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Research Article

A New Generalization of Exponential-Exponential Distribution Using Kumaraswamy-G Family of Distribution: Theory and Application to Cancer Data

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Abstract

In recent years so many discrete and continuous distributions are derived in the literature using the technique of different family of distributions to generalize or extend. The aim of this article is to propose a new generalization of exponential-exponential distribution by the method of Kumaraswamy G (Kw-G) family of distributions. The new proposed distribution is a three-parameter distribution and is termed as Kumaraswamy Exponential-Exponential (Kw EED) distribution and derived its probability and survival functions to study the behavior of the plot by various values of parameter. We discussed the properties, incomplete moments, Shannon and Renyi entropy and order statistics. The Lorenz and Bonferroni curve functions are derived with the help of quantile function. Maximum likelihood estimation is used for the estimation of the model parameter. The stochastic ordering of the probability distribution is also included in this work. A simulation study is used to find the parameter values. The superiority and adaptability of the suggested model is described by comparing it with other models using some model criterions with the help of cancer data. The findings reveals that the proposed model is better fit than other compared models.

Keywords

Kw-G Family of Distributions, Exponential-Exponential Distribution, Survival Function, Order Statistic, Maximum Likelihood Estimation

1. Introduction

The statistics literature is filled with numerous lifetime distributions. New methods for constructing significant lifetime distributions are the main focus of the authors in recent years. There has been an increased interest in defining new generators or generalized classes of univariate continuous distributions by introducing additional shape parameter (s) to a baseline model. The extended distributions have attracted several statisticians to develop new models because the computational and analytical facilities available in programming software's such as R, Maple and Mathematica can easily tackle the problems involved in computing special functions in these extended distributions. The examples of well-known model generators are Kumaraswamy-G (Kw-G) by [1], exponentiated T-X by [2], transformed-transformer (T-X)

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(Weibull-X and gamma-X) by [3]. The T-X$^{\theta}$ family, by [4] covers discretization procedures and defines discrete sub-families under that framework including discrete T-X models.

In many applied areas like lifetime analysis, finance, insurance and biology, there is a clear need for extended forms of the classical distributions, i.e., new distributions more flexible to model real data. Recent developments focus on new techniques by adding parameters to existing distributions for building classes of more flexible distributions. Following this idea, [1] introduced an interesting method by adding two new parameters to a parent distribution to model data with a high degree of skewness and kurtosis. The five-parameter Kumaraswamy modified Weibull (Kw MW) distribution which seems to be superior over the MW model for certain applications are discussed in [4]. An extended version of Kumaraswamy inverse Weibull distribution and its properties is discussed in [5]. Beta generated Kumaraswamy-G family of distributions, in which an infinite mixture of the Kumaraswamy- G distribution is discussed in [6]. The Kumaraswamy Marshal Olkin family of distributions are proposed and discussed by [7]. Kumaraswamy Weibull distribution and its application to failure data was proposed by [8]. Therefore, in this article we proposed a new three parameter Exponential-Exponential distribution by [9] and applying Kumaraswamy G family of distributions. Using two data sets, from biomedical origin have demonstrated the novel distribution's applicability in real world scenarios. The Kumaraswamy odd log-logistic family is a flexible generator-based class of distributions with diverse hazard rate shapes and applications to lifetime data which is discussed by [21]. Some of the Kumaraswamy G family of distributions were recently developed and explains its application in the field of insurance, radiation data, cancer data, heavy rainfall and risk data etc. These applications and development of theory, properties related to each distribution are discussed in [10-15].

The paper is organized as follows. Section 2 presents the definition and some important properties of Kumaraswamy Exponential-Exponential distribution. Section 3 discusses certain reliability aspects of the distribution, while some statistical properties of the distribution and generating functions are considered in Section 4. The distribution of first order and nth order statistics is discussed in section 5 and Section 6 discusses the entropy measures. The quantile function, Lorenz curve and Bonferroni curve were discussed in section 7. We discuss the stochastic ordering in section 8. The parameter estimation by the method of maximum likelihood and Fisher Information matrix is included in section 9. In Section 10, we simulated different random sample data with various values of the parameter and check the validity with MSE and Bias. In the next subsection of the real-life application, the validity of the model is illustrated using cancer data sets and the efficiency of the Kw EE distribution is compared with that of various other distributions with the help of certain information measures.

2. Kumaraswamy Exponential-Exponential Distribution

If G (x) is the cumulative distribution function (cdf) of a baseline model, then the Kumaraswamy generalized (Kw-G) family by [1] has cdf:

$$F(x) = 1 - \{1 - G(x)^{\alpha}\}^{\beta} \tag{1}$$

where x ≥ 0, and α, β ≥ 0.

The probability density function (pdf) corresponding to (1) is given by

$$f(x) = \alpha\beta\ g(x)\ G(x)^{\alpha-1}\ \{1 - G(x)^{\alpha}\}^{\beta-1} \tag{2}$$

where x ≥ 0, and α, β ≥ 0.

Each new Kw-G distribution can be obtained from a specified G distribution. For α = β = 1, the G distribution is a basic exemplar of the Kw-G family with a continuous crossover towards cases with different shapes. One major benefit of equation (2) is its ability of fitting skewed data that cannot be properly fitted by existing distributions. Further, it allows for greater flexibility of its tails and can be widely applied in many areas of reliability and biology.

The PDF and CDF of the Exponential-Exponential distribution (EED) by [9] is given

$$f(x) = \lambda^2 \exp(-\lambda^2 x) \tag{3}$$

$$F(x) = 1 - \exp(-\lambda^2 x) \tag{4}$$

Where $x > 0, \lambda > 0$.

The new distribution is obtained by incorporating (3) and (4) in equations (1) and (2).

2.1. Definition

A continuous random variable X, $X \geq 0$ with cdf

$$F(x) = 1 - \{1 - (1 - \exp(-\lambda^2 x))^{\alpha}\}^{\beta}, \tag{5}$$

for β > 0, α > 0 and λ > 0

is said to have the "Kumaraswamy Exponential-Exponential Distribution (Kw EED)". Here after we denote the distribution as Kw EED for convenience.

The corresponding pdf is

$$f(x) = \alpha\beta\lambda^2\ \exp(-\lambda^2 x)\ (1 - \exp(-\lambda^2 x))^{\alpha-1} \{1 - (1 - \exp(-\lambda^2 x))^{\alpha}\}^{\beta-1}, \tag{6}$$

for $\alpha > 0, \beta > 0, \lambda > 0.$

2.2. General Expansion for Density Function

According to [1], the general expansion of the Kumaraswamy G family of distribution, for $\beta > 0$ real non-integer, the form of the distribution

$$\{1 - G(x)^{\alpha}\}^{\beta-1} = \sum_{i=0}^{\infty}(-1)^i \binom{\beta-1}{i} G(x)^{\alpha_i} \qquad (7)$$

where the binomial coefficient [16] is defined for any real. Using the above expansion and formula (7), we can rewrite the Kw EED density as follows

$$f(x) = \alpha\beta\lambda^2 \sum_{i=0}^{\infty}(-1)^i \binom{\beta-1}{i} \exp(-\lambda^2 x) \; (1 - \exp(-\lambda^2 x))^{\alpha(i+1)-1} \qquad (8)$$

3. Reliability Analysis

In this section the expressions for the survival function, hazard function, reverse hazard function and Mills ratio are presented. The survival function or reliability function

$$S(x) = \{1 - (1 - \exp(-\lambda^2 x))^{\alpha}\}^{\beta} \qquad (9)$$

The hazard function

$$h(x) = \frac{\alpha\beta\lambda^2 \exp(-\lambda^2 x) \left(1 - \exp(-\lambda^2 x)\right)^{\alpha-1}}{[1 - (1 - \exp(-\lambda^2 x))^{\alpha}]} \qquad (10)$$

The reverse hazard function

$$h_r(x) = \frac{\alpha\beta\lambda^2 \exp(-\lambda^2 x) \left(1 - \exp(-\lambda^2 x)\right)^{\alpha-1} \left\{1 - \left(1 - \exp(-\lambda^2 x)\right)^{\alpha}\right\}^{\beta-1}}{1 - \{1 - (1 - \exp(-\lambda^2 x))^{\alpha}\}^{\beta}} \qquad (11)$$

The Mills Ratio

$$\frac{1}{h_r(x)} = \frac{1 - \left\{1 - \left(1 - \exp(-\lambda^2 x)\right)^{\alpha}\right\}^{\beta}}{\alpha\beta\lambda^2 \exp(-\lambda^2 x) \; (1 - \exp(-\lambda^2 x))^{\alpha-1} \; \{1 - (1-\exp(-\lambda^2 x))^{\alpha}\}^{\beta-1}} \qquad (12)$$

4. Statistical Properties

Here we present the incomplete moments, moment generating function and characteristic function of the proposed distribution.

Theorem 4.1. The r[th] moment about the origin of Kw EE distribution is given by

$$E(X^r) = \mu_r' = \frac{\alpha\beta}{\lambda^{2r}} \sum_{i=0}^{\infty}(-1)^i \binom{\beta-1}{i} \int_0^1 ln\left(\frac{1}{1-t}\right)^r \frac{t^{\alpha(i+1)-1}}{1-t} \; dt, \; r = 1, 2, 3, \ldots \qquad (13)$$

Proof:

The r[th] moment about the origin of Kw EE distribution is given by:

$$E(X^r) = \int_0^{\infty} x^r \; f(x) \; dx$$

$$E(X^r) = \alpha\beta\lambda^2 \sum_{i=0}^{\infty}(-1)^i \binom{\beta-1}{i} \int_0^{\infty} x^r \exp(-\lambda^2 x) \; (1 - \exp(-\lambda^2 x))^{\alpha(i+1)-1} dx$$

by substituting, $t = 1 - \exp(-\lambda^2 x) \Rightarrow$

$$x = -\frac{1}{\lambda^2}\ln(1-t)$$

$$E(X^r) = \frac{\alpha\beta}{\lambda^{2r}} \sum_{i=0}^{\infty}(-1)^i \binom{\beta-1}{i} \int_0^1 ln\left(\frac{1}{1-t}\right)^r \frac{t^{\alpha(i+1)-1}}{1-t} \; dt$$

Clearly, we can see that it is an incomplete moment. The r[th] moment of Kw EED involves complex multiple nested summations and integrals combining binomial, power of logarithms and beta type integrals. We can calculate the moments numerically with the help of R software.

Theorem 4.2. The Moment Generating Function and characteristic function of Kw EE distribution are given by

$$M_X(t) = \alpha\beta \sum_{k=0}^{\infty}(-1)^k \binom{\beta-1}{k} \frac{\Gamma(\alpha(k+1))\Gamma\left(-\frac{t}{\lambda^2}\right)}{\Gamma\left(\alpha(k+1)-\frac{t}{\lambda^2}\right)} \text{ and}$$ 5)

$$\phi_X(t) = \alpha\beta \sum_{k=0}^{\infty}(-1)^k \binom{\beta-1}{k} \frac{\Gamma(\alpha(k+1))\Gamma\left(-\frac{it}{\lambda^2}\right)}{\Gamma\left(\alpha(k+1)-\frac{it}{\lambda^2}\right)}.$$

Proof:

The moment generating function of Kw EE distribution is given by:

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x)\, dx$$

by substituting, $u = 1 - e^{-\lambda^2 x} \Rightarrow x = -\frac{1}{\lambda^2}\ln(1-u)$

$$M_X(t) = \alpha\beta \int_0^1 u^{\alpha-1} (1-u)^{-\frac{t}{\lambda^2}-1} (1-u^{\alpha})^{\beta-1}$$

Using binomial series expansion,

$$(1-u^{\alpha})^{\beta-1} = \sum_{k=0}^{\infty}(-1)^k \binom{\beta-1}{k} u^{\alpha k}$$

$$M_X(t) = \alpha\beta \sum_{k=0}^{\infty}(-1)^k \binom{\beta-1}{k} \int_0^1 u^{\alpha(k+1)-1} (1-u)^{-\frac{t}{\lambda^2}-1} du$$

Since, $\int_0^1 u^{\alpha(k+1)-1} (1-u)^{-\frac{t}{\lambda^2}-1} du = B\left(\alpha(K+1), -\frac{t}{\lambda^2}\right)$

$$M_X(t) = \alpha\beta \sum_{k=0}^{\infty}(-1)^k \binom{\beta-1}{k} \frac{\Gamma(\alpha(k+1))\Gamma\left(-\frac{t}{\lambda^2}\right)}{\Gamma\left(\alpha(k+1)-\frac{t}{\lambda^2}\right)} \quad (14)$$

The corresponding characteristic function is

$$\phi_X(t) = \alpha\beta \sum_{k=0}^{\infty}(-1)^k \binom{\beta-1}{k} \frac{\Gamma(\alpha(k+1))\Gamma\left(-\frac{it}{\lambda^2}\right)}{\Gamma\left(\alpha(k+1)-\frac{it}{\lambda^2}\right)} \quad (1$$

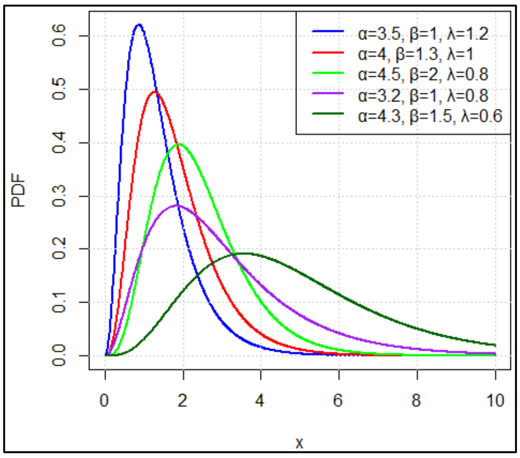

Figure 1. *The PDF Plot of Kw EED with Various Values of the Parameters α, β and λ.*

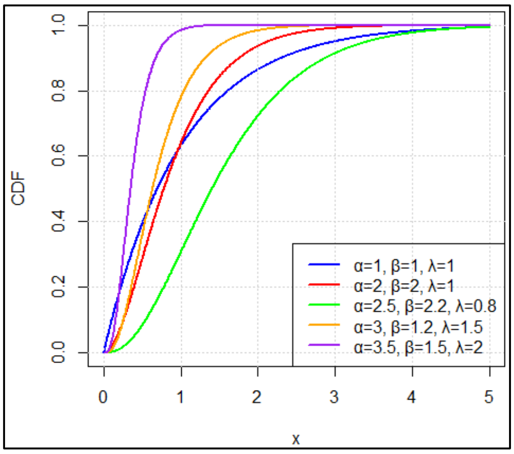

Figure 2. *The CDF Plot of Kw EED with Various Values of the Parameters α, β and λ.*

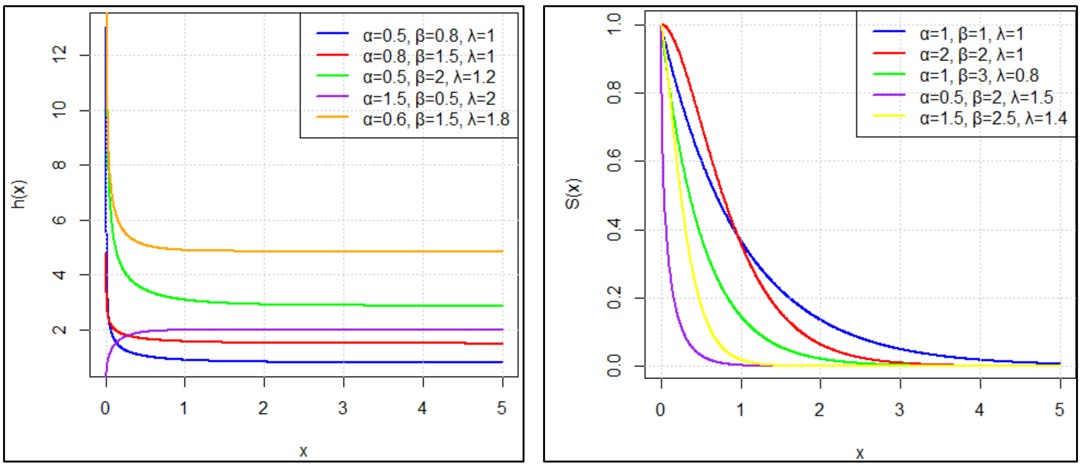

Figure 3. *The Survival Function (Left) and Hazard Function (Right) of Kw EED with Various Values of the Parameter α, β and λ.*

5. Distribution of Order Statistics

Order statistics make their appearance in many statistical theory and practice. We know that if $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, \ldots, X_n$ from a continuous population with cdf $F_x(x)$ and pdf $f_x(x)$, then the pdf of r^{th} order statistics $X_{(r)}$ is

$$f_{X(r)}(x) = \frac{n!}{(r-1)!(n-r)!} . f_x(x). [F_x(x)]^{r-1} \ [1 - F_x(x)]^{n-r} \quad (16)$$

By substituting the pdf and cdf of Kw EE distribution in equation (16) we obtain densities of the first order and n^{th} order statistics.

The first order statistic $X_{(1)}$ is

$$f_{X(1)}(x) = n\alpha\beta\lambda^2 \exp(-\lambda^2 x) \ (1 - \exp(-\lambda^2 x))^{\alpha\beta(n-1)+\alpha-1}$$

$$\{1 - (1 - \exp(-\lambda^2 x))^{\alpha}\}^{\beta-1}$$

Also, the n^{th} order statistic $X_{(n)}$ *is*

$$f_{X(n)}(x) = n\alpha\beta\lambda^2 \exp(-\lambda^2 x) \ \left(1 - e^{-\lambda^2 x}\right)^{\alpha-1} \left\{1 - \left(1 - e^{-\lambda^2 x}\right)^{\alpha}\right\}^{n\beta-1}$$

6. Entropy Measures

The entropy of a random variable X is a measure of variation of the uncertainty. It plays important role in various fields, including probability and statistics, physics communication theory and economics by quantifying diversity and unpredictability. A system with higher entropy indicates greater uncertainty and more uniform outcomes. In this section we include Renyi entropy and Shannon entropy.

6.1. Shannon Entropy

The Shannon entropy of the proposed distribution is

$$S_{\mu} = -\int_0^{\infty} f(x) \ln\big(f(x)\big)\, dx$$

$$S_{\mu} = -\int_0^{\infty} \alpha\beta\lambda^2 \ \exp(-\lambda^2 x) \ \ (1 - \exp(-\lambda^2 x))^{\alpha-1} \ \{1 - (1 - \exp(-\lambda^2 x))^{\alpha}\}^{\beta-1} \ln\big(\alpha\beta\lambda^2 \ \exp(-\lambda^2 x) \ (1 - \exp(-\lambda^2 x))^{\alpha-1} \ \{1 - (1 - \exp(-\lambda^2 x))^{\alpha}\}^{\beta-1}\big)\, dx$$

Let's, simplify $\ln\big(f(x)\big)$ of equation (19)

$$\ln\big(f(x)\big) = \ln(\alpha\beta\lambda^2) - \lambda^2 x + (\alpha-1)\ln(1 - \exp(-\lambda^2 x)\ ) + (\beta-1) \ \ln(1 - (1 - \exp(-\lambda^2 x))^{\alpha})$$

Converting each logarithm into series. We obtain

$$\ln(1 - \exp(-\lambda^2 x)\ ) = -\sum_{k=1}^{\infty} \frac{e^{-\lambda^2 kx}}{k}$$

$$\ln(1 - (1 - \exp(-\lambda^2 x))^{\alpha}) = -\sum_{j=1}^{\infty} \frac{(1 - \exp(-\lambda^2 x))^{\alpha j}}{j}$$

$$\ln\big(f(x)\big) = \ln(\alpha\beta\lambda^2) - \lambda^2 x - (\alpha-1) \ \sum_{k=1}^{\infty} \frac{\exp(-\lambda^2 kx)}{k} - (\beta-1) \ \sum_{j=1}^{\infty} \frac{(1 - \exp(-\lambda^2 x))^{\alpha j}}{j}$$

Substituting all the values in equation (19), we obtain the expression for Shannon entropy of Kw EE distribution as,

$$S_{\mu} = -\ln(\alpha\beta\lambda^2) + \lambda^2 E(x) + (\alpha-1) \ \sum_{k=1}^{\infty} \frac{\mathrm{E}(\exp(-\lambda^2 kx))}{k} \ (\beta-1)\sum_{j=1}^{\infty} \frac{E\ (1 - \exp(-\lambda^2 x))^{\alpha j}}{j} \quad (17)$$

6.2. Renyi Entropy

For a given probability distribution, Renyi entropy is given by,

$$R_{\mu}\big(f(x)\big) = \frac{1}{1-\mu} \ln \int_0^{\infty} [f(x)]^{\mu} \ dx$$

$$f(x)^{\mu} = (\alpha\beta\lambda^2)^{\mu} \ \exp(-\lambda^2 \mu x) \ \ (1 - \exp(-\lambda^2 x))^{\mu(\alpha-1)} \ \{1 - (1 - \exp(-\lambda^2 x))^{\alpha}\}^{\mu(\beta-1)}$$

expanding each term of the above equation using binomial expansion

$$(1-\exp(-\lambda^2 x))^{\mu(\alpha-1)} = \sum_{m=0}^{\infty}\binom{\mu(\alpha-1)}{m}(-1)^m \exp(-\lambda^2 m x)$$

$$\{1-(1-\exp(-\lambda^2 x))^{\alpha}\}^{\mu(\beta-1)} = \sum_{n=0}^{\infty}\binom{\mu(\beta-1)}{n}(-1)^n(1-\exp(-\lambda^2 x))^{\alpha n}$$

$$\text{Let } u = 1-\exp(-\lambda^2 x)$$

$$(1-u^{\alpha})^{\mu(\beta-1)} = \sum_{n=0}^{\infty}(-1)^n\binom{\mu(\beta-1)}{n}u^{\alpha n} \text{ and } (1-\exp(-\lambda^2 x))^{\alpha n} = \sum_{r=0}^{\infty}\binom{\alpha n}{r}(-1)^r \exp(-r\lambda^2 x)$$

$$\int_0^{\infty} f(x)^{\mu} dx = (\alpha\beta\lambda^2)^{\mu}\sum_{m,n,r=0}^{\infty}(-1)^{m+n+r}\binom{\mu(\alpha-1)}{m}\binom{\mu(\beta-1)}{n}\binom{\alpha n}{r}$$

$$\int_0^{\infty}\exp(-(\mu+m+r)\lambda^2 x\, d$$

The Renyi entropy of Kw EE distribution is

$$R_{\mu}(f(x)) = \frac{1}{1-\mu}\ln\left\{\frac{(\alpha\beta\lambda^2)^{\mu}}{\lambda^2}\sum_{m,n,r=0}^{\infty}\frac{(-1)^{m+n+r}\binom{\mu(\alpha-1)}{m}\binom{\mu(\beta-1)}{n}\binom{\alpha n}{r}}{\mu+m+r}\right\} \quad (18)$$

7. Quantile Function, Lorenz Curve and Bonferroni Curve

The Bonferroni and Lorenz curve have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medical science.

In addition to being used in economics to examine income and poverty, the Bonferroni and Lorenz curves are also used in other domains such as reliability, demography, insurance, and medical science. Here we derived Lorenz curve using quantile function which is described by [17], the quantile version of Lorenz curve.

For a given distribution function, the quantile function is given by $Q(z) = F^{-1}(z)$.

The quantile function of Kw EE distribution is

$$Q(z) = \frac{1}{\alpha\lambda^2}\ln\left[\frac{1}{1-z^{\frac{1}{\beta}}}\right] \quad (19)$$

The Lorenz curve function of Kw EE distribution is

$$L(p) = \frac{1}{\mu}\int_0^p Q(z)dz$$

$$L(p) = \frac{1}{\mu}\int_0^p \frac{1}{\alpha\lambda^2}\ln\left[\frac{1}{1-z^{\frac{1}{\beta}}}\right] dz$$

$$L(p) = \frac{1}{\mu\alpha\lambda^2}\sum_{n=1}^{\infty}\frac{p^{1+\frac{n}{\beta}}}{n(1+\frac{n}{\beta})} \quad (20)$$

The Bonferroni curve function of Kw EE distribution is

$$B(p) = \frac{L(p)}{p}$$

$$B(p) = \frac{1}{\mu\alpha\lambda^2}\sum_{n=1}^{\infty}\frac{p^{\frac{n}{\beta}}}{n(1+\frac{n}{\beta})} \quad (21)$$

8. Stochastic Ordering

The stochastic ordering of a non-negative continuous random variable is a tool for comparing the behaviour of components. Let X and Y be two random variables. If the random variable X is said to be smaller than another random variable Y in the

1) Stochastic order ($X \leq_{st} Y$), if $F_X(x) \geq F_Y(x)$ for all x.
2) Hazard rate order ($X \leq_{hr} Y$), if $F_X(x) \geq F_Y(x)$ for all x.
3) Likelihood ratio order ($X \leq_{lr} Y$), if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

The following stochastic ordering relationships due to [18] are well known for establishing stochastic ordering for continuous distributions.

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y$$

Theorem 9.1: Let $X \sim KwEE\ (\alpha_1, \beta_1, \lambda_1)$ and $Y \sim KwEE\ (\alpha_2, \beta_2, \lambda_2)$, then,

If $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, and $\lambda_1 > \lambda_2$, then $X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y$

If $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$, and $\lambda_1 = \lambda_2$, then $X \leq_{lr} Y \Rightarrow X \leq_{hr}$

Y ⇒ $X \leq_{st}$ Y *Proof:*

$$\frac{f_X(x)}{f_Y(x)} = \frac{\alpha_1\beta_1\lambda_1^2 \exp(-\lambda_1^2 x)\ (1-\exp(-\lambda_1^2 x))^{\alpha_1-1}\ \{1-(1-\exp(-\lambda_1^2 x))^{\alpha_1}\}^{\beta_1-1}}{\alpha_2\beta_2\lambda_2^2\ \exp(-\lambda_2^2 x)\ (1-\exp(-\lambda_2^2 x))^{\alpha_2-1}\ \{1-(1-\exp(-\lambda_2^2 x))^{\alpha_2}\}^{\beta_2-1}}$$

Let $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and taking log on both sides

$$\ln\frac{f_X(x)}{f_Y(x)} = \ln\lambda_1^2 - \ln\lambda_2^2 - \lambda_1^2 x + \lambda_2^2 x + (\alpha-1)[\ln(1-\exp(-\lambda_1^2 x)) - \ln(1-\exp(-\lambda_2^2 x))] + (\beta-1)[\ln(1-(1-\exp(-\lambda_1^2 x))^{\alpha}) - \ln(1-(1-\exp(-\lambda_2^2 x))^{\alpha})]$$

Differentiating the above ratio w.r.to x and yields

$$\frac{d}{dx}\ ln\ \frac{f_X(x)}{f_Y(x)} = -\lambda_1^2 + \lambda_2^2 + (\alpha-1)\left[\frac{\lambda_1^2\exp(-\lambda_1^2 x)}{(1-\exp(-\lambda_1^2 x))} - \frac{\lambda_2^2\exp(-\lambda_2^2 x)}{(1-\exp(-\lambda_2^2 x))}\right] + (\beta\text{-}1)\left[\frac{\alpha\lambda_1^2(1-\exp(-\lambda_1^2 x))^{\alpha}}{1-(1-\exp(-\lambda_1^2 x))^{\alpha}} - \frac{\alpha\lambda_2^2(1-\exp(-\lambda_2^2 x))^{\alpha}}{1-(1-\exp(-\lambda_2^2 x))^{\alpha}}\right]$$

Since $\lambda_1 > \lambda_2 \Rightarrow -\lambda_1^2 + \lambda_2^2 < 0$

$\frac{d}{dx}\ ln\ \frac{f_X(x)}{f_Y(x)} < 0 \ \Rightarrow\ \frac{f_X(x)}{f_Y(x)}$ is decreasing.

This means that $X \leq_{lr}$ Y, and hence, $X \leq_{lr}$ Y ⇒ $X \leq_{hr}$ Y ⇒ $X \leq_{st}$ Y

Similarly (ii) also can be easily proved.

9. Parameter Estimation and Fisher Information Matrix

In this section we estimate the parameter of Kw EE distribution using method of maximum likelihood. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from Kw EE distribution, then the likelihood function is,

$$L(x;\ \alpha,\beta,\lambda) = \prod_{i=1}^{n} f(x;\ \alpha,\beta,\lambda)$$

$$L(x;\ \alpha,\beta,\lambda) = \prod_{i=1}^{n}\ \alpha\ \beta\lambda^2\ \exp(-\lambda^2 x_i)\ [1-\exp(-\lambda^2 x_i)]^{\alpha-1}\{1-(1-\exp(-\lambda^2 x_i))^{\alpha}\}^{\beta-1}$$

$$L(x;\ \alpha,\beta,\lambda) = (\alpha\beta\ \lambda^2)^n\ \prod_{i=1}^{n}\left[\begin{matrix}1-\\ \exp(-\lambda^2 x_i)\end{matrix}\right]^{\alpha-1} \exp(-\lambda^2 x_i)\ \ \{1-(1-\exp(-\lambda^2 x_i))^{\alpha}\}^{\beta-1} \tag{22}$$

The log likelihood function is given by

$$l(x;\ \alpha,\beta,\lambda) = \ln L(x;\ \alpha,\beta,\lambda)$$

$$l(x;\ \alpha,\beta,\lambda) = n\ln\alpha + 2n\ln\lambda + n\ln\beta + (\alpha-1)\sum_{i=1}^{n}\ln(1-\exp(-\lambda^2 x_i)) - \lambda^2\sum_{i=1}^{n} x_i + (\beta-1)\sum_{i=1}^{n}\ln\{1-(1-\exp(-\lambda^2 x_i))^{\alpha}\} \tag{23}$$

Differentiating the above equation (23) partially w.r.to α, β and λ, and equating to zero, we get the following system of normal equations.

$$\frac{\partial l}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left(1-e^{-\lambda^2 x_i}\right) - \sum_{i=1}^{n}\frac{\alpha\ (\beta-1)(1-\exp(-\lambda^2 x_i))^{\alpha-1}}{1-\left(1-e^{-\lambda^2 x_i}\right)^{\alpha}} = 0 \tag{24}$$

$$\frac{\partial l}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\ln[1-(1-\exp(-\lambda^2 x_i))^{\alpha}] = 0 \tag{25}$$

$$\frac{\partial l}{\partial\lambda} = \frac{2n}{\lambda} - 2\lambda\sum_{i=1}^{n} x_i + 2\lambda\ (\alpha-1)\ \sum_{i=1}^{n}\frac{x_i\exp(-\lambda^2 x_i)}{1-\exp(-\lambda^2 x_i)} + 2\lambda\alpha\ (\beta-1)\ \sum_{i=1}^{n}\frac{x_i\ \exp(-\lambda^2 x_i)\ (1-\exp(-\lambda^2 x_i))^{\alpha-1}}{1-(1-\exp(-\lambda^2 x_i))^{\alpha}} = 0 \tag{26}$$

We denote the MLE's of (α,β,λ) by $(\hat{\alpha},\hat{\beta},\hat{\lambda})$, where these estimations are derived by solving simultaneously the equations $\frac{\partial l}{\partial\alpha} = 0, \frac{\partial l}{\partial\beta} = 0$ and $\frac{\partial l}{\partial\lambda} = 0$. Since the normal equation are non-linear with respect to the parameters, so we cannot obtain the explicit expression of $(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$. Newton Raphson method is applied to solve these equations. Also, any software like R, might be used to solve and obtain the MLE's. Moreover, using the normal equations, we can represent the 3x3 Fisher Information Matrix, $I(\theta)$ as follows,

$$I(\theta) = = \begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha\lambda} \\ I_{\beta\alpha} & I_{\beta\beta} & I_{\beta\lambda} \\ I_{\lambda\alpha} & I_{\lambda\beta} & I_{\lambda\lambda} \end{bmatrix}$$

Where $\theta = (\alpha, \beta, \lambda)^T, I_{ii} = \frac{\partial^2 l}{\partial i^2}$; $I_{ij} = \frac{\partial^2 l}{\partial i \partial j}$ for any $i, j = \alpha, \beta, \lambda$

10. Simulation and Real-Life Applications

10.1. Simulation

In this section, a simulation study is performed to test the performance of MLEs. We generate a random sample of size n = 30, 50, 100, 150, 200 and 500 from Kw EE distribution for the values of $\alpha = 2, \beta = 3, \lambda = 1$ and $\alpha = 0.8, \beta = 0.5, \lambda = 1.5$ with replication 100 times using R software. It is observed that the mean squared error (MSE) and average bias decreases when the sample size increases. The result obtained from the tables shows that as the sample size increases, biases and MSEs of the parameters become smaller. This result aligns with the first-order asymptotic theory. The simulation results for different parameter values of Kw EE distribution are presented in Tables 1 and 2.

Table 1. *Average Values of the Bias and MSE's for the Kw EE Distribution for* $\alpha = 2, \beta = 3, \lambda = 1$.

N	Parameter	Bias	MSE
	α	7.4437	21.5929
n = 30	β	37.2459	58.7246
	λ	1.4856	1.3495
	α	3.1838	4.6425
n = 50	β	24.9155	38.4766
	λ	1.1316	0.8980
	α	2.2827	0.7503
n = 100	β	16.0142	27.1794
	λ	1.0781	0.6820
	α	2.3929	1.5467
n = 150	β	13.0328	21.0196
	λ	1.0686	0.6661
	α	2.1293	0.4410
n = 200	β	9.0911	15.0587
	λ	1.0299	0.4513
	α	2.0669	0.26728
n = 500	β	5.6998	6.91500
	λ	1.0379	0.38181

Table 2. *Average Values of the Bias and MSE's for the Kw EE Distribution for* $\alpha = 0.8, \beta = 0.5, \lambda = 1.5$.

n	Parameter	Bias	MSE
	α	0.9771	0.6737
n = 30	β	7.1557	12.3392
	λ	1.6701	1.2721
	α	0.8411	0.3449
n = 50	β	4.3800	8.0791
	λ	1.6552	1.1610
	α	0.8124	0.16566
n = 100	β	3.3075	5.9981
	λ	1.6601	1.0187
	α	0.8135	0.1539
n = 150	β	2.6383	5.0550
	λ	1.5654	0.9287
	α	0.8088	0.1058
n = 200	β	2.3060	3.5545
	λ	1.4566	0.8770
	α	0.8040	0.0718
n = 500	β	2.1125	3.0409
	λ	1.3951	0.7897

10.2. Real-Life Application

For illustrating the superiority and practical applications of the Kw EED (α, β, λ) in bio medical fields, we consider two real life data sets. Further, we consider several other distributions proposed in the literature to compare. In order to compare the performance, Kumaraswamy Exponential Exponential distribution (Kw EED) with Kumaraswamy Gamma Distribution (Kw GD), Exponentiated Gamma Exponential Distribution (EGED), Gamma Distribution (GD) and exponential-exponential distribution (EED) and we are using the Criterion values like AIC (Akaike information criterion), AICC (corrected Akaike information criterion), BIC (Bayesian information criterion) and HQIC (Hannan-Quinn Information Criterion). The better distribution corresponds to lesser AIC, AICC, BIC and HQIC values.

The formulae for calculation of AIC, BIC, AICC and HQIC are;

AIC $= 2k - 2\log L$, AICC $= AIC + \frac{2k\ (k+1)}{n-\ k-1}$, BIC $= k\log n - 2\log L$ and

HQIC = $2k\log\ (\log(n)) - 2\log L$.

where n is the sample size, k is the number of parameters, and L denotes the likelihood function and C we take it as 2. Any probability model having smaller value of AIC, BIC and $-LogL$ being the best model to fit the data set. AIC, BIC, AICC, -Log L, and HIQC values are lower in Kw EED than in the Kw G, EGE, GD and EED according to Table 4 and Table 6. Our conclusion is that the Exponentiated EED provides a better fit than other distribution.

Dataset 1: This data set includes the life expectancy (in years) of 40 blood cancer (leukemia) patients from one of Saudi Arabia's Ministry of Health facilities, as described in [19].

0.315, 0.496, 0.616, 1.145, 1.208, 1.263, 1.414, 2.025, 2.036, 2.162, 2.211, 2.370, 2.532, 2.693, 2.805, 2.910, 2.912, 3.192, 3.263, 3.348, 3.348, 3.427, 3.499, 3.534, 3.767, 3.751, 3.858, 3.986, 4.049, 4.244, 4.323, 4.381, 4.392, 4.397, 4.647, 4.753, 4.929, 4.973, 5.074, 5.381.

Data Set 2: The dataset on the remission periods (in months) of 36 bladder cancer patients described in and provided by [20].

0.08, 0.2, 0.4, 0.5, 0.51, 0.81, 0.87, 0.9, 1.05, 1.19, 1.26, 1.35, 1.4, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.02, 3.25, 3.31, 3.36, 3.36.

Table 3. *Descriptive Statistics for Data Set 1.*

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
0.315	2.19	3.348	3.141	4.264	5.381

Table 4. *ML Estimates of the Parameters with Measures for Model Selection for Data Set 1.*

Models	MLE's	-LL	AIC	BIC	AICc	HQIC
Kw EED	$\hat{\alpha}$ = 2.507 $\hat{\beta}$=13948.71 $\hat{\lambda}$ = 0.07995	-69.643	145.285	150.352	145.952	147.117
Kw GD	$\hat{\alpha}$ = 47.3176 $\hat{\theta}$ = 1.0974 $\hat{a}$ = 0.05455 $\hat{b}$ = 109.9767	-70.032	148.064	154.819	149.207	150.506
EGED	$\hat{\alpha}$ = 10.7088 $\hat{\beta}$ = 1.5291 $\hat{\theta}$ = 3.3031 $\hat{c}$ = 0.00259	-70.515	149.0304	155.786	150.1733	151.473
GD	$\hat{\alpha}$ = 3.4648 $\hat{\theta}$ = 0.9065	-73.549	151.097	154.475	151.422	152.319
EED	$\hat{\lambda}$ = 0.5643	-85.778	173.556	175.245	173.662	174.167

Table 5. *Descriptive Statistics for Data Set 2.*

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
0.080	1.155	2.080	1.940	2.705	3.360

Table 6. ML Estimates of the Parameters with Measures for Model Selection for Data Set 2.

Models	MLE's	-LL	AIC	BIC	AICc	HQIC
Kw EED	$\hat{\alpha}$ = 1.9724 $\hat{\beta}$=4338.57 $\hat{\lambda}$ = 0.0814	-51.4505	108.9011	113.6517	109.6511	110.5592
Kw GD	$\hat{\alpha}$ = 37.5684 $\hat{\theta}$ = 1.2443 $\hat{a}$ = 0.0527 $\hat{b}$ = 76.7400	-51.714	111.428	117.762	112.718	113.639
EGED	$\hat{\alpha}$ = 11.3502 $\hat{\beta}$ = 1.0162 $\hat{\theta}$ = 3.1417 $\hat{c}$ = 0.0041	-50.7967	109.5936	115.9277	110.8839	111.8043
GD	$\hat{\alpha}$ = 2.3097 $\hat{\theta}$ = 0.8399	-54.0924	112.2848	115.3518	112.5484	113.2904
EED	$\hat{\lambda}$ = 0.7179	-59.8567	121.7135	123.2971	121.8312	122.2662

We have obtained maximum likelihood estimators of the parameters of Kw EE distribution (α, β, λ) by using R software in case of the above two data. We have computed some information criterion such as AIC, BIC and AICC for each fitted model. The numerical values are presented in the following Tables 3-6 (included in the last). For graphical comparison we have plotted the cumulative plots and empirical distribution plots shown in Figures 4 and 5 using R software.

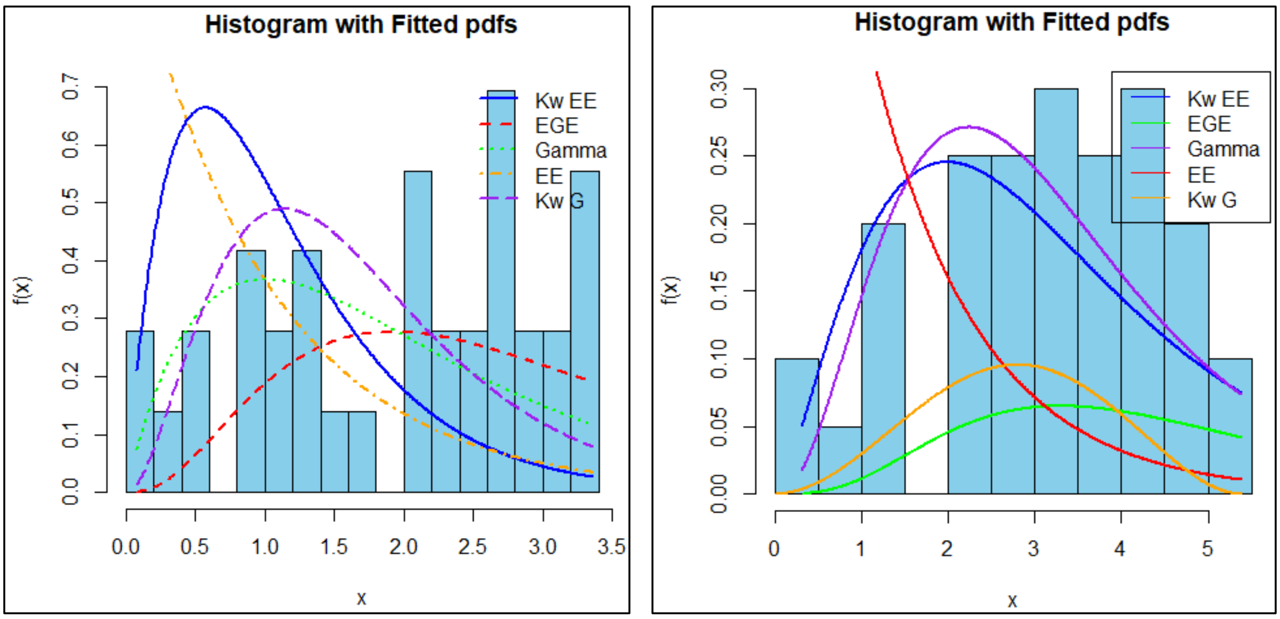

Figure 4. Histogram and Fitted Pdfs for the Data Sets 1 (left) & 2 (right).

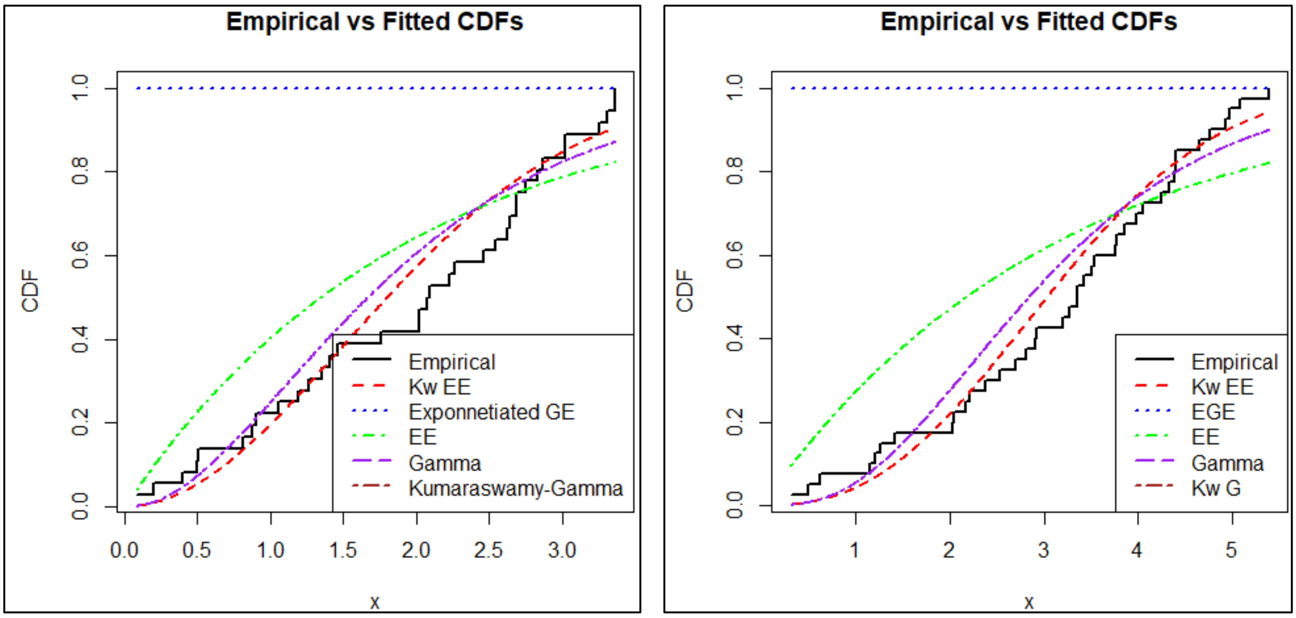

Figure 5. *Empirical CDF and Fitted CDFs for the data sets 1 (left) & 2 (right).*

11. Conclusion

In this paper we proposed a new three parameter distribution which is a generalization of Kumaraswamy G family of distributions. Some statistical and reliability concepts have been studied. The maximum likelihood approach is used to estimate the parameter. The results of the data analysis indicate that the Kw EE distribution has the lowest AIC, BIC, AICC and HQIC values. Therefore, the proposed model is more superior and flexible match than the other competing models. Hence, the Kw EE distribution can be seen as an important distribution in modelling real-life data.

Abbreviations

EED	Exponential-Exponential distribution
Kw EED	Kumaraswamy Exponential-Exponential Distribution
CDF	Cumulative Distribution Function
PDF	Probability Density Function
AIC	Akaike information criterion
BIC	Bayesian Information Criterion
AICC	Corrected Akaike Information Criterion
HQIC	Hannan-Quinn Information Criterion

Author Contributions

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Mani Vijayakumar: Conceptualization, Data curation, Methodology, Supervision, Validation, Writing – review & editing

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Conflicts of Interest

The authors declare no conflicts of interest.

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