has a cylindrical shape and a temperature $T_j(t)$. Readers interested in the detailed formulation of the model can refer to [1, 5, 7, 8 9].

The total energy q_j received by cell j through convection and radiation from the N_j burning cells is modeled by the following differential equation:

$$\begin{cases} \dfrac{dT_j(t)}{dt} = \displaystyle\sum_{i=1}^{N_j} S_{ij,\theta}(T_j(t)) \\ T_j(0) = T_\infty \end{cases} \tag{1}$$

where the source term is given by:

$$\sum_{i=1}^{N_j} S_{ij,\theta}(T_j(t)) = a_{fb} q_j^{SR} + \varepsilon_b T_b q_j^{IR} + k_{fl} K_1 (T_{fl} - T_j) + k_b K_2 (T_b - T_j) - \varepsilon_{fb} K_3 (T_j^4 - T_\infty^4) \tag{2}$$

Definition 2.1 (Fundamental parameter θ). The vectors $\theta = (a_{fb}, T_{fl}, \varepsilon_b, \varepsilon_{fb}, k_{fl}, k_b, T_b)^\top \in \Omega \subset \mathbb{R}^7$ and $K = (K_1, K_2, K_3)^\top \in \Omega \subset \mathbb{R}^3$ are parameters containing constants that have a significant influence on fire propagation. The source function is defined as:

$$\sum_{i=1}^{N_j} S_{ij,\theta} : [T^\infty, T_{fl}] \times [t_0, t_{\max}] \to \mathbb{R}, \quad (T_j, t) \mapsto \sum_{i=1}^{N_j} \frac{1}{\rho_j C_{pj} \phi_j} q_{ij} \tag{3}$$

where:

1. ρ_j : density of the fuel in cell j
2. C_{pj} : specific heat capacity of the fuel
3. ϕ_j : geometric shape factor of the cell
4. q_{ij} : heat flux received from the neighboring burning cell i

These physical parameters determine the fire propagation dynamics in the model.

The domain of definition is $T_j \in [T^\infty, T_{fl}]$ and $t \in [t_0, t_{\max}]$, where T^∞ is the ambient temperature and T_{fl} is the ignition temperature [10,11].

2.2. Existence and Uniqueness for the ODE

2.2.1. Theoretical Background

Theorem 2.1 (Cauchy-Lipschitz Theorem for ODEs). Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function, locally Lipschitz with respect to its second variable. Then, for any initial condition (t_0, y_0), there exists a unique maximal solution to the Cauchy problem:

$$\begin{cases} y'(t) = F(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

defined on a maximal interval.

2.2.2. Lipschitz Continuity of the Source Term

Theorem 2.2. The function $F(T_j) := \sum_{i=1}^{N_j} S_{ij,\theta}(T_j)$ is Lipschitz continuous in T_j on the compact interval $[T^\infty, T_{fl}]$.

Proof: Consider $T_{1j}, T_{2j} \in [T^\infty, T_{fl}]$. From (2), we have:

$$\begin{aligned} &|F(T_{1j}) - F(T_{2j})| \\ &= \big|\big[k_{fl}K_1(T_{fl} - T_{1j}) + k_bK_2(T_b - T_{1j}) - \varepsilon_{fb}K_3(T_{1j}^4 - T_\infty^4)\big] \\ &\quad - \big[k_{fl}K_1(T_{fl} - T_{2j}) + k_bK_2(T_b - T_{2j}) - \varepsilon_{fb}K_3(T_{2j}^4 - T_\infty^4)\big]\big| \\ &= \big|k_{fl}K_1(T_{2j} - T_{1j}) + k_bK_2(T_{2j} - T_{1j}) + \varepsilon_{fb}K_3(T_{2j}^4 - T_{1j}^4)\big| \end{aligned}$$

Applying the mean value inequality to the function $f(x) = x^4$ on $[T^\infty, T_{fl}]$:

$$|T_{2j}^4 - T_{1j}^4| = |f'(c)| \cdot |T_{2j} - T_{1j}| = 4c^3|T_{2j} - T_{1j}| \leq 4T_{fl}^3|T_{2j} - T_{1j}|$$

where $c \in [\min(T_{1j}, T_{2j}), \max(T_{1j}, T_{2j})] \subset [T^\infty, T_{fl}]$.

Let us now bound the expression:

$$\begin{aligned} |F(T_{1j}) - F(T_{2j})| &\leq k_{fl}K_1|T_{2j} - T_{1j}| + k_bK_2|T_{2j} - T_{1j}| \\ &\quad + \varepsilon_{fb}K_3 \cdot 4T_{fl}^3|T_{2j} - T_{1j}| \\ &= \left(k_{fl}K_1 + k_bK_2 + 4\varepsilon_{fb}K_3T_{fl}^3\right)|T_{2j} - T_{1j}| \end{aligned} \tag{4}$$

Thus, F is Lipschitz continuous with constant:

$$L = k_{fl}K_1 + k_bK_2 + 4\varepsilon_{fb}K_3T_{fl}^3 \qquad 0 < L < 1$$

2.2.3. Existence and Uniqueness Theorem

Theorem 2.3. The ODE (1) admits a unique solution $T_j \in C^1([t_0, t_{\max}])$ taking values in $[T^\infty, T_{fl}]$.

Proof: Applying the Cauchy-Lipschitz theorem, by construction, F is continuous on $[T^\infty, T_{fl}]$ (as a sum of continuous functions). The previous proof established that F is Lipschitz continuous on this compact interval.

The Cauchy-Lipschitz theorem thus guarantees the existence of a unique maximal local solution. Since F is Lipschitz continuous on the compact interval $[T^\infty, T_{fl}]$, this solution is global on $[t_0, t_{\max}]$ and remains within $[T^\infty, T_{fl}]$ by the physical construction of the model.

3. New Propagation Model via a PDE

3.1. Functional Spaces

In this mathematical analysis, we use several essential functional spaces to rigorously formulate our problem and to prove the existence and uniqueness of solutions.

Definition 3.1 (Space L^∞). Set $\Omega \subset \mathbb{R}^n$ a measurable domain. The space of essentially bounded functions is defined by:

$$L^\infty(\Omega) = \Big\{ f : \Omega \to \mathbb{R} \text{ mesurable } \Big| \exists C > 0,\ |f(x)| \leq C, \text{ pour presque tout } x \in \Omega \Big\} \tag{5}$$

with the norm $\|f\|_{L^\infty(\Omega)} = \inf\{C \geq 0 \mid |f(x)| \leq C \text{ p.p. on } \Omega\}$.

Definition 3.2 (Sobolev space $W^{1,\infty}$). The Sobolev space $W^{1,\infty}(\Omega)$ is defined as:

$$W^{1,\infty}(\Omega) = \{f \in L^\infty(\Omega) \mid D^\alpha f \in L^\infty(\Omega), |\alpha| \leq 1\}$$

$$W^{1,\infty}(\Omega) = \{f \in L^\infty(\Omega) \mid \nabla f \in L^\infty(\Omega; \mathbb{R}^n)\}$$

with the norm $\|f\|_{W^{1,\infty}(\Omega)} = \max\left\{\|f\|_{L^\infty(\Omega)}, \|\nabla f\|_{L^\infty(\Omega)}\right\}$.

These functional spaces are particularly well suited to the study of our fire propagation model, as they precisely capture the regularity required for the velocity field, the initial temperature, the boundary conditions, and the solution of the problem.

3.2. Existence and Uniqueness Analysis of the Transport? Reaction PDE Model

We consider, on a domain $\Omega \subset \mathbb{R}^2$ and for $t \in [0, t_{\max}]$, the equation

$$\partial_t T(\vec{x}, t) + \vec{u}(\vec{x}, t) \cdot \nabla T(\vec{x}, t) = F(T(\vec{x}, t)), \tag{6}$$

where $F(T) = \sum_{i=1}^{N_j} S_{ij,\theta}(T)$ is the source term defined in §2.

Assumptions

(H1) $\vec{u} \in W^{1,\infty}(\Omega \times [0, t_{\max}]; \mathbb{R}^2)$ (bounded and Lipschitz continuous in $\vec{x}$, measurable in t).

(H2) $F : [T^\infty, T_{fl}] \to \mathbb{R}$ is Lipschitz continuous with constant L, $0 < L < 1$ (see the Lipschitz Theorem in Section 3.2).

(H3) *Interval invariance*: $F(T^\infty) \geq 0$ and $F(T_{fl}) \leq 0$.

(H4) Bounded data: $T_0 \in L^\infty(\Omega)$ with $T^\infty \leq T_0(\vec{x}) \leq T_{fl}$ almost everywhere.

The well-posed problem is:

$$\partial_t T(\vec{x}, t) + \vec{u}(\vec{x}, t) \cdot \nabla T(\vec{x}, t) = F(T(\vec{x}, t)) \quad \text{dans } \Omega \times (0, t_{\max}),$$

with the initial condition:

$$T(\vec{x}, 0) = T_0(\vec{x}) \quad \text{dans } \Omega,$$

where $F(T) = \sum_{i=1}^{N_j} S_{ij,\theta}(T)$ is the source term, and under assumptions (H1)–(H4).

3.2.1. Method of Characteristics

The method of characteristics transforms the PDE into a system of ordinary differential equations (ODE) along curves called characteristics.

The characteristics are defined by the solutions of:

$$\frac{d\vec{X}}{dt} = \vec{u}(\vec{X}(t), t),$$

with the initial condition:

$$\vec{X}(0) = \vec{x}_0 \quad \text{for } \vec{x}_0 \in \Omega.$$

Under assumption (H1), the vector field $\vec{u}$ is Lipschitz continuous in $\vec{x}$ and measurable in t. By the Cauchy-Lipschitz theorem (Caratheodory version), there exists a unique absolute solution $\vec{X}(t; \vec{x}_0)$ for almost every $\vec{x}_0$.

Along the characteristic $\vec{X}(t; \vec{x}_0)$, the PDE becomes:

$$\frac{d}{dt} T(\vec{X}(t; \vec{x}_0), t) = F(T(\vec{X}(t; \vec{x}_0), t)),$$

with the initial condition:

$$T(\vec{X}(0; \vec{x}_0), 0) = T_0(\vec{x}_0).$$

Thus, we obtain an ODE for T along each characteristic:

$$\frac{d\phi}{dt} = F(\phi), \quad \phi(0) = T_0(\vec{x}_0),$$

where $\phi(t) = T(\vec{X}(t; \vec{x}_0), t)$.

indent Under assumption (H2), F is Lipschitz continuous with constant $L < 1$. By the Cauchy-Lipschitz theorem, the ODE admits a unique solution $\phi(t; T_0(\vec{x}_0))$ for each initial condition.

Moreover, assumption (H3) ensures the invariance of the interval $[T^\infty, T_{fl}]$: if $\phi(0) \in [T^\infty, T_{fl}]$, then $\phi(t) \in [T^\infty, T_{fl}]$ for all $t \geq 0$. This follows from the fact that $F(T^\infty) \geq 0$ and $F(T_{fl}) \leq 0$, which implies that the flux is directed inward toward the interval.

For all $(\vec{x}, t) \in \Omega \times [0, t_{\max}]$, we define the solution $T(\vec{x}, t)$ by inverting the characteristic flow:

1. Set $\vec{Y}(s; \vec{x}, t)$ the solution of :

$$\frac{d\vec{Y}}{ds} = \vec{u}(\vec{Y}(s), s), \quad \vec{Y}(t) = \vec{x}.$$

2. we denote $\vec{x}_0 = \vec{Y}(0; \vec{x}, t)$ the initial point of the characteristic passing through $(\vec{x}, t)$.
3. Then, we set:

$$T(\vec{x}, t) = \phi(t; T_0(\vec{x}_0)),$$

where ϕ is the solution of the ODE $\frac{d\phi}{ds} = F(\phi)$ with $\phi(0) = T_0(\vec{x}_0)$.

Under assumption (H1), the flow $\vec{Y}$ is Lipschitz continuous in $\vec{x}$ and invertible. Thus, the construction is well-defined.

3.2.2. Proof of Existence and Uniqueness

The function T defined above satisfies:

1. $T(\vec{x}, 0) = T_0(\vec{x})$ by construction.
2. Along any characteristic, T satisfies $\partial_t T + \vec{u} \cdot \nabla T = F(T)$ by the chain rule.
3. Thanks to (H4) and (H3), $T_0(\vec{x}_0) \in [T^\infty, T_{fl}]$ almost everywhere, hence $T(\vec{x}, t) \in [T^\infty, T_{fl}]$ almost everywhere by interval invariance.

Thus, T is a bounded solution of the PDE with the initial condition.

Suppose that T_1 and T_2 are two bounded solutions of the PDE with the same initial condition. For almost every $\vec{x}_0 \in \Omega$, along the characteristic $\vec{X}(t;\vec{x}_0)$, the functions $t \mapsto T_1(\vec{X}(t), t)$ and $t \mapsto T_2(\vec{X}(t), t)$ satisfy the same ODE:

$$\frac{d\phi}{dt} = F(\phi), \quad \phi(0) = T_0(\vec{x}_0).$$

By the uniqueness of the ODE solution (assumption (H2)), we have $T_1(\vec{X}(t), t) = T_2(\vec{X}(t), t)$ for all t. Since the characteristics cover $\Omega \times [0, t_{\max}]$ (invertibility of the flow), we conclude that $T_1 = T_2$ p.p .

Under assumptions (H1)–(H4), the transport–reaction PDE admits a unique solution $T \in L^\infty(\Omega \times [0, t_{\max}])$ given by the method of characteristics. Moreover, this solution satisfies $T^\infty \leq T(\vec{x}, t) \leq T_{fl}$ almost everywhere in $\Omega \times [0, t_{\max}]$.

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Abbreviations

PDE	Partial Differential Equation
ODE	Ordinary Differential Equation

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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