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Research Article

Numerical Inference on the Inverse Weibull Model Parameters Based on Dual Generalized Hybrid Progressive Censoring Data

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Abstract

In parameter estimation techniques, several methods exist for estimating the distribution parameters in life data analysis. However, some of them are less efficient than Bayes' method, despite its subjectivity to prior information other than data that can mislead subsequent inferences. Thus, the main objective of this study is to present optimal numerical iteration techniques, such as the Picard and the Runge-Kutta methods, which are more efficient than Bayes' method. The proposed methods have been applied to the inverse Weibull distribution parameters and compared to the Bayes' method based on the informative gamma prior and the non-parametric kernel and characteristic priors, via an extensive Monte Carlo simulation study through the absolute average bias and the mean squared errors for the parameter estimators. The simulation results indicated that the Picard and Runge-Kutta methods provide better estimates and outperform the Bayes' method based on the dual generalized progressive hybrid censoring data. Finally, it has been shown that the inverse Weibull distribution gives a good fit to new areas of dataset applications, such as flood data and reactor pump data. We have analyzed and illustrated the proposed methods using these datasets to confirm the simulation results.

Keywords

Bayesian Estimation, Characteristic Prior, Informative Prior, Kernel Prior, Picard's Method, Runge- Kutta Method

1. Introduction

Burkschat et al. [4] have introduced DGOS: the dual generalized order statistics as the Lower Generalized Order Statistics, which is a combined mechanism for studying the random variables arranged in descending order. This technique was introduced as the inverse image of GOS: the generalized order statistics introduced by Kamps [15]. The DGOS has been used by many authors considering different estimation procedures for lifetime distributions based on complete and progressive data, see Ahsanullah ([1, 2]) and Maswadah ([20, 21]). The IWD: the inverse Weibull distribution is the one that has a small part of applications based on the DGOS despite its flexibility for describing the lifetime variables. It is useful for modelling and analyzing lifetime data in extreme weather and extreme data. The density func-

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tion and the cumulative distribution function for the IWD are given, respectively, as follows:

$$f(x;\alpha,\beta) = \alpha\beta x^{-\alpha-1}\, exp(-\beta x^{-\alpha}), \qquad x > 0 \qquad (1)$$

$$F(x;\alpha,\beta) = exp(-\beta x^{-\alpha}), \qquad x > 0 \qquad (2)$$

$\alpha, \beta > 0$ are the shape and scale parameters, respectively.

This model has been proposed as a model in the analysis of life test data and played an important role in many applications, including the dynamic components of diesel engines and several data sets such as the breakdown times of the insulating fluids subject to the action of a constant tension. Furthermore, it can be used in the reliability engineering discipline and various failure characteristics, such as infant mortality and wear-out periods. It can also be used to determine the cost effectiveness and maintenance periods of reliability centered maintenance activities. It includes two well-known distributions such as the inverse exponential distribution for $\alpha = 1$ and the inverse Rayleigh distribution for $\alpha = 2$. Keller et al. ([16, 17]) derived this model based on physical considerations of some failures of mechanical components subject to degradation phenomena. Erto [13] introduced other physical failure processes leading to this distribution. Moreover, it has been shown that the IWD gave a good fit to life testing data reported in Nelson [31]. Maswadah [20] provided the IWD with a new area of applications such as weather phenomena (flood, drought, rainfall, etc.), which ensures its usefulness in modeling extreme value data.

The estimation procedures in classical and Bayesian approaches to this distribution have been studied extensively in the literature. Calabria and Pulcini [5] investigated the statistical properties of the MLEs: the maximum likelihood estimators of the parameters and reliability of a complete sample. Erto [13] used LS: the least-square method to obtain the estimators of the parameters and reliability. Calabria and Pulcini [6] derived the MLEs and the LS of the parameters. Calabria and Pulcini [7] derived the Bayes estimator of the parameters and reliability. Calabria and Pulcini [8] derived the prediction for some future variables. Khan et al. ([18, 19]) provided some theoretical analysis and derived the MLEs for the IWD parameters. Miljenko et al. [29] derived the least squares estimates for the three-parameter IWD. In recent decades, some authors have used technical information about real systems and converted it into degrees of belief about the model parameters that improved the accuracy of the estimators, see Calabria and Pulcini [8] and Sultan [34].

Recently, Cho et al. ([9-11]) have suggested GPHCS: the generalized progressive hybrid censoring scheme, which has a great part of applications with different lifetime distributions, such as the inverse Weibull distribution, see Mohie El-Din et al. [30] and Maswadah ([25-28]). Therefore, in this work analogous to the dual generalized order statistics, DGPHCS: the dual generalized progressive hybrid censoring scheme has been introduced for estimating the inverse Weibull distribution parameters based on numerical iteration techniques, such as the Picard and the Runge-Kutta methods. This scheme can be described as follows:

Consider a life-testing experiment in which n identical units $X_1, X_2, \ldots\ldots, X_n$ are placed on test. For $T \in (0,\infty)$ and integers k and m are pre-fixed such that $k < m$ with $R_1, R_2, \ldots, R_m$ are the random removal units that are fixed at the beginning of the experiment such that $N = m + \sum_{i=1}^{m} R_i$. Generally, at the time of the i_th failure, R_i units are randomly removed from the remaining surviving units $S_i = n - i - \sum_{j=1}^{i-1} R_j$, where $i = 1,2,\ldots,m$. This process continues until, immediately following the terminated time $T^* = \min\{X_{k:m:n}, \max\{X_{m:n:n}, T\}\}$, where at this time all the remaining surviving units are removed from the experiment according to the following cases. Let D denote the number of observed failures up to the time T. Thus, we have one of the following types of observations, see Figure 1:

Case I: $X_{1:m:n} \geq \ldots \geq X_{k:m:n} > X_{k+1:m:n} > \ldots > X_{D:m:n}$

If $X_{m:m:n} \leq T < X_{k:m:n}$.

Case II: $X_{1:m:n} \geq \ldots \geq X_{k:m:n} > X_{k+1:m:n} > \ldots > X_{m:m:n}$

If $T < X_{m:m:n} < X_{k:m:n}$.

Case III: $X_{1:m:n} \geq \ldots \geq X_{D:m:n} > X_{D+1:m:n} > \ldots > X_{k:m:n}$

If $X_{m:m:n} < X_{k:m:n} < T$.

Note that for Case I, $X_{D:m:n} > T > X_{D+1:m:n}$ and $X_{D+1:m:n}, \ldots\ldots, X_{m:m:n}$ are not observed.

For Case III, $T > X_{k:m:n} > X_{m:m:n}$. and $X_{k+1:m:n}, \ldots\ldots, X_{m:m:n}$ are not observed.

Thus, given the DGPHCS, the likelihood function for the three different cases can be written in a unified form as follows:

$$L(\theta; x) = C \prod_{i=1}^{n} f(x_{i,m,n})\, [F(x_{i,m,n})]^{R_i} [F(T)]^{\delta R_T^*}, \qquad (3)$$

where $C = \prod_{i=1}^{n} \sum_{j=i}^{m} (R_j + 1)$,

$$n = \begin{cases} D, & \delta = 1 \quad , \quad if \quad X_{m:m:n} \leq \quad T < \quad X_{k:m:n} \\ m, & \delta = 0 \quad , \quad if \quad T < \quad X_{m:m:n} \leq \quad X_{k:m:n} \\ k, & \delta = 0 \quad , \quad if \quad X_{m:m:n} < \quad X_{k:m:n} < \quad T \end{cases}$$

and

$$\underline{X} = \begin{cases} (X_{1:m:n}, X_{2:m:n}, \ldots, X_{D:m:n}) \quad , & if \quad X_{m:m:n} \le \ T < \ X_{k:m:n} \\ (X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}) \quad , & if \quad T < \ X_{m:m:n} \le \ X_{k:m:n}\,, \\ (X_{1:m:n}, X_{2:m:n}, \ldots, X_{D:m:n}, X_{D+1:m:n} \ldots, X_{k:m:n}) & if \quad X_{m:m:n} < \ X_{k:m:n} < \ T \end{cases}$$

for $F^{-1}(0) > x_{1:m:n} > x_{2:m:n} > \ldots, > x_{n:m:n} > F^{-1}(1)$ of R^n.

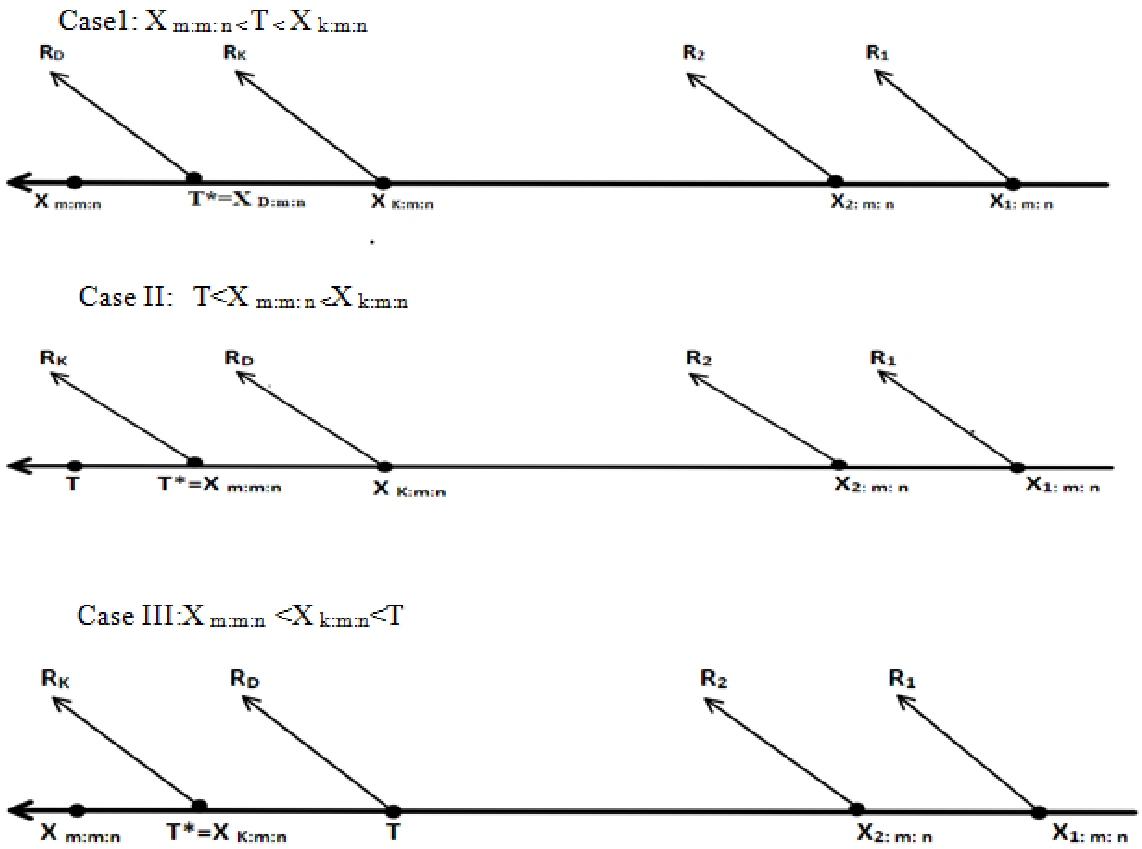

Figure 1. *Schematic representation of the dual generalized progressive hybrid censoring scheme (DGPHCS).*

2. Estimation Methods

2.1. Picard's Method

Theoretically, it is known that the traditional log-likelihood function, $\mathrm{H}(\theta; \mathrm{x})$, depends on the unknown parameter $\theta = (\alpha, \beta)$ and the sample data $\underline{X} = (\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_N)$, which can be used to derive the MLE $\hat{\theta}(x)$ of θ, by solving the stationary equation $\frac{\partial \mathrm{H}(\theta;\mathrm{x})}{\partial \theta}|_{\hat{\theta}(x)} = 0$. Thus, based on the dependence of the MLE on the sample data, we can apply the implicit function theorem to the stationary equation by taking the total derivative with respect to any $x \in \underline{X}$, with considering all partial derivatives as well as the total derivatives are assumed to be evaluated at some known value of $\hat{\theta}(x_0) = \theta_0$, we obtain the following equation:

$$\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\partial \mathrm{H}(\theta|\mathrm{X})}{\partial \theta}\right) = \frac{\partial^2 \mathrm{H}(\theta|\mathrm{X})}{\partial \theta\, \partial \mathrm{x}}\Big|_{\theta=\hat{\theta}} + \frac{\partial^2 \mathrm{H}(\theta|\mathrm{X})}{\partial \theta^2}\Big|_{\theta=\hat{\theta}} \frac{\mathrm{d}\hat{\theta}}{\mathrm{dx}} = 0. \tag{4}$$

Solving (4) we obtain the first derivative of $\hat{\theta}$ with respect to $x \in \underline{X}$ at $\theta = \hat{\theta}$ as:

$$\frac{\mathrm{d}\hat{\theta}(x)}{\mathrm{dx}} = -\left(\frac{\partial^2 \mathrm{H}(\theta|\mathrm{X})}{\partial \theta^2}\Big|_{\theta=\hat{\theta}}\right)^{-1} \frac{\partial^2 \mathrm{H}(\theta|\mathrm{X})}{\partial \theta\, \partial \mathrm{x}}\Big|_{\theta=\hat{\theta}}. \tag{5}$$

Thus, we can write (5) as the following form:

$$\frac{\mathrm{d}\hat{\theta}(x)}{\mathrm{dx}} = \mathrm{f}(\mathrm{x}, \hat{\theta}), \quad \text{at} \ \ \hat{\theta}(x_0) = \theta_0, \tag{6}$$

where $\mathrm{f}(\mathrm{x}, \hat{\theta}) = -\left(\frac{\partial^2 \mathrm{H}(\theta|\mathrm{X})}{\partial \theta^2}\Big|_{\theta=\hat{\theta}}\right)^{-1} \frac{\partial^2 \mathrm{H}(\theta|\mathrm{X})}{\partial \theta\, \partial \mathrm{x}}\Big|_{\theta=\hat{\theta}}$.

The equation (6) is a first order ordinary differential equation in $\hat{\theta}$. Using any numerical technique, such as the fourth order Runge-Kutta and Picard methods, we can find the approximate solution given a trial set of parameter values and initial conditions. If the initial conditions are unavailable, they must be appended to the parameter $\hat{\theta}$ as quantities that fit the optimized solution.

The Picard estimator for the parameter α from (6) for any $x \in \underline{X}$ can be obtained as follows:

The general iteration rule for Picard's method for the parameter α (say), can be derived by integrating (6) with respect to x from x_0 to x^* as:

$$\hat{\alpha}(x^*) = \hat{\alpha}(x_0) + \int_{x_0}^{x^*} f\left(x, \hat{\alpha}_0, \hat{\beta}\right) dx$$

$$= \alpha_0 + \int_{\alpha_0}^{\alpha^*} f\left(x, \hat{\alpha}_0, \hat{\beta}\right) \frac{dx}{d\hat{\beta}} d\hat{\beta}$$

Using the differential equation for the parameter, $\frac{d\hat{\beta}}{dx} = g\left(x, \hat{\alpha}_0, \hat{\beta}_0\right)$, we get the recurrence relation for the parameter α as follows:

$$\alpha_{i+1}^* = \alpha_0 + \int_{\alpha_0}^{\alpha^*} \left[\frac{f(x, \hat{\alpha}_i, \hat{\beta})}{g(x, \hat{\alpha}_i \hat{\beta})}\right] d\hat{\beta}, \quad i = 0,1,2,3, .. \quad (7)$$

where α_0 is the initial point and α^* is the value for which the desired solution should be optimized.

Similarly, the Picard estimator for β can be derived from the following recurrence relation:

$$\beta_{i+1}^*(x) = \beta_0 + \int_{\beta_0}^{\beta^*} \left[\frac{f(x, \hat{\alpha}, \hat{\beta}_i)}{f(x, \hat{\alpha}, \hat{\beta}_i)}\right\} d\hat{\alpha}, \quad i = 0,1,2,3, .. \quad (8)$$

It is important to mention that the Picard's method has been used for estimating many distribution parameters, see Maswadah [28].

2.2. Runge-Kutta Method

The fourth order Runge-Kutta recurrence solution for (6) can be written as follows:

$$\hat{\theta}_{i+1} = \hat{\theta}_i + (K_1 + 2K_2 + 2K_3 + K_4)/6, \text{ for } i = 0,1,2, \ldots, \quad (9)$$

where

$$K_1 = hf(x_i, \hat{\theta}_i), \; K_2 = hf(x_i + h/2, \hat{\theta}_i + K_1/2),$$

$$K_3 = hf(x_i + h/2, \hat{\theta}_i + K_2/2) \text{ and } K_4 = hf(x_i + h, \hat{\theta}_i + K_3).$$

In both methods h is the step height with a small value say (1E-02) and $\hat{\theta}(x_0) = \theta_0$, is the initial value for $\hat{\theta}$. The iterative process is continued using (7), (8) and (9) until two consecutive numerical solutions are almost the same, that is if $|\theta_{i+1}^* - \theta_i^*| < 1E - 05$, for $i = 0,1,2, \ldots$.

It is important to notice that the Runge-Kutta method has been used for estimating many distribution parameters, see Maswadah ([25-28]).

For the inverse Weibull model, the log-likelihood function of the DGPHCS (3) can be derived as follows:

$$L(\theta; \underline{x}) = C(\alpha\beta)^n \prod_{i=1}^n x_i^{-\alpha-1} e^{-\beta x_i^{-\alpha}} \left[e^{-\beta x_i^{-\alpha}}\right]^{R_i} \left[e^{-\beta T^{-\alpha}}\right]^{R_T^* \delta}$$

$$= C(\alpha\beta)^n exp\left[-(\alpha+1)\sum_{i=1}^n lnx_i - \beta\left(\sum_{i=1}^n (1+R_i) x_i^{-\alpha} + \delta R_T^* T^{-\alpha}\right)\right]$$

Thus, the log likelihood function is given by

$$H = H(\alpha, \beta; x) = K + n\, ln(\alpha\beta) - (\alpha+1)\sum_{i=1}^n ln\, x_i - \beta\left[\sum_{i=1}^n (1+R_i) x_i^{-\alpha} + \delta R_T^* T^{-\alpha}\right].$$

The derivatives of H can be concluded from the corresponding derivatives in Appendix II.

2.3. Bayes Method

In this section, the Bayes estimations will be derived based on the informative gamma prior and the non-parametric priors, such as the characteristic and kernel priors. These non-parametric prior distributions don't contain hyperparameters, which can mislead to subsequent inferences.

1) The Informative Gamma Prior

We consider the unknown parameters α and β have independent gamma prior distributions with a joint probability density function defined as the following:

$$h(\alpha, \beta) \propto \alpha^{a-1} \beta^{c-1} e^{-b\alpha - d\beta} \quad (10)$$

The hyper-parameter a, b, c and d are assumed to be known and positives to reflect the prior belief about the unknown parameters.

2) The Nonparametric Kernel Prior

For deriving the kernel prior, we introduce the bivariate kernel density estimator for the unknown probability density function $g(\alpha, \beta)$ with support on $(0, \infty)$, which is defined as follows:

$$\hat{g}(\alpha, \beta) = \frac{1}{nh_1h_2} \sum_{i=1}^n K\left(\frac{\alpha - \hat{\alpha}_i}{h_1}, \frac{\beta - \hat{\beta}_i}{h_2}\right), \quad (11)$$

$h_i, i = 1,2$ are called the bandwidths or smoothing parameters, which are chosen such that $h_i \to 0$ and $nh_i \to \infty$ as $n \to \infty$, where n is the sample size. The influence of the smoothing parameter h is critical because it determines the amount of smoothing. However, the optimal choice for h_i, which minimizes the mean squared errors is given by $h_i = 1.06 S_i n^{-0.2}$, where S_i is the sample standard deviation. The optimal choice for the kernel function $K(.,.)$ can be used as the bivariate standard normal distribution for the parameters α and β. Based on the properties of the MLEs of the parameters, which are converging in probability with the original parameters, the kernel prior estimate can be derived. It is worthwhile to mention that this kernel prior has been used for some distributions, see Ahsanullah et al. [3] and Maswadah ([22-24]).

3) The Nonparametric Characteristic Prior

The characteristic function CF is the Fourier transform of the cumulative distribution function CDF, and hence there is a one-to-one correspondence between the CF and the CDF. Thus, the CF is fully characterizing the distribution of the underlying random variable. Since the CF can be estimated using the empirical characteristic function ECF, which retains all the information in the sample, it plays an increasing and important role in econometrics and finance, see Feuerverger [14]. Thus, based on the CF for two random variables and its inversion formula for the probability density function. The characteristic prior for the parameters α and β as follows:

$$\hat{q}(\alpha,\beta) = \frac{1}{4n\pi^2}\sum_{i=1}^{n}\frac{1}{|(\alpha-\hat{\alpha}_i)(\hat{\beta}_i-\beta)|}, \text{ see Appendix I.} \quad (12)$$

Thus, using the joint priors (10), (11) and (12) with the likelihood function of the DGPHCS (3) the posterior density for the unknown parameters α and β can be written in a unified form as follows:

$$f(\alpha,\beta|\underline{X}) = KQ(\alpha,\beta)L(X;\alpha,\beta)$$

where $Q(\alpha,\beta)$ is the general prior, which can be define as:

$$Q(\alpha,\beta) = g(\alpha,\beta)q(\alpha,\beta)h(\alpha,\beta) = \hat{g}^{p_1}(\alpha)\hat{g}^{p_2}(\beta)\hat{q}^{s_1}(\alpha)q^{s_2}(\beta)\alpha^{a-1}\beta^{c-1}e^{-b\alpha-d\beta},$$

which have the following special cases:

1. For the informative prior (10): $p_1 = p_2 = 0$, $s_1 = s_2 = 0$.
2. For the kernel prior (11): $p_1 = p_2 = 1$ and $a = c = 1, \ b = d = 0$.
3. For the characteristic prior (12): $p_1 = p_2 = 0$, $s_1 = s_2 = a = c = 1, \ b = d = 0$.

Thus, the posterior density can be written as follows:

$$f(\alpha,\beta|x) = K\hat{g}^{p_1}(\alpha)\hat{g}^{p_2}(\beta)\hat{q}^{s_1}(\alpha)q^{s_2}(\beta)\alpha^{n+a-1}\beta^{n+c-1}$$

$$\times exp[-\alpha b - (\alpha+1)\sum_{i=1}^{n} ln(x_i)].$$

$$\times exp[-\beta(d + \sum_{i=1}^{n}(R_i+1)x_i^{-\alpha} + \delta R_T^* T^{-\alpha}). \quad (13)$$

Based on (13) we can use the Tierney and Kadane approximation method to approximate all the Bayes estimators for the unknown parameters. Tierney and Kadane [35] introduced an easily computable approximation for the posterior mean of a non-negative parameter or more generally, of a smooth function of the parameter that is non-zero on the interior of the parameter space. For detail, let $q(\alpha,\beta)$ be a smooth, positive function in the parameter space. The posterior expectation of $q(\alpha,\beta)$ can be obtained as

$$q^* = E(q(\alpha,\beta)|x) = \frac{\int_0^\infty\int_0^\infty e^{nH^*(\alpha,\beta)}d\alpha d\beta}{\int_0^\infty\int_0^\infty e^{nH(\alpha,\beta)}d\alpha d\beta}, \quad (14)$$

where $H(\alpha,\beta) = \ln f(\alpha,\beta|x)/n$, and
$H^*(\alpha,\beta) = H(\alpha,\beta) + \ln q(\alpha,\beta)/n$.

For (α,β) the Bayes estimator using Tierney and Kadane approximation for $q(\alpha,\beta)$ can be obtained as follows:

$$q^* = \sqrt{|\Sigma^*|/|\Sigma|}\exp[n[H^*(\alpha,\beta) - H(\alpha,\beta)],$$

where $(\hat{\alpha},\hat{\beta})$ and $(\hat{\alpha}^*,\hat{\beta}^*)$ maximize the $H(\hat{\alpha},\hat{\beta})$ and $H^*(\hat{\alpha}^*,\hat{\beta}^*)$, respectively.

$$\text{Let } |\Sigma| = \begin{vmatrix} H_{11} & H_{12} \\ H_{21} & H_{22}\end{vmatrix}^{-1}, \text{ and } |\Sigma*| = \begin{vmatrix} H_{11}^* & H_{12}^* \\ H_{21}^* & H_{22}^*\end{vmatrix}^{-1}$$

denote the minus of inverse of Hessians of $H(\alpha,\beta)$ and $H^*(\alpha,\beta)$ at $(\hat{\alpha},\hat{\beta})$ and $(\hat{\alpha}^*,\hat{\beta}^*)$ respectively.

Using (13) we can define the log posterior density $H = H(\alpha,\beta)$ as follows:

$$H(\alpha,\beta|\underline{x}) = [p_1\ln\hat{g}_1(\alpha) + p_2\ln\hat{g}_2(\beta) + s_1\ln\hat{q}_1(\alpha)$$

$$+s_2\ln\hat{q}_2(\beta) + (n+a-1)\ln\alpha$$

$$+(n+c-1)\ln\beta - \alpha b - (\alpha+1)\sum_{i=1}^{n}\ln(x_i)$$

$$-\beta[d + \sum_{i=1}^{n}(1+R_i)x_i^{-\alpha} + \delta R_T^* T^{-\alpha})].$$

The derivatives of $H(\alpha,\beta)$ and $H^*(\alpha,\beta)$ have been derived in Appendix II.

3. Simulation Study

The simulation studies are carried out for studying the performance of the Picard, Runge-Kutta and Bayes methods through AVB: the absolute average bais and MSE: the mean squared error, which are given respictively, as follows:

$$AVB = \frac{1}{L}\sum_{i=1}^{L}|\hat{\theta}_i - \theta|, \text{ and } MSE(\theta^*) = \frac{\sum_{i=1}^{L}(\theta_i-\theta^*)^2}{L}$$

where $\theta^*$is the point estimate of the unknown parameter θ and L is the number of replications.

In our simulation study, we choose different combinations of the hyperparameters of α and β, say $a = c = 0.5$, and $b = d = 0.3$, for which the prior distribution will be a decreasing function for each parameter, respectively. Based on these hyperparameters, we generated from the gamma distribution two values for the parameter $\alpha = (1, 2)$ and two values for the parameter $\beta = (2, 3)$. Using the above values of the parameters for generating different samples from the IWD with sizes $n = 20, 40,$ and 60 to represent small, moderate, and large sizes. We choose two values for the termination time T=(0.75, 3). To assess the performance of these estimates, the AVB and the MSE for each sample were calculated using 1000 replications.

From the simulation results in Tables 1, 2, 3 and 4, some points are quite clear based on these estimates and the others have been summarized in the following main points:

1) It is clear that, in general, the point estimates based on the Picard and R-K methods have the smallest estimated values of AVB and MSE as compared to the estimates based on the Bayes method with the three different priors.
2) The estimated values of the AVB and MSE increase as the value of α increases and decrease as the value of β increases.
3) The estimated values of the AVB and MSE decrease with decreasing the hyperparameters of the informative priors and increasing the sample size and the termination time of the experiment T as expected.
4) The Bayes estimated values of the AVB and MSE based on the characteristic and kernel priors are less than the ones based on the informative gamma prior.

As a conclusion, it appears that the point estimates based on the Picard and R-K methods compete and outperform the Bayes method based on the different priors.

Table 1. *The AVB and the MSEs in parentheses for parameter* $\boldsymbol{\alpha}$ *using the Picard, Runge Kutta (R-K), and Bayes methods at T=0.75 with m = (n/2 and 3n/4) and k= (m/2 and 3m/4).*

n	m	k	α	β	Picard Estimates	R-K Estimates	Bayes estimates: Gamma Prior	Chara-Prior	Kernel Prior
20	10	5	1	2	0.0977(0.0097)	0.1134(0.0129)	0.1831(0.0336)	0.1517(0.0230)	0.1735(0.0301)
				3	0.1115(0.0127)	0.1076(0.0116)	0.1966(0.0466)	0.1512(0.0229)	0.1670(0.0279)
			2	2	0.1282(0.0164)	0.3199(0.1041)	0.3772(0.1426)	0.3126(0.0977)	0.3479(0.1211)
				3	0.1347(0.0182)	0.2804(0.0795)	0.4078(0.2459)	0.3083(0.0951)	0.3354(0.1125)
		8	1	2	0.0977(0.0097)	0.1133(0.0129)	0.1832(0.0336)	0.1517(0.0230)	0.1735(0.0301)
				3	0.1113(0.0126)	0.1078(0.0116)	0.1996(0.0558)	0.1512(0.0229)	0.1670(0.0279)
			2	2	0.1283(0.0165)	0.3185(0.1031)	0.3765(0.1420)	0.3124(0.0976)	0.3476(0.1209)
				3	0.1341(0.0180)	0.2815(0.0800)	0.4017(0.1744)	0.3084(0.0951)	0.3356(0.1126)
	15	8	1	2	0.0962(0.0094)	0.1179(0.0140)	0.1724(0.0297)	0.1521(0.0231)	0.1694(0.0287)
				3	0.1109(0.0125)	0.1088(0.0118)	0.1701(0.0290)	0.1513(0.0229)	0.1646(0.0271)
			2	2	0.1236(0.0153)	0.3165(0.1016)	0.3461(0.1198)	0.3123(0.0975)	0.3383(0.1144)
				3	0.1291(0.0167)	0.2800(0.0791)	0.3333(0.1111)	0.3082(0.0950)	0.3280(0.1076)
		11	1	2	0.0964(0.0094)	0.1172(0.0138)	0.1725(0.0298)	0.1521(0.0231)	0.1694(0.0287)
				3	0.1108(0.0125)	0.1087(0.0118)	0.1699(0.0289)	0.1513(0.0229)	0.1645(0.0271)
			2	2	0.1238(0.0154)	0.3143(0.1002)	0.3457(0.1196)	0.3120(0.0974)	0.3380(0.1143)
				3	0.1291(0.0167)	0.2801(0.0791)	0.3334(0.1111)	0.3083(0.0950)	0.3280(0.1076)
40	20	10	1	2	0.0927(0.0087)	0.1136(0.0129)	0.1644(0.0270)	0.1517(0.0230)	0.1622(0.0263)
				3	0.1063(0.0114)	0.1079(0.0116)	0.1630(0.0266)	0.1512(0.0228)	0.1589(0.0252)
			2	2	0.1233(0.0152)	0.3121(0.0981)	0.3329(0.1108)	0.3117(0.0971)	0.3281(0.1076)
				3	0.1292(0.0167)	0.2775(0.0774)	0.3250(0.1056)	0.3078(0.0948)	0.3202(0.1026)
		15	1	2	0.0923(0.0086)	0.1138(0.0130)	0.1644(0.0270)	0.1517(0.0230)	0.1622(0.0263)

n	m	k	α	β	Picard Estimates	R-K Estimates	Bayes estimates Gamma Prior	Chara-Prior	Kernel Prior
				3	0.1062(0.0114)	0.1080(0.0117)	0.1630(0.0266)	0.1512(0.0229)	0.1589(0.0252)
			2	2	0.1233(0.0152)	0.3113(0.0976)	0.3327(0.1107)	0.3116(0.0971)	0.3280(0.1076)
				3	0.1290(0.0167)	0.2776(0.0774)	0.3250(0.1056)	0.3079(0.0948)	0.3203(0.1026)
	30	15	1	2	0.0919(0.0085)	0.1161(0.0135)	0.1628(0.0265)	0.1519(0.0231)	0.1614(0.0260)
				3	0.1070(0.0116)	0.1092(0.0119)	0.1612(0.0260)	0.1513(0.0229)	0.1585(0.0251)
			2	2	0.1238(0.0154)	0.3217(0.1042)	0.3314(0.1099)	0.3127(0.0978)	0.3278(0.1075)
				3	0.1302(0.0170)	0.2856(0.0819)	0.3236(0.1047)	0.3087(0.0953)	0.3203(0.1026)
		23	1	2	0.0911(0.0084)	0.1176(0.0139)	0.1623(0.0264)	0.1521(0.0231)	0.1611(0.0260)
				3	0.1072(0.0116)	0.1108(0.0123)	0.1609(0.0259)	0.1514(0.0229)	0.1585(0.0251)
			2	2	0.1258(0.0158)	0.3296(0.1093)	0.3319(0.1102)	0.3135(0.0983)	0.3283(0.1078)
				3	0.1328(0.0177)	0.2943(0.0870)	0.3249(0.1056)	0.3096(0.0959)	0.3214(0.1033)
60	30	15	1	2	0.0920(0.0085)	0.1135(0.0129)	0.1605(0.0258)	0.1517(0.0230)	0.1591(0.0253)
				3	0.1048(0.0111)	0.1080(0.0117)	0.1598(0.0255)	0.1512(0.0228)	0.1567(0.0246)
			2	2	0.1241(0.0154)	0.3143(0.0992)	0.3275(0.1072)	0.3119(0.0973)	0.3241(0.1050)
				3	0.1301(0.0169)	0.2782(0.0776)	0.3212(0.1032)	0.3078(0.0948)	0.3171(0.1006)
		23	1	2	0.0919(0.0085)	0.1136(0.0129)	0.1605(0.0258)	0.1517(0.0230)	0.1591(0.0253)
				3	0.1046(0.0110)	0.1081(0.0117)	0.1599(0.0256)	0.1512(0.0228)	0.1567(0.0246)
			2	2	0.1240(0.0154)	0.3150(0.0997)	0.3276(0.1073)	0.3120(0.0973)	0.3242(0.1051)
				3	0.1298(0.0169)	0.2789(0.0780)	0.3213(0.1033)	0.3079(0.0948)	0.3172(0.1006)
	45	23	1	2	0.0908(0.0083)	0.1165(0.0136)	0.1589(0.0252)	0.1519(0.0231)	0.1582(0.0250)
				3	0.1050(0.0111)	0.1089(0.0119)	0.1577(0.0249)	0.1512(0.0229)	0.1561(0.0244)
			2	2	0.1225(0.0150)	0.3158(0.1001)	0.3242(0.1051)	0.3120(0.0974)	0.3223(0.1039)
				3	0.1283(0.0165)	0.2810(0.0792)	0.3174(0.1007)	0.3081(0.0949)	0.3158(0.0997)
		34	1	2	0.0913(0.0084)	0.1167(0.0136)	0.1590(0.0253)	0.1520(0.0231)	0.1582(0.0250)
				3	0.1062(0.0114)	0.1101(0.0121)	0.1577(0.0249)	0.1513(0.0229)	0.1562(0.0244)
			2	2	0.1251(0.0157)	0.3294(0.1090)	0.3261(0.1063)	0.3135(0.0983)	0.3240(0.1050)
				3	0.1318(0.0174)	0.2919(0.0855)	0.3197(0.1022)	0.3093(0.0957)	0.3175(0.1008)

Table 2. *The AVB and the MSEs in parentheses for parameter α using the Picard, R-K, and Bayes methods at T=3 with m = (n/2 and 3n/4) and k= (m/2 and 3m/4).*

n	m	k	α	β	Picard Estimates	R-K Estimates	Bayes estimates Gamma Prior	Chara-Prior	Kernel Prior
20	10	5	1	2	0.0811(0.0067)	0.1046(0.0109)	0.1455(0.0776)	0.1509(0.0228)	0.2567(0.1102)
				3	0.0913(0.0086)	0.1025(0.0105)	0.1378(0.0191)	0.1506(0.0227)	0.1887(0.0376)
			2	2	0.1135(0.0129)	0.2519(0.0639)	0.5169(0.6370)	0.3054(0.0933)	0.3512(0.1235)
				3	0.1199(0.0145)	0.2379(0.0569)	0.4438(0.6048)	0.3039(0.0924)	0.3360(0.1129)

n	m	k	α	β	Picard Estimates	R-K Estimates	Bayes estimates: Gamma Prior	Bayes estimates: Chara-Prior	Bayes estimates: Kernel Prior
		8	1	2	0.0951(0.0092)	0.1095(0.0120)	0.1991(0.0402)	0.1514(0.0229)	0.1761(0.0310)
				3	0.1082(0.0120)	0.1053(0.0111)	0.2479(0.1897)	0.1510(0.0228)	0.1678(0.0282)
			2	2	0.1276(0.0163)	0.2618(0.0689)	0.3723(0.1390)	0.3064(0.0939)	0.3411(0.1164)
				3	0.1318(0.0174)	0.2518(0.0636)	0.4076(0.1972)	0.3053(0.0932)	0.3325(0.1105)
		8	1	2	0.0952(0.0092)	0.1092(0.0119)	0.2024(0.0457)	0.1513(0.0229)	0.1760(0.0310)
				3	0.1083(0.0120)	0.1053(0.0111)	0.2631(0.5345)	0.1510(0.0228)	0.1678(0.0282)
			2	2	0.1281(0.0164)	0.2601(0.0679)	0.3709(0.1380)	0.3062(0.0937)	0.3406(0.1160)
	15			3	0.1318(0.0174)	0.2517(0.0636)	0.4302(0.3520)	0.3053(0.0932)	0.3324(0.1105)
		11	1	2	0.1023(0.0105)	0.1131(0.0128)	0.1793(0.0322)	0.1517(0.0230)	0.1719(0.0296)
				3	0.1127(0.0129)	0.1088(0.0119)	0.1837(0.0339)	0.1513(0.0229)	0.1666(0.0278)
			2	2	0.1415(0.0201)	0.2692(0.0728)	0.3525(0.1243)	0.3071(0.0943)	0.3362(0.1131)
				3	0.1454(0.0212)	0.2614(0.0686)	0.3550(0.1262)	0.3063(0.0938)	0.3302(0.1090)
		10	1	2	0.0767(0.0059)	0.1040(0.0108)	0.1361(0.0188)	0.1508(0.0227)	0.2348(0.2390)
				3	0.0845(0.0072)	0.1023(0.0105)	0.1454(0.0211)	0.1505(0.0227)	0.2313(0.2259)
			2	2	0.1129(0.0128)	0.2485(0.0619)	0.3933(0.1935)	0.3050(0.0930)	0.3323(0.1105)
	20			3	0.1176(0.0139)	0.2382(0.0568)	0.3995(0.3506)	0.3039(0.0924)	0.3233(0.1046)
		15	1	2	0.0905(0.0082)	0.1088(0.0118)	0.1760(0.0312)	0.1513(0.0229)	0.1659(0.0275)
				3	0.1028(0.0107)	0.1050(0.0110)	0.2137(0.3842)	0.1509(0.0228)	0.1607(0.0258)
			2	2	0.1269(0.0161)	0.2547(0.0650)	0.3385(0.1146)	0.3056(0.0934)	0.3259(0.1062)
40				3	0.1305(0.0171)	0.2471(0.0611)	0.3528(0.1304)	0.3048(0.0929)	0.3208(0.1029)
		15	1	2	0.0903(0.0082)	0.1090(0.0119)	0.1759(0.0311)	0.1513(0.0229)	0.1658(0.0275)
				3	0.1033(0.0108)	0.1049(0.0110)	0.2065(0.0731)	0.1509(0.0228)	0.1606(0.0258)
			2	2	0.1265(0.0160)	0.2559(0.0656)	0.3393(0.1152)	0.3057(0.0935)	0.3262(0.1064)
	30			3	0.1317(0.0174)	0.2450(0.0601)	0.3460(0.1206)	0.3046(0.0928)	0.3203(0.1026)
		23	1	2	0.1099(0.0121)	0.1117(0.0125)	0.1638(0.0268)	0.1516(0.0230)	0.1615(0.0261)
				3	0.1146(0.0132)	0.1089(0.0119)	0.1638(0.0268)	0.1513(0.0229)	0.1591(0.0253)
			2	2	0.1618(0.0262)	0.2441(0.0596)	0.3231(0.1044)	0.3046(0.0928)	0.3190(0.1018)
				3	0.1617(0.0262)	0.2453(0.0602)	0.3229(0.1043)	0.3047(0.0928)	0.3169(0.1004)
		15	1	2	0.0829(0.0069)	0.1045(0.0109)	0.2109(0.1310)	0.1509(0.0228)	0.1672(0.0296)
				3	0.0943(0.0090)	0.1025(0.0105)	0.1937(0.2346)	0.1506(0.0227)	0.1587(0.0252)
			2	2	0.1125(0.0127)	0.2488(0.0620)	0.3758(0.1572)	0.3050(0.0930)	0.3255(0.1060)
	30			3	0.1167(0.0136)	0.2383(0.0569)	0.3722(0.2935)	0.3039(0.0923)	0.3185(0.1015)
60		23	1	2	0.0897(0.0081)	0.1087(0.0118)	0.1751(0.0620)	0.1513(0.0229)	0.1618(0.0262)
				3	0.1015(0.0104)	0.1051(0.0110)	0.1873(0.0491)	0.1509(0.0228)	0.1578(0.0249)
			2	2	0.1253(0.0157)	0.2619(0.0687)	0.3298(0.1088)	0.3063(0.0938)	0.3214(0.1033)
				3	0.1335(0.0178)	0.2433(0.0592)	0.3278(0.1075)	0.3044(0.0927)	0.3155(0.0995)
	45	23	1	2	0.0895(0.0080)	0.1089(0.0119)	0.1651(0.0283)	0.1513(0.0229)	0.1619(0.0262)

n	m	k	α	β	Picard Estimates	R-K Estimates	Bayes estimates Gamma Prior	Chara-Prior	Kernel Prior
				3	0.1018(0.0104)	0.1051(0.0110)	0.1792(0.0356)	0.1509(0.0228)	0.1578(0.0249)
			2	2	0.1251(0.0157)	0.2619(0.0686)	0.3297(0.1087)	0.3063(0.0938)	0.3214(0.1033)
				3	0.1329(0.0177)	0.2444(0.0597)	0.3289(0.1082)	0.3045(0.0927)	0.3157(0.0996)
		34	1	2	0.1095(0.0120)	0.1113(0.0124)	0.1597(0.0255)	0.1515(0.0230)	0.1584(0.0251)
				3	0.1135(0.0129)	0.1088(0.0118)	0.1598(0.0255)	0.1512(0.0229)	0.1567(0.0246)
			2	2	0.1675(0.0281)	0.2394(0.0573)	0.3163(0.1001)	0.3042(0.0925)	0.3142(0.0987)
				3	0.1652(0.0273)	0.2416(0.0584)	0.3161(0.0999)	0.3044(0.0926)	0.3128(0.0978)

Table 3. *The AVB and the MSEs in parentheses for parameter β using the Picard, R-K, and Bayes methods at T=0.75 with m = (n/2 and 3n/4), and k= (m/2 and 3m/4).*

n	m	k	α	β	Picard Estimates	R-K Estimates	Bayes estimates Gamma prior	Chara-Prior	Kernel Prior
20	10	5	1	2	0.0946(0.0090)	0.1709(0.0296)	0.4335(0.1883)	0.2976(0.0885)	0.3624(0.1313)
				3	0.1249(0.0156)	0.2698(0.0732)	0.7915(0.6591)	0.4473(0.2001)	0.5141(0.2643)
			2	2	0.1022(0.0104)	0.1072(0.0130)	0.4309(0.1860)	0.2915(0.0850)	0.3536(0.1250)
				3	0.1364(0.0186)	0.1820(0.0360)	0.8130(0.8099)	0.4391(0.1928)	0.5032(0.2532)
		8	1	2	0.0947(0.0090)	0.1712(0.0296)	0.4344(0.1890)	0.2976(0.0886)	0.3626(0.1315)
				3	0.1251(0.0157)	0.2690(0.0727)	0.7896(0.6701)	0.4473(0.2001)	0.5139(0.2641)
			2	2	0.1021(0.0104)	0.1083(0.0132)	0.4301(0.1853)	0.2916(0.0851)	0.3536(0.1250)
				3	0.1366(0.0187)	0.1788(0.0349)	0.8161(0.7925)	0.4389(0.1926)	0.5031(0.2531)
	15	8	1	2	0.0991(0.0099)	0.1611(0.0264)	0.3766(0.1420)	0.2966(0.0880)	0.3441(0.1184)
				3	0.1303(0.0170)	0.2654(0.0709)	0.6113(0.3741)	0.4469(0.1997)	0.5018(0.2518)
			2	2	0.1047(0.0110)	0.1035(0.0122)	0.3690(0.1362)	0.2912(0.0848)	0.3387(0.1147)
				3	0.1371(0.0188)	0.1738(0.0329)	0.5592(0.3127)	0.4382(0.1921)	0.4873(0.2375)
		11	1	2	0.0988(0.0098)	0.1630(0.0270)	0.3782(0.1431)	0.2968(0.0881)	0.3448(0.1189)
				3	0.1304(0.0170)	0.2656(0.0710)	0.6095(0.3718)	0.4469(0.1997)	0.5016(0.2516)
			2	2	0.1045(0.0109)	0.1068(0.0127)	0.3693(0.1364)	0.2914(0.0849)	0.3389(0.1149)
				3	0.1371(0.0188)	0.1734(0.0327)	0.5593(0.3129)	0.4382(0.1921)	0.4873(0.2375)
40	20	10	1	2	0.0976(0.0095)	0.1690(0.0287)	0.3560(0.1268)	0.2973(0.0884)	0.3336(0.1113)
				3	0.1288(0.0166)	0.2674(0.0717)	0.5606(0.3145)	0.4470(0.1998)	0.4882(0.2383)
			2	2	0.1047(0.0110)	0.1063(0.0121)	0.3422(0.1171)	0.2915(0.0850)	0.3241(0.1051)
				3	0.1393(0.0194)	0.1774(0.0332)	0.5241(0.2747)	0.4386(0.1923)	0.4750(0.2256)
		15	1	2	0.0978(0.0096)	0.1683(0.0285)	0.3558(0.1266)	0.2973(0.0884)	0.3335(0.1112)
				3	0.1289(0.0166)	0.2669(0.0715)	0.5598(0.3136)	0.4470(0.1998)	0.4880(0.2382)
			2	2	0.1046(0.0110)	0.1072(0.0123)	0.3423(0.1172)	0.2916(0.0850)	0.3242(0.1051)
				3	0.1396(0.0195)	0.1775(0.0329)	0.5241(0.2747)	0.4385(0.1923)	0.4750(0.2256)

n	m	k	α	β	Picard Estimates	R-K Estimates	Bayes estimates Gamma prior	Chara-Prior	Kernel Prior
60	30	15	1	2	0.0998(0.0100)	0.1628(0.0268)	0.3437(0.1182)	0.2967(0.0881)	0.3277(0.1074)
				3	0.1311(0.0172)	0.2628(0.0693)	0.5356(0.2869)	0.4466(0.1995)	0.4833(0.2336)
			2	2	0.1044(0.0109)	0.0997(0.0109)	0.3333(0.1111)	0.2909(0.0846)	0.3197(0.1022)
				3	0.1393(0.0194)	0.1689(0.0302)	0.5076(0.2576)	0.4378(0.1917)	0.4705(0.2214)
		23	1	2	0.1011(0.0103)	0.1591(0.0256)	0.3388(0.1148)	0.2964(0.0878)	0.3251(0.1057)
				3	0.1319(0.0174)	0.2573(0.0665)	0.5252(0.2759)	0.4461(0.1990)	0.4803(0.2307)
			2	2	0.1035(0.0107)	0.0978(0.0104)	0.3295(0.1086)	0.2906(0.0845)	0.3175(0.1008)
				3	0.1401(0.0196)	0.1634(0.0282)	0.5029(0.2529)	0.4373(0.1913)	0.4684(0.2194)
	30	15	1	2	0.0973(0.0095)	0.1693(0.0288)	0.3388(0.1148)	0.2973(0.0884)	0.3247(0.1054)
				3	0.1281(0.0164)	0.2671(0.0715)	0.5312(0.2823)	0.4470(0.1998)	0.4802(0.2306)
			2	2	0.1042(0.0109)	0.1063(0.0118)	0.3284(0.1079)	0.2915(0.0850)	0.3163(0.1001)
				3	0.1400(0.0196)	0.1797(0.0333)	0.5049(0.2549)	0.4387(0.1924)	0.4682(0.2192)
		23	1	2	0.0973(0.0095)	0.1693(0.0288)	0.3388(0.1148)	0.2973(0.0884)	0.3247(0.1054)
				3	0.1282(0.0164)	0.2668(0.0713)	0.5322(0.2834)	0.4470(0.1998)	0.4803(0.2307)
			2	2	0.1042(0.0109)	0.1056(0.0117)	0.3285(0.1079)	0.2914(0.0849)	0.3163(0.1001)
				3	0.1402(0.0197)	0.1782(0.0327)	0.5051(0.2552)	0.4385(0.1923)	0.4681(0.2191)
	45	23	1	2	0.1003(0.0101)	0.1622(0.0265)	0.3269(0.1068)	0.2966(0.0880)	0.3183(0.1013)
				3	0.1316(0.0173)	0.2639(0.0698)	0.5082(0.2582)	0.4466(0.1995)	0.4753(0.2259)
			2	2	0.1051(0.0111)	0.1030(0.0112)	0.3201(0.1025)	0.2912(0.0848)	0.3122(0.0975)
				3	0.1382(0.0191)	0.1724(0.0308)	0.4843(0.2346)	0.4378(0.1917)	0.4627(0.2141)
		34	1	2	0.0998(0.0100)	0.1620(0.0264)	0.3266(0.1067)	0.2966(0.0880)	0.3181(0.1012)
				3	0.1315(0.0173)	0.2601(0.0678)	0.5027(0.2528)	0.4463(0.1992)	0.4733(0.2240)
			2	2	0.1037(0.0108)	0.0971(0.0100)	0.3178(0.1010)	0.2906(0.0844)	0.3105(0.0964)
				3	0.1406(0.0198)	0.1653(0.0284)	0.4837(0.2340)	0.4374(0.1913)	0.4616(0.2131)

Table 4. *The AVB and the MSEs in parentheses for parameter β using The Picard, R-K, and Bayes methods at T=3 with m = (n/2 and 3n/4) and k= (m/2 and 3m/4).*

n	m	k	α	β	Picard Estimates	R-K Estimates	Bayes estimates Gamma Prior	Chara-Prior	Kernel Prior
20	10	5	1	2	0.0984(0.0098)	0.1837(0.0339)	0.6804(0.6774)	0.2986(0.0892)	0.6804(0.6774)
				3	0.1236(0.0153)	0.2828(0.0802)	0.7087(0.7177)	0.4484(0.2010)	0.7087(0.7177)
			2	2	0.1124(0.0127)	0.1406(0.0209)	0.4113(0.1694)	0.2946(0.0868)	0.4113(0.1694)
				3	0.1349(0.0182)	0.2133(0.0483)	0.5449(0.2970)	0.4419(0.1953)	0.5449(0.2970)
		8	1	2	0.0937(0.0088)	0.1775(0.0317)	0.3818(0.1458)	0.2982(0.0889)	0.3818(0.1458)
				3	0.1233(0.0152)	0.2760(0.0765)	0.5278(0.2786)	0.4479(0.2006)	0.5278(0.2786)
			2	2	0.1024(0.0105)	0.1531(0.0238)	0.3696(0.1366)	0.2958(0.0875)	0.3696(0.1366)

n	m	k	α	β	Picard Estimates	R-K Estimates	Bayes estimates Gamma Prior	Chara-Prior	Kernel Prior
				3	0.1347(0.0182)	0.2186(0.0490)	0.5160(0.2663)	0.4425(0.1958)	0.5160(0.2663)
40	15	8	1	2	0.0935(0.0088)	0.1785(0.0320)	0.3820(0.1459)	0.2982(0.0890)	0.3820(0.1459)
				3	0.1233(0.0152)	0.2762(0.0766)	0.5275(0.2783)	0.4479(0.2006)	0.5275(0.2783)
			2	2	0.1021(0.0104)	0.1549(0.0243)	0.3695(0.1365)	0.2960(0.0876)	0.3695(0.1365)
				3	0.1347(0.0182)	0.2185(0.0489)	0.5161(0.2664)	0.4424(0.1958)	0.5161(0.2664)
		11	1	2	0.0937(0.0088)	0.1745(0.0306)	0.3568(0.1273)	0.2979(0.0887)	0.3568(0.1273)
				3	0.1258(0.0158)	0.2667(0.0714)	0.5087(0.2587)	0.4471(0.1999)	0.5087(0.2587)
			2	2	0.0973(0.0095)	0.1599(0.0258)	0.3512(0.1234)	0.2965(0.0879)	0.3512(0.1234)
				3	0.1324(0.0175)	0.2258(0.0516)	0.5006(0.2506)	0.4431(0.1964)	0.5006(0.2506)
	20	10	1	2	0.0994(0.0099)	0.1850(0.0343)	0.6416(0.5363)	0.2987(0.0892)	0.6416(0.4361)
				3	0.1238(0.0153)	0.2830(0.0802)	0.3526(0.1324)	0.4482(0.2009)	0.4253(0.3842)
			2	2	0.1123(0.0126)	0.1464(0.0219)	0.3757(0.1413)	0.2951(0.0871)	0.3757(0.1413)
				3	0.1367(0.0187)	0.2145(0.0473)	0.5200(0.2705)	0.4420(0.1953)	0.5200(0.2705)
		15	1	2	0.0945(0.0089)	0.1784(0.0319)	0.3579(0.1282)	0.2982(0.0889)	0.3579(0.1282)
				3	0.1233(0.0152)	0.2770(0.0769)	0.5082(0.2583)	0.4480(0.2007)	0.5082(0.2583)
			2	2	0.1026(0.0105)	0.1588(0.0253)	0.3459(0.1196)	0.2963(0.0878)	0.3459(0.1196)
				3	0.1345(0.0181)	0.2259(0.0514)	0.4968(0.2468)	0.4431(0.1963)	0.4968(0.2468)
	30	15	1	2	0.0946(0.0090)	0.1778(0.0317)	0.3571(0.1276)	0.2982(0.0889)	0.3571(0.1276)
				3	0.1232(0.0152)	0.2775(0.0772)	0.5078(0.2579)	0.4480(0.2007)	0.5078(0.2579)
			2	2	0.1028(0.0106)	0.1574(0.0249)	0.3461(0.1198)	0.2962(0.0877)	0.3461(0.1198)
				3	0.1338(0.0179)	0.2306(0.0535)	0.4967(0.2467)	0.4435(0.1967)	0.4967(0.2467)
		23	1	2	0.0919(0.0084)	0.1798(0.0324)	0.3319(0.1101)	0.2984(0.0890)	0.3319(0.1101)
				3	0.1261(0.0159)	0.2681(0.0720)	0.4859(0.2361)	0.4472(0.2000)	0.4859(0.2361)
			2	2	0.0929(0.0086)	0.1812(0.0328)	0.3282(0.1077)	0.2985(0.0891)	0.3282(0.1077)
				3	0.1284(0.0165)	0.2563(0.0657)	0.4811(0.2315)	0.4460(0.1990)	0.4811(0.2315)
60	30	15	1	2	0.0924(0.0086)	0.1861(0.0347)	0.3981(0.1956)	0.2989(0.0893)	0.3981(0.1956)
				3	0.1213(0.0147)	0.2857(0.0817)	0.5197(0.2702)	0.4487(0.2013)	0.5197(0.2702)
			2	2	0.1126(0.0127)	0.1462(0.0216)	0.3602(0.1298)	0.2951(0.0871)	0.3602(0.1298)
				3	0.1375(0.0189)	0.2144(0.0468)	0.5087(0.2588)	0.4419(0.1953)	0.5087(0.2588)
		23	1	2	0.0949(0.0090)	0.1788(0.0320)	0.3449(0.1191)	0.2982(0.0889)	0.3449(0.1191)
				3	0.1234(0.0152)	0.2767(0.0767)	0.4963(0.2463)	0.4479(0.2006)	0.4963(0.2463)
			2	2	0.1034(0.0107)	0.1518(0.0232)	0.3336(0.1113)	0.2957(0.0874)	0.3336(0.1113)
				3	0.1330(0.0177)	0.2362(0.0559)	0.4854(0.2356)	0.4440(0.1972)	0.4854(0.2356)
	45	23	1	2	0.0951(0.0091)	0.1782(0.0318)	0.3450(0.1193)	0.2982(0.0889)	0.3450(0.1193)
				3	0.1234(0.0152)	0.2769(0.0768)	0.4961(0.2461)	0.4479(0.2007)	0.4961(0.2461)
			2	2	0.1034(0.0107)	0.1519(0.0232)	0.3335(0.1112)	0.2957(0.0874)	0.3335(0.1112)
				3	0.1333(0.0178)	0.2338(0.0548)	0.4855(0.2357)	0.4438(0.1970)	0.4855(0.2357)

n	m	k	α	β	Picard Estimates	R-K Estimates	Bayes estimates Gamma Prior	Chara-Prior	Kernel Prior
		34	1	2	0.0918(0.0084)	0.1805(0.0326)	0.3234(0.1046)	0.2984(0.0891)	0.3234(0.1046)
				3	0.1260(0.0159)	0.2683(0.0721)	0.4780(0.2285)	0.4472(0.2000)	0.4780(0.2285)
			2	2	0.0920(0.0085)	0.1843(0.0340)	0.3203(0.1026)	0.2989(0.0893)	0.3203(0.1026)
				3	0.1276(0.0163)	0.2614(0.0683)	0.4734(0.2241)	0.4466(0.1994)	0.4734(0.2241)

4. Real Data Analysis

In this section, we studied two real data sets to demonstrate the performance of the proposed methods on the IW model in practice and to illustrate that this distribution can be considered a good lifetime model for some new areas of applications, compared to many known distributions such as the Weibull distribution. We have fitted these data sets using some goodness of fit tests such as the Kolmogorov-Smirnov (K-S), Anderson-Darling (A-D), and Chi-Square (CH_2) tests for significance level tests equal to 0.05.

4.1. Flood Data Application

Consider the data given by Dumonceaux and Antle [12], which represent the maximum flood levels (in millions of cubic feet per second) of the Susquehanna River at Harrisburg, Pennsylvania over 20 four-year periods (1890-1969) as:

0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.324, 0.269, 0.740, 0.418, 0.412, 0.494, 0.416, 0.338, 0.392, 0.484, 0.265.

Maswadah [20] has fitted these data to the IWD.

We found the inverse Weibull model is a good fit for this dataset, as shown in Table 1 and Figure 2a. For studying the behavior of the flood levels based on this datset, we find the estimates for the parameters that represent the shape and scale of the flood level for 20 four-year periods. We found the Picard, R-K and Bayes estimates for α lie in the interval [3.5, 4.2] and β lie in the interval [0.009, 0.01]. We noticed that the Picard, R-K, and Bayes estimates for α are greater than one, while for β they are close to zero and that indicate the graph is approximately symmetric, see Figure 2b. These estimates indicate that the maximum flood levels are more stable in this period of years.

4.2. Reactor Pumps Data Application

A real data set for secondary nuclear pumps has been analyzed to illustrate the proposed methods. An important aspect of nuclear energy is safety. One of the most severe accidents in nuclear power generation is the loss of coolant, where the re-circulating coolant of the pressurized water reactor may flash into steam. Under such conditions, the reactor cooling pumps become unable to generate the same head as that of the single-phase flow case. Thus, the secondary reactor pump is a feed water pump that takes from the desecrator storage tank feed water pressured up by the booster pump and pushes it into the steam generator through the high-pressure heater. Accordingly, the main feed pump must be high-temperature and high-pressure pump since it requires the head to be larger than the pressure inside the steam generator. The secondary circulation pump differs slightly in design and has been developed specifically for cooling at higher temperatures. The following data set represents the times between the failures of the secondary reactor pumps. Singh et al. ([32, 33]) have discussed the classical and Bayesian estimation methods under the Type-II censoring scheme of this data set. The times between failures of 23 secondary reactor pumps are as follows:

2.160, 0.746, 0.402, 0.954, 0.491, 6.560, 4.992, 0.347, 0.150, 0.358, 0.101, 1.359, 3.465, 1.060, 0.614, 1.921, 4.082, 0.199, 0.605, 0.273, 0.070, 0.062, 5.320.

We found the inverse Weibull model is a good fit for this dataset, as shown in Table 1 and Figure 3a. For studying the reliability of these reactor pumps based on this datset, we find the estimates for the parameters that represent the shape and scale of the failures between pumps using our model to determine the behavior of the failure pumps. We noticed that the Picard, R-K, and Bayes estimates for α lie in the interval [0.6, 0.75] and β lie in the interval [0.39, 0.42]. These estimates indicate that the above dataset is heavily right-skewed, and that means the failure rate decreases with increasing time, see Figure 3b. Thus, we conclude that decreasing the reliability of safety mechanisms with increasing time.

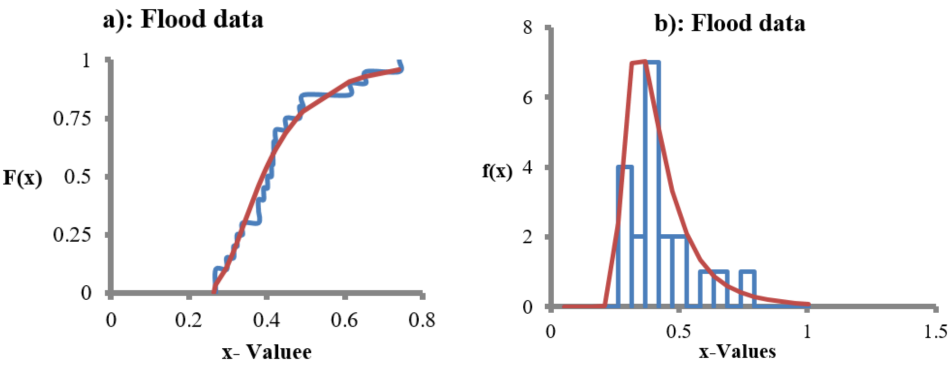

Figure 2. a) The Empirical CDF and the fitted CDF. for the flood data. b) The Histogram and the fitted PDF. for the flood data.

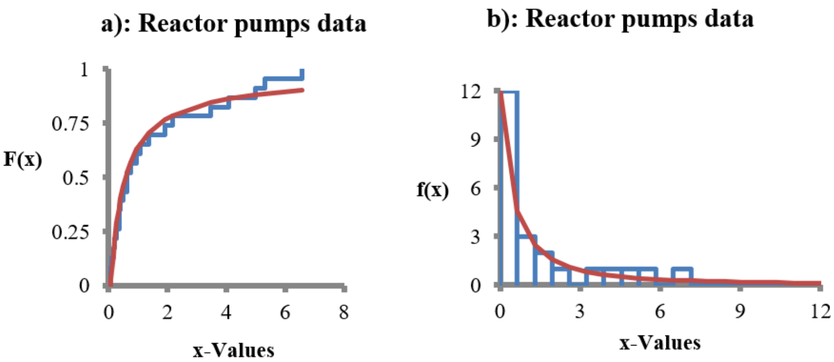

Figure 3. a) The Empirical CDF and the fitted CDF for the reactor pumps data. b) The Histogram and the fitted PDF. for the reactor pumps data.

Table 5. The critical and calculated values for the K-S, A-D and CH2 tests and their powers (p-values) for the inverse Weibull model. The MLE's for the parameters for these data sets have been calculated.

Data	The Tests	Calculated value	Critical value	The p-values	MLES	
					α	β
Flood N=20	K-S	0.6976	0.8482	0.2138	4.3141	0.0119
	A-D	0.3104	0.7414	0.5899		
	CH2	3.5552	31.1109	0.3294		
Reactor pumps N=23	K-S	0.4741	0.8528	0.8113	0.7832	0.4463
	A-D	0.3443	0.7472	0.4915		
	CH2	10.270	31.5744	0.1223		

Table 6. The estimate (Est.) and the MSEs for the parameter α and β based on the Picard, R-K and Bayes methods for the DGHPCS: for $m = 3n/4,\ k = 3m/4$.

Sam.	T	Par.	Picard Estimate		R-K Estamite		Charact. Prior		Gamma Prior		Kernel Prior	
			Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE
Data 4.1	0.25	α	4.1412	0.0298	3.9213	0.1539	3.6551	0.4337	3.5358	0.6051	3.6195	0.4819

Sam.	T	Par.	Picard Estimate		R-K Estamite		Charact. Prior		Gamma Prior		Kernel Prior	
			Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE
N=20		β	0.0112	0.0001	0.0153	0.0002	0.0101	0.0003	0.0101	0.0003	0.0101	0.0003
	0.65	α	4.1411	0.0297	3.3666	0.4192	3.6334	0.4629	3.4723	0.7079	3.5907	0.5227
		β	0.0110	0.0001	0.0133	0.0002	0.0099	0.00004	0.0098	0.0004	0.0099	0.0004
	0.15	α	0.7486	0.0012	0.7271	0.0031	0.6627	0.0145	0.6456	0.0189	0.6471	0.0185
Data 4.2		β	0.3958	0.0025	0.4493	0.0011	0.3804	0.0043	0.3702	0.0058	0.3708	0.0057
N=23	5.0	α	0.7402	0.0018	0.7288	0.0029	0.6629	0.0144	0.6408	0.0203	0.6431	0.0196
		β	0.3922	0.0029	0.4446	0.0012	0.3798	0.0044	0.3661	0.0064	0.3671	0.0063

The results in Table 5 indicate that the inverse Weibull distribution is a good fit for these datasets, as shown in Figures 2a, 3a, where the calculated values for the goodness of fit test statistics are less than the critical values and the power of the tests is greater than the significance level of the tests, which is equal to 0.05.

The results in Table 6 indicate that the estimated MSE values based on the Picard and Runge-Kutta methods are less than those based on the Bayes' method, based on the three different priors for large values of T with considering the MLEs as the true values of the parameters. Thus, the results of these datasets ensure the simulation results.

5. Conclusions

In this study, the Picard, the Runge-Kutta, and the Bayes methods have been used for estimating the inverse Weibull distribution parameters based on the dual generalized progressive hybrid censoring data. The simulation and real dataset results have indicated that the Bayes method has biased estimates and remains noticeable even when the sample sizes are too large. However, the Picard and R-K estimates are highly unbiased and much more efficient than the Bayes estimates, even when using the informative prior. Moreover, the results based on the non-parametric characteristic and kernel priors are more efficient than the results based on the informative Gamma prior, due to choosing the hyperparameters, which sometimes can mislead to subsequent inferences.

Abbreviations

DGOS	The Dual Generalized Order Statistics
GOS	Generalized Order Statistics
IWD	Inverse Weibull Distribution
MLE	Maximum Likelihood Estimators
GPHCS	Generalized Progressive Hybrid Censoring Scheme
DGPHCS	The Dual Generalized Progressive Hybrid Censoring Scheme
AVB	The Absolute Average Bias
MSE	The Root Mean Squared Error
R-K	Runge-Kutta

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Author Contributions

Mohamed Maswadah: He did the simulations and approved the final manuscript.

Alia A. Alkhathami: He reviewed the final manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Appendix

Appendix I: The Characteristic Prior

The joint characteristic function for two random variables X and Y can be defined as

$$\emptyset_{X,Y}(t_1, t_2) = E\left(e^{i(t_1X+t_2Y)}\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(t_1x+t_2y)} f(x, y)dxdy,$$

and the inversion formula for the PDF is as follows:

$$f(x, y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(t_1x+t_2y)} \emptyset_{X,Y}(t_1, t_2)dt_1dt_2, \quad (15)$$

The empirical characteristic function ECF is the sample counter part of the CF and defined by

$$\emptyset_N(t_1,t_2) =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(t_1x+t_2y)} f_N(x,y)dxdy = \frac{1}{N}\sum_{i=1}^{N} e^{i(t_1x_i+t_2y_i)}$$

It is known that the ECF converges almost surely to the population characteristic function, that is

$$\emptyset_N(t_1,t_2) \xrightarrow{\text{almost surly}} \emptyset_{X,Y}(t_1,t_2)$$

Since α and β are random variables in the Bayesian concept, therefore from the inversion formula using the ECF as estimation for the CF we get from (15)

$$\hat{g}(\alpha,\beta) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(t_1\alpha+t_2\beta)}\,\widehat{\emptyset}_N(t_1,t_2)dt_1dt_2$$

$$= \frac{1}{4N\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(t_1\alpha+t_2\beta)}\sum_{i=1}^{N} e^{i(t_1\hat{\alpha}_i+t_2\hat{\beta}_i)}\,dt_1dt_2$$

$$= \frac{1}{4N\pi^2}\sum_{i=1}^{N}\frac{1}{|(\alpha-\hat{\alpha}_i)(\beta-\hat{\beta}_i)|}.$$

Thus, $\hat{g}(\alpha,\beta)$ is the joint characteristic prior function for α and β.

Appendix II: The Derivatives of Tierny and Kadane Method

The log of the posterior density function (13) can be derived as follows:

$$H(\alpha,\beta|\underline{x}) = [p_1\ln\hat{g}_1(\alpha) + p_2\ln\hat{g}_2(\beta) + s_1\ln\hat{q}_1(\alpha) + s_2\ln\hat{q}_2(\beta) + (n+a-1)\ln\alpha$$

$$+(n+c-1)\ln\beta - \alpha b - (\alpha+1)\sum_{i=1}^{n}\ln(x_i)$$

$$-\beta[d + \sum_{i=1}^{n}(1+R_i)x_i^{-\alpha} + \delta R_T^{*}T^{-\alpha})].$$

$$H_1 = \frac{\partial H}{\partial\alpha} = [p_1\frac{\hat{g}_1'(\alpha)}{\hat{g}_1(\alpha)} + s_1\frac{q_1'(\alpha)}{\hat{q}_1(\alpha)} + \frac{n+a-1}{\alpha} - b - \sum_{i=1}^{n}\ln x_i$$

$$+\beta[\sum_{i=1}^{n}(1+R_i)x_i^{-\alpha}\ln x_i + \delta R_T^{*}T^{-\alpha}\ln T],$$

$$H_{11} = \frac{\partial^2 H}{\partial\alpha^2} = [p_1\frac{\hat{g}_1(\alpha)\hat{g}_1''(\alpha)-\hat{g}'^2_1(\alpha)}{\hat{g}_1^2(\alpha)} + s_1\frac{\hat{q}_1(\alpha)\hat{q}_1''(\alpha)-\hat{q}'^2_1(\alpha)}{\hat{q}_1^2(\alpha)} - \frac{n+a-1}{\alpha^2}$$

$$-\beta[\sum_{i=1}^{n}(1+R_i)x_i^{-\alpha}(\ln x_i)^2 + \delta R_T^{*}T^{-\alpha}(\ln T)^2],$$

$$H_2 = \frac{\partial H}{\partial\beta} = [p_2\frac{\hat{g}_2'(\beta)}{\hat{g}_2(\beta)} + s_2\frac{q_2'(\beta)}{\hat{q}_2(\beta)} + \frac{n+c-1}{\beta} - d - (\sum_{i=1}^{n}(1+R_i)x_i^{-\alpha} + \delta R_T^{*}T^{-\alpha})],$$

$$H_{22} = \frac{\partial^2 H}{\partial\beta^2} = [p_2\frac{\hat{g}_2(\beta)g_2''(\beta)-\hat{g}_2'^2(\beta)}{\hat{g}_2^2(\beta)} + s_2\frac{\hat{q}_2(\beta)q_2''(\beta)-\hat{q}_2'^2(\beta)}{\hat{q}_2^2(\beta)} - \frac{N+c-1}{\beta^2},$$

$$H_{12} = \frac{\partial^2 H}{\partial\beta\partial\alpha} = \sum_{i=1}^{n}(1+R_i)x_i^{-\alpha}\ln(x_i) + \delta R_T^{*}T^{-\alpha}\ln T$$

$$\frac{\partial^2 H}{\partial\beta\partial x} = \alpha\sum_{i=1}^{n}(1+R_i)x_i^{-\alpha-1}$$

$$\frac{\partial^2 H}{\partial\alpha\partial x} = -\sum_{i=1}^{n}\frac{1}{x_i} - \alpha\beta\sum_{i=1}^{n}(1+R_i)x_i^{-\alpha-1}[\ln x_i + 1]$$

where the r^{th} derivative of the kernel density estimation can be defined as follows:

$$\frac{d^r\hat{g}_1(\alpha)}{d\alpha^r} = \hat{g}_1{}^r(\alpha) = \frac{1}{nh_1^{r+1}}\sum_{i=1}^{n}K^r\left(\frac{\alpha-\hat{\alpha}_i}{h_1}\right), \qquad (16)$$

where r =0,1,2,3, -----.

Using the Gaussian kernel and (16), we have

$$\hat{g}_1(\alpha) = \frac{1}{nh_1\sqrt{2\pi}}\sum_{i=1}^{n} e^{-0.5(\frac{\alpha-\hat{\alpha}_i}{h_1})^2},$$

$$\hat{g}_1'(\alpha) = -\frac{1}{nh_1^2\sqrt{2\pi}}\sum_{i=1}^{n}\left(\frac{\alpha-\hat{\alpha}_i}{h_1}\right)e^{-0.5(\frac{\alpha-\hat{\alpha}_i}{h_1})^2},$$

$$\hat{g}_1''(\alpha) = \frac{1}{nh_1^3\sqrt{2\pi}}\sum_{i=1}^{n}[(\frac{\alpha-\alpha_i}{h_1})^2 - 1]\,e^{-0.5(\frac{\alpha-\alpha_i}{h_1})^2}.$$

Similarly for the kernel priors $\hat{g}_2(\beta)$.

For the parameter α the Characteristic prior and their derivatives can be derived as follows:

$$\hat{q}_1(\alpha) = \frac{1}{2\pi n}\sum_{i=1}^{n}\frac{1}{|(\alpha-\hat{\alpha}_i)|},\ \hat{q}_1'(\alpha) = -\frac{1}{2\pi n}\sum_{i=1}^{n}\frac{1}{(\alpha-\hat{\alpha}_i)^2}$$

and $\hat{q}_1''(\alpha) = \frac{1}{\pi n}\sum_{i=1}^{n}\frac{1}{(\alpha-\hat{\alpha}_i)^3}$.

Similarly for the characteristic priors $\hat{q}_2(\beta)$.

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