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Generated Fuzzy Quasi-ideals in Ternary Semigroups

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Abstract: Here in this paper, we provide characterizations of fuzzy quasi-ideal in terms of level and strong level subsets. Along with it, we provide expression for the generated fuzzy quasi-ideal generated by level subsets and strong level subsets of the given fuzzy set in explicit manner. We also establish the existence for a generated fuzzy quasi-ideal in ternary semigroup by showing that the non-empty intersection of an arbitrary family of fuzzy quasi-ideals is again a fuzzy quasi-ideal. This paper introduces and explores the concept of generated fuzzy quasi-ideals in ternary semigroups, extending classical algebraic notions into the fuzzy domain. A fuzzy quasi-ideal is defined as a fuzzy set that satisfies specific conditions analogous to those of crisp quasi-ideals under the ternary operation. A foundational result established in this work is that the non-empty intersection of any family of fuzzy quasi-ideals in a ternary semigroup remains a fuzzy quasi-ideal, reinforcing the internal consistency of the structure. Furthermore, we explore key properties of these generated fuzzy quasi-ideals, including their relationships with level sets and strong level subsets. The central focus of the paper is on how fuzzy quasi-ideals can be generated by arbitrary fuzzy sets within a ternary semigroup. We establish methods for constructing the smallest fuzzy quasi-ideal containing a given fuzzy set, along with expressions for this generated structure in terms of the quasi-ideals generated by its level and strong level subsets. Through constructive proofs, we demonstrate the existence by showing the non-empty intersection of an arbitrary family of fuzzy quasi-ideals in a ternary semigroup is itself a fuzzy quasi-ideal and uniqueness of such generated fuzzy quasi-ideals. The findings contribute to a deeper understanding of the internal structure of ternary semigroups and provide a foundational framework for further research in fuzzy algebraic systems. In summary, this work establishes a comprehensive framework for generated fuzzy quasi-ideals in ternary semigroups, revealing their structural properties, generation mechanisms, and theoretical importance. These results contribute meaningfully to the study of fuzzy algebraic systems and open new avenues for further research in fuzzy ternary algebra.

Keywords: Ternary Semigroups, Arbitrary Intersection, Level Subsets, Strong Level Subsets, Generated Fuzzy Ideals

1. Introduction

Fuzzy mathematics, particularly fuzzy logic and fuzzy set theory, offers a flexible approach to addressing real-world problems characterized by ambiguity, uncertainty, and partial truths. Unlike classical mathematics, which operates on strict binary logic (true or false, yes or no), fuzzy logic allows for a spectrum of truth values between 0 and 1, better reflecting the nuances of human perception and language. This adaptability makes it invaluable across various domains: household appliances like washing machines adjust settings based on inputs such as "lightly soiled" rather than rigid categories; in healthcare, it supports diagnostic systems that interpret vague symptoms like "slight fever" or "moderate pain"; and in AI, it enables machines to reason in uncertain or partially visible environments. Fuzzy logic is also used to process imprecise data in weather forecasting, model subjective financial terms like "moderate risk", and manage complex control systems in industries where precise measurement is challenging. It plays a role in transportation-optimizing traffic signals and cruise control, as well as in agriculture, where fuzzy systems assess qualitative data like "dry soil" or "warm day" to guide irrigation. Even human emotions, such as happiness or anxiety, which lack clear boundaries, can be effectively modeled through fuzzy sets, highlighting the system's broad capability to simulate and manage the kind of imprecision that traditional logic cannot.

Fuzzy logic plays a crucial role across various domains by enabling systems to handle ambiguity and uncertainty in a manner similar to human reasoning. In Natural Language Processing (NLP), it aids in interpreting the nuanced and often ambiguous nature of human language, supporting tasks such as sentiment analysis, machine translation, and information retrieval by assigning degrees of meaning to words and phrases. In image processing, fuzzy techniques are employed for tasks like image segmentation, edge detection, noise reduction, and pattern recognition, effectively managing imprecise boundaries and object classification under uncertainty. Consumer electronics showcase fuzzy logic in everyday applications such as air conditioners, rice cookers, camcorders, and microwave ovens, where it helps optimize control based on variable inputs. In the automotive sector, it enhances safety and performance in systems like anti-lock braking, automatic transmission, cruise control, and intelligent traffic management by processing imprecise sensor data. Aerospace applications include altitude control and satellite attitude regulation, where conditions are highly dynamic and difficult to model precisely. For complex systems lacking exact mathematical models, fuzzy logic provides an intuitive and efficient alternative for control and decision-making. Overall, fuzzy mathematics serves as a powerful framework for building intelligent systems that operate effectively in ambiguous and uncertain environments, bridging the gap between human intuition and machine precision.

The theory of ternary algebraic system was introduced by Lehmer [8] in 1932. A ternary semigroup, an algebraic concept with origins traced to Banach, was demonstrated through his example to not always simplify into a standard semigroup. Los [9] showed that any ternary semigroup may be embedded in an ordinary semigroup in such a way that the ternary operation in the ternary semigroup is an extension of the binary operation in the ordinary semigroup. Sioson [12] studied ideal theory in ternary semigroups and gave the definitions of ideals. He also introduced the notion of regular ternary semigroups and characterized them by using the notion of quasi-ideals. In [11], Santiago developed the theory of ternary semigroups and semiheaps. Hewitt. and Zuckerman [3] studied about Ternary opertions in ternary semigroups. Iampan [4] defined lateral ideals of ternary semigroup.

In 1995, Dixit and Dewan [1] derived expressions for various types of ideals in a ternary semigroup, including left, right, lateral ideals, as well as bi-ideals and quasi-ideals, all generated by a subset of a ternary semigroup. Dutta et al. [2] established expressions for left (lateral, right) ideals and the ideal generated by an element in a ternary semigroup. Later, in 2012, Kar and Sarkar [5] contributed by providing expressions for left (right, lateral) ideals and the ideal generated by an element within a ternary semigroup. Additionally, in the same year, Kar and Sarkar [6] introduced expressions for generated fuzzy left (right, lateral) ideals and fuzzy ideals by a fuzzy set within a ternary semigroup.

Kar and Sarkar [6] provided set theortic definition of fuzzy quasi-ideal in ternary semigroups in 2012. Srivastava [13] defined pointwise definition of fuzzy quasi-ideals in ternary semigroups and provided equivalence between these two defintions. Here in this paper, we provide characterizations of fuzzy quasi-ideal in terms of level and strong level subsets. Along with it, we provide expression for the generated fuzzy quasi-ideal in terms of quasi-ideal generated by level subsets and strong level subsets of the given fuzzy set in explicit manner, using the construction given by Kumar [7]. We also establish the existence for a generated fuzzy quasi-ideal in ternary semigroup by showing that the non-empty intersection of an arbitrary family of fuzzy quasi-ideals is again a fuzzy quasi-ideal.

Quasi-Ideals are specific types of ideals (generalizing left, right, and bi-ideals) that arise in ternary semigroups and capture particular multiplicative properties. Combining all concepts leads to fuzzy-generated quasi-ideals, which explore how these specific ideal structures behave under the lens of fuzzy membership. This is a natural approach in abstract algebra, seeking to generalize and refine existing concepts. In essence, studying fuzzy generated quad-ideals in ternary semigroups is a pursuit of mathematical generalization and abstraction, driven by the desire to build robust frameworks that can handle increasing complexity and uncertainty. While direct real-world applications of "fuzzy generated quasi-ideals" might not be immediately obvious to a non-specialist, the underlying principles and tools developed in this area contribute to the broader mathematical landscape that supports advancements in areas like artificial intelligence, data analysis, and complex system modelling.

2. Preliminaries

A *ternary semigroup* is a non-empty set S together with a ternary operation which has the property of association: $(abc)de = a(bcd)e = ab(cde)$, $\forall\ a, b, c, d, e \in S$. A non-empty set T of a ternary semigroup S is called a *subsemigroup* of S if $T^3 \subseteq T$. A non-empty subset Q of a ternary semigroup S is called a *quasi-ideal* if $SSQ \cap (SQS \cup SSQSS) \cap QSS \subseteq Q$.

We recall the definition of a fuzzy set f on a non-set X defined by Zadeh [14] as a mapping $f : X \to [0, 1]$. Let S be a ternary semigroup. Define a binary operation $* : F(S) \times F(S) \times F(S) \to F(S)$ as follows:

$$(f_1 * f_2 * f_3)(x) = \begin{cases} \sup\limits_{x=x_1*x_2*x_3} [\min\{f_1(x_1), f_2(x_2), f_2(x_3)\}] \\ 0, \qquad\qquad \text{if } x \text{ not expressed as } x = x_1 * x_2 * x_3 \end{cases}$$

Definition 2.1 ([5]). A fuzzy set f of a ternary semigroup S is called a *fuzzy subsemigroup* of S if $f(xyz) \geq \min\{f(x), f(y), f(z)\}, \forall\, x, y, z \in S$.

Definition 2.2 ([6]). A fuzzy set f of a ternary semigroup S is called a *fuzzy quasi-ideal* of S if $f \circ S \circ S \cap (S \circ f \circ S \cup S \circ S \circ f \circ S \circ S) \cap S \circ S \circ f \subseteq f$.

Definition 2.3 ([13]). A fuzzy set f of a ternary semigroup S is called a fuzzy quasi-ideal if $\forall\, z \in S$,

$$f(z) \geq \min\left\{\sup_{z=x_1x_2x_3} f(x_3), \max\left\{\sup_{z=x_1x_2x_3} f(x_2), \quad \sup_{z=x_1x_2x_3x_4x_5} f(x_3)\right\}, \sup_{z=x_1x_2x_3} f(x_1)\right\}.$$

The author in [13] established the following equivalence between Definitions 2.2 and 2.3.

Theorem 2.1 ([13]). Let f be a fuzzy set in a ternary semigroup S. Then f is a fuzzy quasi-ideal of S if and only if $f \circ S \circ S \cap (S \circ f \circ S \cup S \circ S \circ f \circ S \circ S) \cap S \circ S \circ f \subseteq f$.

3. Arbitrary Intersection of Fuzzy Ideals in a Ternary Semigroup

Theorem 3.1. The non-empty intersection of an arbitrary family of fuzzy subsemigroups of a ternary semigroup S is a fuzzy subsemigroups of S.

Proof. Let $\{f_i\}_{i\in\delta}$ be family of fuzzy subsemigroup of S and let $f = \bigcap_{i\in\delta} f_i$, where δ is an index set. Let $x, y, z \in S$. Then

$$\begin{aligned} f(xyz) &= \bigcap_{i\in\delta} f_i(xyz) \\ &= \inf_{i\in\delta} f_i(xyz) \\ &\geq \inf_{i\in\delta} \min\{f_i(x), f_i(y), f_i(z)\} \\ &= \min\left\{\inf_{i\in\delta} f_i(x), \inf_{i\in\delta} f_i(y), \inf_{i\in\delta} f_i(z)\right\} \\ &= \min\left\{\bigcap_{i\in\delta} f_i(x), \bigcap_{i\in\delta} f_i(y), \bigcap_{i\in\delta} f_i(z)\right\}. \end{aligned}$$

Thus f is a fuzzy subsemigroup of S.

Theorem 3.2. The non-empty intersection of an arbitrary family of fuzzy quasi-ideals of a ternary semigroup S is a fuzzy quasi-ideal of S.

Proof. Let $\{f_i\}_{i\in\delta}$ be family of fuzzy quasi-ideals of S and let $f = \bigcap_{i\in\delta} f_i$, where δ is an index set. Now for $z \in S$,

$$\begin{aligned} f(z) &= \bigcap_{i\in\delta} f_i(z) \\ &\geq \inf_{i\in\delta} \min\left\{\sup_{z=x_1x_2x_3} f_i(x_3), \max\left\{\sup_{z=x_1x_2x_3} f_i(x_2), \quad \sup_{z=x_1x_2x_3x_4x_5} f_i(x_3)\right\}, \sup_{z=x_1x_2x_3} f_i(x_1)\right\} \\ &= \min\left\{\inf_{i\in\delta} \sup_{z=x_1x_2x_3} f_i(x_3), \inf_{i\in\delta} \max\left\{\sup_{z=x_1x_2x_3} f_i(x_2), \quad \sup_{z=x_1x_2x_3x_4x_5} f_i(x_3)\right\}, \inf_{i\in\delta} \sup_{z=x_1x_2x_3} f_i(x_1)\right\} \\ &= \min\left\{\sup_{z=x_1x_2x_3} \inf_{i\in\delta} f_i(x_3), \max\left\{\sup_{z=x_1x_2x_3} \inf_{i\in\delta} f_i(x_2), \quad \sup_{z=x_1x_2x_3x_4x_5} \inf_{i\in\delta} f_i(x_3)\right\}, \sup_{z=x_1x_2x_3} \inf_{i\in\delta} f_i(x_1)\right\} \\ &= \min\left\{\sup_{z=x_1x_2x_3} \bigcap_{i\in\delta} f_i(x_3), \max\left\{\sup_{z=x_1x_2x_3} \bigcap_{i\in\delta} f_i(x_2), \quad \sup_{z=x_1x_2x_3x_4x_5} \bigcap_{i\in\delta} f_i(x_3)\right\}, \sup_{z=x_1x_2x_3} \bigcap_{i\in\delta} f_i(x_1)\right\}. \end{aligned}$$

4. Level Subsets and Strong Level Subsets of a Fuzzy Set in a Ternary Semigroup

Definition 4.1. Let f be a fuzzy set defined on a non-empty set X, and $t \in [0, 1)$. The level subset f_t, and the strong level subset $f_t^>$ of f are defined as

$$f_t = \{x \in X \mid f(x) \geq t\} \text{ and } f_t^> = \{x \in X \mid, f(x) > t\}.$$

Example 4.1. Level subsets and strong level subsets are useful for categorizing and analyzing data where the membership isn't black and white, but rather a matter of degree.

Real-Life Example of a Level Subset f_t

Let's consider a fuzzy set defined on a group of students based on their "participation" in a class. The fuzzy set f assigns a value between 0 and 1 to each student, indicating their degree of participation.

Define a fuzzy Set f (Student Participation):

$f(\text{Alice}) = 0.9$ (Very high participation),

$f(\text{Bob}) = 0.7$ (Good participation),

$f(\text{John}) = 0.5$ (Moderate participation),

$f(\text{Don}) = 0.3$ (Low participation),

$f(\text{Eve}) = 0.1$ (Very low participation)

Level Subset: Imagine the teacher wants to identify all students who show "at least moderate participation". They decide that "moderate participation" means a participation score of 0.5 or higher. Therefore, we have to find level subset $f_{0.5}$ where $t = 0.5$.

$$f_{0.5} = \{x \in \text{Students} \mid f(x) \geq 0.5\}$$

i.e.

$$f_{0.5} = \{\text{Alice, Bob, John}\}$$

This level subset identifies the group of students (Alice, Bob, and John) who meet or exceed the participation threshold of 0.5. This could be used, for example, to determine who qualifies for a bonus participation grade or who is eligible to lead a group project.

Real-Life Example of a Strong Level Subset $f_t^{>}$

Let's consider a fuzzy set defined on a set of houses, representing their "desirability" to a potential buyer. The fuzzy set f assigns a value between 0 and 1, with higher values indicating higher desirability.

Define a fuzzy Set f (House Desirability):

$f(\text{House A}) = 0.95$ (Extremely desirable),

$f(\text{House B}) = 0.80$ (Very desirable),

$f(\text{House C}) = 0.60$ (Desirable),

$f(\text{House D}) = 0.40$ (Moderately desirable),

$f(\text{House E}) = 0.80$ (Very desirable),

$f(\text{House F}) = 0.50$ (Moderately desirable).

Strong Level Subset: The buyer wants to identify only the houses that are "exceptionally desirable", meaning their desirability score must be strictly greater than a certain threshold. They set this threshold at 0.75. Therefore, we have to find strong level subset $f_{0.75}^{>}$ where $t = 0.75$.

$$f_{0.75}^{>} = \{x \in \text{Houses} \mid f(x) > 0.75\}$$

i.e.

$$f_{0.75}^{>} = \{\text{House A, House B, House E}\}$$

This strong level subset identifies House A, House B, and House E as the ones that are considered exceptionally desirable (strictly above 0.75). Notice that House C and House F, while "desirable" or "moderately desirable", are excluded because their scores are not strictly greater than 0.75. This could be used to create a "shortlist" of top-tier properties for immediate viewing. In both examples, the choice between a level subset and a strong level subset depends on whether you want to include items that meet the threshold exactly (level subset) or only those that strictly exceed it (strong level subset).

It is well known that

Lemma 4.1. Let f and g be fuzzy sets in a ternary semigroup S. Then, $\forall\, t \in [0, 1)$

1. $(f \circ g)_t^{>} = f_t^{>} g_t^{>}$.
2. $(f \cap g)_t^{>} = f_t^{>} \cap g_t^{>}$.
3. $(f \cup g)_t^{>} = f_t^{>} \cup g_t^{>}$.

Theorem 4.1. The following statements are equivalent in a ternary semigroup S:

1. f is a fuzzy subsemigroup of S.
2. Each $\emptyset \neq f_t$ of S is a subsemigroup of S.
3. Each $\emptyset \neq f_t^{>}$ of f is a subsemigroup of S.

Proof. (1)⇒(3). Let f be a fuzzy subsemigroup of S. We claim that $f_t^{>} f_t^{>} f_t^{>} \subseteq f_t^{>}$. Let $x \in f_t^{>} f_t^{>} f_t^{>}$. Then $x = a_1a_2a_3$, for some $a_1, a_2, a_3 \in f_t^{>}$. This implies that $f(a_1) > t$, $f(a_2) > t$, $f(a_3) > t$. Since f is a fuzzy subsemigroup, therefore, $f(a_1a_2a_3) \geq \min\{f(a_1), f(a_2), f(a_3)\} > t$. Thus $x = a_1a_2a_3 \in f_t^{>}$. Hence $f_t^{>} f_t^{>} f_t^{>} \subseteq f_t^{>}$.

(3)⇒(2). Let $\emptyset \neq f_t$ be a level subset of f. Then, $f_t = \bigcap\limits_{r<t} f_t^{>}$. Since $\emptyset \neq f_t$, $\emptyset \neq f_t^{>}$ for each $r < t$. By (3) $f_t^{>}$ is a subsemigroup of S. Again, since the non-empty intersection of an arbitrary family of subsemigroups of S is a subsemigroup of S, f_t is a subsemigroup.

(2)⇒(1). Let f_t is a subsemigroup of S and suppose f is not a fuzzy subsemigroup of S. Since f is not a fuzzy subsemigroup of S, $f(xyz) < \min\{f(x), f(y), f(z)\}$, for some $x, y, z \in S$. This implies $f(a) < f(x)$, $f(a) < f(y)$, $f(a) < f(z)$, where $a = xyz$. Choose a real number t such that $f(a) < t < \min\{f(x), f(y), f(z)\}$. Then $a \notin f_t$ and $x, y, z \in f_t$. Therefore, $a = xyz \in f_tf_tf_t$ but $a \notin f_t$. Hence $f_tf_tf_t \nsubseteq f_t$, which contradicts (2).

Theorem 4.2. The following statements are equivalent in a ternary semigroup S:

1. f is a fuzzy quasi-ideal of S.
2. Each $\emptyset \neq f_t$ of f is a quasi-ideal of S.
3. Each $\emptyset \neq f_t^{>}$ of f is a quasi-ideal of S.

Proof. (1)⇒(3). Let f be a fuzzy quasi-ideal of S. Let $f_t^{>}$ be a non-empty strong level subset of S. To show that $f_t^{>}$ is a quasi-ideal of S. If possible let, $f_t^{>}$ is not a quasi-ideal of S. Therefore, $f_t^{>}$ is a subsemigroup of S and $(SSf_t^{>}) \cap (Sf_t^{>}S \cup SSf_t^{>}SS) \cap f_t^{>}SS) \nsubseteq f_t^{>}$, that is,

$$(SSf_t^{>} \cap Sf_t^{>}S \cap f_t^{>}SS) \cup (SSf_t^{>} \cap SSf_t^{>}SS \cap f_t^{>}SS) \nsubseteq f_t^{>}.$$

Therefore, either $SSf_t^{>} \cap Sf_t^{>}S \cap f_t^{>}SS \nsubseteq f_t^{>}$ or $SSf_t^{>} \cap SSf_t^{>}SS \cap f_t^{>}SS \nsubseteq f_t^{>}$.

If $SSf_t^{>} \cap Sf_t^{>}S \cap f_t^{>}SS \nsubseteq f_t^{>}$, then, there exists $z \in SSf_t^{>} \cap Sf_t^{>}S \cap f_t^{>}SS$ but $z \notin f_t^{>}$. Therefore, $z = s_1s_2x_1 = s_3x_2s_4 = x_3s_5s_6$ for some $x_1, x_2, x_3 \in f_t^{>}$ and $s_1, s_2, s_2, s_3, s_4, s_5, s_6 \in S$. Since $x_1, x_2, x_3 \in f_t^{>}$, $f(x_1) > t$, $f(x_2) > t$, $f(x_3) > t$, this implies

$$\sup_{z=s_1s_2x_1} f(x_1) > t, \quad \sup_{z=s_3x_2s_4} f(x_2) > t, \quad \sup_{z=x_3s_5s_6} f(x_3) > t$$

that is, in general,

$$\sup_{z=x_1x_2x_3} f(x_3) > t, \quad \sup_{z=x_1x_2x_3} f(x_2) > t, \quad \sup_{z=x_1x_2x_3} f(x_1) > t.$$

If $SSf_t^> \cap SSf_t^> SS \cap f_t^> SS \nsubseteq f_t^>$, then, there exists $z \in SSf_t^> \cap SSf_t^> SS \cap f_t^> SS$ but $z \notin f_t^>$. Therefore, $z = s_1s_2x_1 = s_3s_4x_2s_5s_6 = x_3s_7s_8$ for some $x_1, x_2, x_3 \in f_t^>$ and $s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8 \in S$. Since $x_1, x_2, x_3 \in f_t^>$, $f(x_1) > t$, $f(x_2) > t$, $f(x_3) > t$, this implies

$$\sup_{z=x_1x_2x_3} f(x_1) > t, \quad \sup_{z=s_3s_4x_2s_5s_6} f(x_2) > t, \quad \sup_{z=x_3s_7s_8} f(x_3) > t$$

that is, in general,

$$\sup_{z=x_1x_2x_3} f(x_3) > t, \quad \sup_{z=x_1x_2x_3x_4x_5} f(x_3) > t, \quad \sup_{z=x_1x_2x_3} f(x_1) > t.$$

Since f is a fuzzy quasi-ideal. Then,

$$f(z) \geq \min\left\{ \sup_{z=x_1x_2x_3} f(x_3), \max\left\{ \sup_{z=x_1x_2x_3} f(x_2), \sup_{z=x_1x_2x_3x_4x_5} f(x_3) \right\}, \sup_{z=x_1x_2x_3} f(x_1) \right\}.$$

Therefore, $f(z) > t$ and hence $z \in f_t^>$, a contradiction as $z \notin f_t^>$.
(3)⇒(2). Follows similar to (3)⇒(2) of Theorem 4.1.
(2)⇒(1). Suppose f is not a fuzzy quasi-ideal of S. Since f is not a fuzzy quasi-ideal of S, therefore, for some $z \in S$

$$f(z) < \min\left\{ \sup_{z=x_1x_2x_3} f(x_3), \max\left\{ \sup_{z=x_1x_2x_3} f(x_2), \sup_{z=x_1x_2x_3x_4x_5} f(x_3) \right\}, \sup_{z=x_1x_2x_3} f(x_1) \right\}.$$

Therefore,

$$\begin{aligned} f(z) &< \sup_{z=x_1x_2x_3} f(x_3), \\ f(z) &< \max\left\{ \sup_{z=x_1x_2x_3} f(x_2), \sup_{z=x_1x_2x_3x_4x_5} f(x_3) \right\}, \\ f(z) &< \sup_{z=x_1x_2x_3} f(x_1). \end{aligned}$$

Now $f(z) < \sup_{z=x_1x_2x_3} f(x_3)$ implies that there exists $x_1, x_2, x_3 \in S$ such that $z = x_1x_2x_3$ and $f(z) < f(x_3)$. Similarly, $f(z) < \sup_{z=x_1x_2x_3} f(x_1)$ implies that there exists $x_1, x_2, x_3 \in S$ such that $z = x_1x_2x_3$ and $f(z) < f(x_1)$. Also,

$$f(z) < \max\left\{ \sup_{z=x_1x_2x_3} f(x_2), \sup_{z=x_1x_2x_3x_4x_5} f(x_3) \right\}$$

implies that either

$$f(z) < \sup_{z=x_1x_2x_3} f(x_2) \ \text{or} \ f(z) < \sup_{z=x_1x_2x_3x_4x_5} f(x_3),$$

that is either there exists $x_1, x_2, x_3 \in S$ such that $z = x_1x_2x_3$ and $f(z) < f(x_2)$ or there exists $x_1, x_2, x_3, x_4, x_5 \in S$ such that $z = x_1x_2x_3x_4x_5$ and $f(z) < f(x_3)$. Choose a real number t such that $f(z) < t < \min\{f(x_1), f(x_2), f(x_3)\}$. Then $z \notin f_t$ and $x_1, x_2, x_3 \in f_t$. Therefore, $z \in (SSf_t \cap Sf_tS \cap f_tSS) \cup (SSf_t \cap SSf_tSS \cap f_tSS) = SSf_tS \cap (Sf_tS \cup SSf_tSS) \cap f_tSS$ but $z \notin f_t$. Hence $SSf_tS \cap (Sf_tS \cup SSf_tSS) \cap f_tSS \nsubseteq f_t$, which contradicts (2).

5. Generated Fuzzy Quasi-ideals in a Ternary Semigroup Without Multiplicative Identity

Definition 5.1. Let f be a fuzzy set in a ternary semigroup S. Then, the smallest fuzzy quasi-ideal generated by f is $\langle \overset{q}{f} \rangle = \bigcap_{g \in FQ(S)} \{g \mid f \subseteq g\}$, where $FQ(S)$ is the set of all fuzzy quasi-ideals of S.

Definition 5.2. Let S be a ternary semigroup and $f \in F(S)$. We define the fuzzy sets f^q and f^{qq} on S as follows:

$$f^q(z) = \sup\{t \mid z \in \langle \overset{q}{f_t} \rangle\} \text{ and } f^{qq}(z) = \sup\{t \mid z \in \langle \overset{q}{f_t^>} \rangle\},$$

where $\langle \overset{q}{f_t} \rangle$ is a quasi-ideal generated by level subset f_t and $\langle \overset{q}{f_t^>} \rangle$ is a quasi-ideal generated by strong level subset $f_t^>$.

Lemma 5.1 ([1]). If A is a non-empty subset of a ternary semigroup S, then $(A \cup SSA) \cap (A \cup SAS \cup SSASS) \cap (A \cup ASS)$ is the smallest quasi-ideal of S containing A.

In order to derive a comparable result within the context of fuzzy theory, we follow a specific method. Throughout this procedure, various forms of the resulting fuzzy ideals are derived.

Lemma 5.2. Let f be a fuzzy set in a ternary semigroup S and $t \in [0, 1)$. Then, $(f^{qq})_t^{>} = \overset{q}{\langle f_t^{>}\rangle}$.

Proof. Let $z \in (f^{qq})_t^{>}$. Then, $f^{qq}(z) > t$ and hence $\sup\{t_i \mid z \in \overset{q}{\langle f_{t_i}^{>}\rangle}\} > t$. Thus, there exists $t_0 > t$ and $z \in (f_{t_0}^{>} \cup SS) \cap (f_{t_0}^{>} \cup Sf_{t_0}^{>}S \cup SSf_{t_0}^{>}SS) \cap (f_{t_0}^{>} \cup f_{t_0}^{>}SS) \subseteq (f_t^{>} \cup SSf_t^{>}) \cap (f_t^{>} \cup Sf_t^{>}S \cup SSf_t^{>}SS) \cap (f_t^{>} \cup f_t^{>}SS)$. Hence, $z \in \overset{q}{\langle f_t^{>}\rangle}$.

Conversely, let $z \in \overset{q}{\langle f_t^{>}\rangle}$. Then $z \in f_t^{>} \cup (SSf_t^{>} \cap (Sf_t^{>}S \cup SSf_t^{>}SS) \cap f_t^{>}SS$. Therefore, either $z \in f_t^{>}$ or $z \in (SSf_t^{>} \cap (Sf_t^{>}S \cup SSf_t^{>}SS) \cap f_t^{>}SS)$. That is either $z \in f_t^{>}$ or $z \in (SSf_t^{>} \cap Sf_t^{>}S \cap f_t^{>}SS) \cup (SSf_t^{>} \cap SSf_t^{>}SS \cap f_t^{>}SS)$.

Suppose $z \in f_t^{>}$. Then $f(z) > t$. Choose t_0 such that $f(z) > t_0 > t$. Then, $z \in f_{t_0}^{>} \subseteq \overset{q}{\langle f_{t_0}^{>}\rangle}$, for some $t_0 > t$.

If $z \in SSf_t^{>} \cap Sf_t^{>}S \cap f_t^{>}SS \cup (SSf_t^{>} \cap SSf_t^{>}SS \cap f_t^{>}SS)$, then either $z \in (SSf_t^{>} \cap Sf_t^{>}S \cap f_t^{>}SS)$ or $z \in (SSf_t^{>} \cap SSf_t^{>}SS \cap f_t^{>}SS)$.

First, let $z \in (SSf_t^{>} \cap Sf_t^{>}S \cap f_t^{>}SS)$. Then $z = s_1s_2a_1 = s_3a_2s_4 = a_3s_5s_6$ for some $a_1, a_2, a_3 \in f_t^{>}$ and $s_1, s_2, s_3, s_4, s_5, s_6 \in S$. Therefore, $f(a_1) > t$, $f(a_2) > t$ and $f(a_3) > t$. Choosing t_1, t_2, t_3 such that $f(a_1) > t_1 < t$, $f(a_2) > t_2 < t$ and $f(a_3) > t_3 < t$, we get $a_1 \in f_{t_1}^{>}$, $a_2 \in f_{t_2}^{>}$ and $a_3 \in f_{t_3}^{>}$. Write $t_4 = \min\{t_1, t_2, t_3\}$. Then $a_1, a_2, a_3 \in f_{t_4}^{>}$. Therefore, $z = s_1s_2a_1 = s_3a_2s_4 = a_3s_5s_6 \in SSf_{t_4}^{>} \cap Sf_{t_4}^{>}S \cap f_{t_4}^{>}SS \subseteq \overset{q}{\langle f_{t_4}^{>}\rangle}$, where $t_4 > t$.

Next let $z \in (SSf_t^{>} \cap SSf_t^{>}SS \cap f_t^{>}SS)$. Then $z = s'_1s'_2a'_1 = s'_3s'_4a'_2s'_5s'_6 = a'_3s'_7s'_8$ for some $a'_1, a'_2, a'_3 \in f_t^{>}$ and $s'_1, s'_2, s'_3, s'_4, s'_5, s'_6, s'_7, s'_8 \in S$. Therefore, $f(a'_1) > t, f(a'_2) > t, f(a'_3) > t$. Choosing t'_1, t'_2, t'_3 such that $f(a'_1) > t'_1 < t$, $f(a'_2) > t'_2 < t$ and $f(a'_3) > t'_3 < t$, we get $a'_1 \in f_{t'_1}^{>}$, $a'_2 \in f_{t'_2}^{>}$ and $a'_3 \in f_{t'_3}^{>}$. Write $t'_4 = \min\{t'_1, t'_2, t'_3\}$. Then $a'_1, a'_2, a'_3 \in f_{t'_4}^{>}$. Therefore, $z = s'_1s'_2a'_1 = s'_3s'_4a'_2s'_5s'_6 = a'_3s'_7s'_8 \in SSf_{t_4}^{>} \cap SSf_{t_4}^{>}SS \cap f_{t_4}^{>}SS \subseteq \overset{q}{\langle f_{t'_4}^{>}\rangle}$, where $t'_4 > t$.

Thus in any case we have, $z \in \overset{q}{\langle f_{t_i}^{>}\rangle}$ for some $t_i > t$ and hence $\sup\{t_i \mid z \in \overset{q}{\langle f_{t_i}^{>}\rangle}\} > t$, that is $z \in (f^{qq})_t^{>}$.

The proofs of subsequent theorems is omitted, as their results follow from arguments similar to those used in Theorems 4.5, 4.7 and Lemma 4.6 of [10].

Theorem 5.1. Let f be a fuzzy set in a ternary semigroup S and $t \in [0, 1)$, then

(1) f^{qq} is the fuzzy quasi-ideal generated by f.

(2) $f^{qq} = f^q$

(3) fuzzy quasi-ideal $\overset{q}{\langle f\rangle}$ generated by f is $\overset{q}{\langle f\rangle}(z) = \sup\{t \mid z \in \overset{q}{\langle f_t^{>}\rangle}\}$.

Lemma 5.3. Let f be a fuzzy set in a ternary semigroup S. Then $\forall\, t \in [0, 1)$, $\overset{q}{\langle f\rangle}_t^{>} = \overset{q}{\langle f_t^{>}\rangle}$.

Proof. By Lemma 5.2 and Theorem 5.1, $(f^q)_t^{>} = \overset{q}{\langle f_t^{>}\rangle}$. Moreover, by Definition 5.2 and Theorem 5.1, we get, $f^{qq} = \overset{q}{\langle f\rangle}$. Using Theorem 5.1, $f^q = \overset{q}{\langle f\rangle}$. This implies $(f^q)_t^{>} = \overset{q}{\langle f\rangle}_t^{>}$. We have already establish that $(f^q)_t^{>} = \overset{q}{\langle f_t^{>}\rangle}$, therefore, $\overset{q}{\langle f\rangle}_t^{>} = \overset{q}{\langle f_t^{>}\rangle}$.

Theorem 5.2. Let f be a fuzzy set in a ternary semigroup S. Then, $(f \cup S \circ S \circ f) \cap (f \cup (S \circ f \circ S \cup S \circ S \circ f \circ S \circ S)) \cap (f \cup f \circ S \circ S)$ is the smallest fuzzy quasi-ideal of S containing f.

Proof. Let $\alpha^q = (f \cup S \circ S \circ f) \cap (f \cup (S \circ f \circ S \cup S \circ S \circ f \circ S \circ S)) \cap (f \cup f \circ S \circ S)$. Then, for any $t \in [0, 1)$,

$$\begin{aligned}
(\alpha^q)_t^{>} &= ((f \cup S\circ S\circ f) \cap (f \cup (S\circ f\circ S \cup S\circ S\circ f\circ S\circ S)) \\
&\quad \cap (f \cup f\circ S\circ S))_t^{>} \\
&= (f \cup S\circ S\circ f)_t^{>} \cap (f \cup (S\circ f\circ S \cup S\circ S\circ f\circ S\circ S))_t^{>} \\
&\quad \cap (f \cup f\circ S\circ S)_t^{>} \qquad \text{(by Lemma 4.1)} \\
&= (f_t^{>} \cup SSf_t^{>}) \cap (f_t^{>} \cup (Sf_t^{>}S \cup SSf_t^{>}SS)) \\
&\quad \cap (f \cup f_t^{>}SS) \qquad \text{(by Lemma 4.1)} \\
&= \overset{q}{\langle f_t^{>}\rangle} \\
&= \overset{q}{\langle f\rangle}_t^{>} \qquad \text{(by Lemma 5.3)}\,.
\end{aligned}$$

Thus $\alpha^q = \overset{q}{\langle f\rangle}$.

6. Generated Fuzzy Quasi-ideals in a Ternary Semigroup with Multiplicative Identity

Lemma 6.1. If A is a non-empty subset of a ternary semigroup S with multiplicative identity, then the quasi-ideal generated by A is $SSASS$.

Lemma 6.2. Let f be a fuzzy set in a ternary semigroup S with multiplicative identity and $t \in [0, 1)$. Then $(f^{qq})_t^{>} = \overset{q}{\langle f_t^{>}\rangle}$.

Proof. Let $z \in (f^{qq})_t^{>}$. Then, $f^{qq}(z) > t$ and hence $\sup\{t_i \mid z \in \overset{q}{\langle f_{t_i}^{>}\rangle}\} > t$. Thus, there exists $t_0 > t$ and $z \in SSf_{t_0}^{>}SS \subseteq SSf_t^{>}SS$. Hence, $z \in \overset{q}{\langle f_t^{>}\rangle}$.

Conversely, let $z \in \overset{q}{\langle f_t^{>}\rangle}$. Then $z \in SSf_t^{>}SS$. Then by part (iv) of Lemma 5.2, $z \in (f^{qq})_t^{>}$ and hence $(f^{qq})_t^{>} = \overset{q}{\langle f_t^{>}\rangle}$.

Note. Theorems 5.1, and Lemma 5.3 also holds for a ternary semigroup with the multiplicative identity.

Theorem 6.1. Let f be a fuzzy set in a ternary semigroup S with multiplicative identity. Then $\overset{q}{\langle f\rangle} = S \circ S \circ f \circ S \circ S$.

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Author Contributions

Ravi Srivastava is the sole author. The author read and approved the final manuscript.

Conflicts of Interest

The author declares no conflicts of interest.

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