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Research Article

A Simple Method to Improve the Multiplicative Consistency of an Interval Fuzzy Preference Relation

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Abstract

In this paper, we propose a method that improves the multiplicative consistency and minimizes indeterminacy (the sum of widths of all interval membership degrees) by using only off-diagonal elements of an interval fuzzy preference relation (IFPR). In addition, this method keeps the initial information as much as possible. To do so, we formulate a concept of the multiplicative consistency that satisfies the additive reciprocity between the related preferences of the IFPR and is invariant under any permutation of objects. Next, the equations which are equivalent to the multiplicative consistency for the IFPR and uses only off-diagonal elements are derived. Based on these equations, the linear models to judge the multiplicative consistency of the IFPR and calculate multiplicatively consistent IFPR minimizing indeterminacy by using only off-diagonal elements are constructed. Based on linear models, we construct an algorithm that calculates the acceptable consistent IFPR keeping the initial information as much as possible and prove that a consistency index of algorithm converges to zero. The proposed method can reduce a large amount of calculations and is correct in judging and improving the multiplicative consistency for the IFPR in comparison with previous results because it uses only off-diagonal elements of the initial IFPR. In addition, a numerical example is provided to show the feasibility and efficiency of the proposed method.

Keywords

Interval Fuzzy Preference Relation (IFPR), Multiplicative Consistency, Off-diagonal Elements, Additive Reciprocity, Permutation of Objects

1. Introduction

Decision making problems based on preference relations could be reached the most appropriate object by comparing all objects over criteria or attributes. Fuzzy preference relations are composed in the way to express preferences of one objet over other object among finite objects. Then all preferences are defined in interval $[0,1]$. In order to select the most appropriate object, it is necessary that the fuzzy preference relation should satisfy a given consistency condition.

Tanino [12] proposed two types of consistency concept for fuzzy preference relations: the additive consistency based on additive transitivity of preferences and the multiplicative consistency based on multiplicative transitivity of preferences. All definitions of consistency for fuzzy preference relations should preserve two basic properties: additive reciprocity between the related preferences and invariance under any permutation of objects.

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In order to deal with indeterminacy information in fuzzy environment, extension of fuzzy preference relations to intervals has been introduced [17]. Extension of fuzzy preference relations to intervals requires also a kind of consistency as well as for FPRs. They should satisfy additive reciprocity between the related interval preferences and invariance under any permutation of objects [4]. Xu and Chen [18] extended Tanino's multiplicative consistency to IFPRs. Xu [17] applied Xu and Chen's multiplicative consistency concept to study group decision making with IFPRs. Genç et al. [2] investigated the multiplicative transitivity for IFPRs based on Tanino's multiplicative consistency and presented a method for determining missing values in incomplete IFPRs. Wu and Chiclana [15] established an approach for group decision-making problems with IFPRs based on trust-consensus analysis using the multiplicative consistency concept [2]. Xia and Xu [16] also proposed the multiplicative consistency of IFPRs using two associated fuzzy preference relations, which is based on the same principle of Liu et al.'s multiplicative consistency [5]. Wang and Chen [13] noted that the multiplicative consistency concepts in the literature [5, 16] do not invariance under a permutation of objects. Then, Wang and Chen [13] provided a geometric transitivity-based multiplicative consistency concept for IFPRs. Later, Wang and Li [14] applied this concept to study group decision making with incomplete IFPRs. However, this concept does not adequate for addressing inconsistent and incomplete IFPRs [7]. Considering this issue, Meng et al. [7] proposed another multiplicative consistency concept for IFPRs in a similar way to that of Meng et al. [6] and provided a programming model-based group decision making method with incomplete and inconsistent IFPRs to avoid the defects in previous methods. Motivated by the multiplicative consistency of Krejčí [3], Zhang [19] and Meng et al. [8] defined a multiplicative consistency for IFPRs. In addition, Zhang [19] considered inclusion relationship among some previous consistency concepts and constructed an optimal log-model to derive multiplicatively consistent IFPR from an IFPR. In addition, investigations of multiplicatively consistent IFPRs were studied in literatures [9-11].

After investigating previous results for multiplicative consistencies of IFPRs, we find some defects:

In researching into the multiplicative consistency of IFPRs, the additive reciprocity between the related interval preferences and invariance of the multiplicative consistency under any permutation of objects did not correctly demonstrate.

The problem that minimizes indeterminacy in models did not consider. In fact, the smaller the width of an interval fuzzy number is the more correct is the interval number.

In particular, the amount of calculations to improve multiplicative consistencies of IFPRs is too large because both the lower limit and the upper limit of interval fuzzy numbers are calculated.

To solve these issues, we formulate a multiplicative consistency that satisfies the additive reciprocity of the IFPR and is invariant under any permutation of objects and derive the equations that are equivalent to definition expressions for the multiplicative consistency and uses only off-diagonal elements of the IFPR. Based on these equations, linear models that judge the multiplicative consistency of the IFPR and calculate multiplicatively consistent IFPR minimizing indeterminacy using only off-diagonal elements are derived. In addition, we propose an algorithm to improve the multiplicative consistency keeping the initial information as much as possible and prove that a consistency index of the algorithm is converged to zero.

This paper is organized as follows. Section 2 reviews previous concepts and models for multiplicative consistencies of IFPRs. In section 3, we formulate the multiplicative consistency of an IFPR and demonstrate that the multiplicative consistency satisfies the additive reciprocity and is invariant under permutation of objects. In addition, the equations that are equivalent to definition's expression for the multiplicative consistency of the IFPR and use only off-diagonal elements are derived. Section 4 constructs the two-stage linear models to calculate multiplicatively consistent IFPR minimizing indeterminacy of interval membership degrees by using only off-diagonal elements of the IFPR. By using models and consistency index, we construct an algorithm that calculates the acceptably multiplicative consistent IFPR keeping the initial information as much as possible and prove that a consistency index of the algorithm converges to zero. Section 5 summarizes the results obtained in the paper.

2. Preliminaries

In this section, the concepts of fuzzy preference relation and interval fuzzy preference relation are recalled and previous multiplicative consistencies are reviewed.

Let $X=\{x_1,x_2,...,x_n\}$ $(n\geq 3)$ be the set of objects and $N=\{1,2,\cdots,n\}$.

Definition 1 [12]. A fuzzy preference relation $B=(b_{ij})_{n\times n}$ ($b_{ij}\in[0,1]$) on the set X is a square matrix that is additive reciprocal $b_{ij}+b_{ji}=1$, $i,j\in N$, where b_{ij} denotes the preference's degree of object x_i over object x_j.

Definition 2 [15]. A interval fuzzy preference relation $R=(r_{ij})_{n\times n}$ $(r_{ij}=[r_{ij}^-,r_{ij}^+])$ on the set X is a square matrix satisfying the following conditions.

$$0\leq r_{ij}^-\leq r_{ij}^+\leq 1,\ \ r_{ij}^-+r_{ji}^+=r_{ij}^++r_{ji}^-=1,\ \ r_{ii}^-=r_{ii}^+=0.5, \quad (1)$$

where $r_{ij}=[r_{ij}^-,r_{ij}^+]$ represents the interval preferred degree of object x_i over object x_j. Then the elements of interval fuzzy preference relation $R=(r_{ij})_{n\times n}$ $(r_{ij}=[r_{ij}^-,r_{ij}^+])$, $r_{t(t+1)}$, $t=1,\cdots,n-1$ are called off-diagonal elements.

Consistency modeling transitivity of the preferences among any three objects verifies corretness of preference's judgmens.

Definition 3 [12]. A fuzzy preference relation $B=(b_{ij})_{n\times n}$ is the multiplicative consistency if it satisfies the following multiplicative transitivity:

$$b_{ij}b_{jk}b_{ki}=b_{ik}b_{kj}b_{ji},\ i,j,k\in N. \tag{2}$$

Fuzzy preference relation $B=(b_{ij})_{n\times n}$ is multiplicatively consistent if and only if its elements are represented by a weight vector $w=(w_1,w_2,\cdots,w_n)$ ($w_i\in[0,1]$, $\sum_{i=1}^{n}w_i=1$) as follows:

$$b_{ij}=\frac{w_i}{w_i+w_j},\ i,j\in N. \tag{3}$$

Definition 4 [15]. For IFPR $R=(r_{ij})_{n\times n}$ $(r_{ij}=[r_{ij}^-,r_{ij}^+])$, if there exists a weight vector $w=(w_1,w_2,\cdots,w_n)$ ($w_i\in[0,1]$, $\sum_{i=1}^{n}w_i=1$) such that

$$r_{ij}^-\le\frac{w_i}{w_i+w_j}\le r_{ij}^+,\ i,j\in N \tag{4}$$

then R is called multiplicatively consistent.

Liu et al. [5] defined a multiplicative consistency of the IFPR by applying the multiplicative consistency of a fuzzy preference relation.

Definition 5 [5]. For IFPR $R=(r_{ij})_{n\times n}$ $(r_{ij}=[r_{ij}^-,r_{ij}^+])$, R is multiplicatively consistent if there are multiplicatively consistent fuzzy preference relations $P=(p_{ij})_{n\times n}$ and $Q=(q_{ij})_{n\times n}$ satisfying Definition 3:

$$p_{ij}=\begin{cases}r_{ij}^+,\ i<j\\0.5,\ i=j\\r_{ij}^-,\ i>j\end{cases},\quad q_{ij}=\begin{cases}r_{ij}^-,\ i<j\\0.5,\ i=j\\r_{ij}^+,\ i>j\end{cases}. \tag{5}$$

In addition, Wu and Chiclana [15] defined the multiplicative consistency of an IFPR.

Definition 6 [15]. Let $R=(r_{ij})_{n\times n}$ $(r_{ij}=[r_{ij}^-,r_{ij}^+])$ be an IFPR. R is multiplicatively consistent if the following conditions are satisfied:

$$\frac{r_{ij}}{r_{ji}}=\frac{r_{ik}}{r_{ki}}\frac{r_{kj}}{r_{jk}},\ i,j,k\in N,\ i<k<j. \tag{6}$$

Eq. (6) are equivalent to the following equations:

$$\frac{r_{ij}^-}{r_{ji}^+}=\frac{r_{ik}^-}{r_{ki}^+}\frac{r_{kj}^-}{r_{jk}^+},\ \frac{r_{ij}^+}{r_{ji}^-}=\frac{r_{ik}^+}{r_{ki}^-}\frac{r_{kj}^+}{r_{jk}^-},\ i,j,k\in N,\ i<k<j. \tag{7}$$

Definitions 4 and 5 are equivalent but do not invariance under a permutation of objects. Thus, although the initial IFPR is multiplicatively consistent according Definition 4 and 5, the multiplicative consistency is changed after acting some permutation of objects.

Different from the above concepts, Wang and Li [14] presented the following multiplicative consistency concept.

Definition 7 [14]. Let $R=(r_{ij})_{n\times n}$ $(r_{ij}=[r_{ij}^-,r_{ij}^+])$ be an IFPR. R is multiplicatively consistent if the following conditions are satisfied:

$$\frac{r_{ij}}{r_{ji}}\frac{r_{jk}}{r_{kj}}\frac{r_{ki}}{r_{ik}}=\frac{r_{ik}}{r_{ki}}\frac{r_{kj}}{r_{jk}}\frac{r_{ji}}{r_{ij}},\ i,j,k\in N. \tag{8}$$

They further offered the following equivalent multiplicative consistency condition:

$$r_{ij}^-r_{ij}^+r_{jk}^-r_{jk}^+r_{ki}^-r_{ki}^+=r_{ik}^-r_{ik}^+r_{kj}^-r_{kj}^+r_{ji}^-r_{ji}^+,\ i,j,k\in N. \tag{9}$$

Definition 8 [7]. Let $R=(r_{ij})_{n\times n}$ $(r_{ij}=[r_{ij}^-,r_{ij}^+])$ be an IFPR. R is multiplicatively consistent if there exists its associated multiplicative consistent quasi-IFPR.

$$\bar{b}_{ij}\bar{b}_{jk}\bar{b}_{ki}=\bar{b}_{ji}\bar{b}_{ik}\bar{b}_{kj},\ i,j,k\in N, \tag{10}$$

where $\begin{cases}\bar{b}_{ij}=[r_{ij}^-,r_{ij}^+]\\\bar{b}_{ji}=[r_{ji}^+,r_{ji}^-]\end{cases}$ or $\begin{cases}\bar{b}_{ij}=[r_{ij}^+,r_{ij}^-]\\\bar{b}_{ji}=[r_{ji}^-,r_{ji}^+]\end{cases}$, $i,j\in N$.

Analyzing inclusion relations among Definition 4, 5 and 6, Zhang [19] demonstrated that Definition 4 is the most comprehensive. Motivated by Krejčí's multiplicative consistency, he defined a multiplicative consistency of IFPRs.

Definition 9 [19]. Let $R=(r_{ij})_{n\times n}$ $(r_{ij}=[r_{ij}^-,r_{ij}^+])$ be an IFPR. R is multiplicatively consistent if $\forall\ b_{ij}\in[r_{ij}^-,r_{ij}^+]$, $\exists\ b_{ik}\in[r_{ik}^-,r_{ik}^+]\ \wedge\ \exists\ b_{kj}\in[r_{kj}^-,r_{kj}^+]$:

$$\frac{b_{ij}}{1-b_{ij}}=\frac{b_{ik}b_{kj}}{(1-b_{ik})(1-b_{kj})},\ i,j,k\in N. \tag{11}$$

Zhang [19] built an optimal model to improve the multiplicative consistency of the IFPR.

Model 1. $F=\min\sum_{i=1}^{n-1}\sum_{j=1}^{n}(c_{ij}^-+c_{ij}^++d_{ij}^-+d_{ij}^+)$

$$s.t.\begin{cases} \ln\frac{r_{ij}^{-}}{1-r_{ij}^{-}}+c_{ij}^{+}-c_{ij}^{-}\geq \ln\frac{r_{ik}^{-}}{1-r_{ik}^{-}}+c_{ik}^{+}-c_{ik}^{-}+\ln\frac{r_{kj}^{-}}{1-r_{kj}^{-}}+c_{kj}^{+}-c_{kj}^{-},\ i,j,k\in N,\ i<j \\ \ln\frac{r_{ij}^{+}}{1-r_{ij}^{+}}+d_{ij}^{+}-d_{ij}^{-}\leq \ln\frac{r_{ik}^{+}}{1-r_{ik}^{+}}+d_{ik}^{+}-d_{ik}^{-}+\ln\frac{r_{kj}^{+}}{1-r_{kj}^{+}}+d_{kj}^{+}-d_{kj}^{-},\ i,j,k\in N,\ i<j \\ \ln\frac{r_{ij}^{-}}{1-r_{ij}^{-}}+c_{ij}^{+}-c_{ij}^{-}\leq \ln\frac{r_{ij}^{+}}{1-r_{ij}^{+}}+d_{ij}^{+}-d_{ij}^{-},\ i,j\in N,\ i<j \\ c_{ij}^{+}-c_{ij}^{-}+d_{ji}^{+}-d_{ji}^{-}=0,\ i,j\in N,\ i<j \\ d_{ij}^{+}-d_{ij}^{-}+c_{ji}^{+}-c_{ji}^{-}=0,\ i,j\in N,\ i<j \\ c_{ij}^{+},\ c_{ij}^{-},\ d_{ij}^{+},\ d_{ij}^{-}\geq 0,\ i,j\in N \end{cases}.$$

3. The Multiplicative Consistency of IFPRs

In this section, we formulate a concept of a multiplicative consistency for IFPRs and prove the additive reciprocity between the related preferences and invariance of the multiplicative consistency under any permutation of objects. In addition, the equations that are equivalent to the multiplicative consistency and uses only off-diagonal elements of IFPRs are derived.

The first, a multiplicative consistency for IFPRs is formulated.

Definition 10. Let $R=(r_{ij})_{n\times n}$ $(r_{ij}=[r_{ij}^{-},r_{ij}^{+}])$ be an IFPR. R is multiplicatively consistent if $\forall$ $b_{ij}\in[r_{ij}^{-},r_{ij}^{+}]$, $\exists$ $b_{ik}\in[r_{ik}^{-},r_{ik}^{+}]$ $\wedge$ $\exists$ $b_{kj}\in[r_{kj}^{-},r_{kj}^{+}]$:

$$b_{ij}=\frac{b_{ik}b_{kj}}{b_{ik}b_{kj}+(1-b_{ik})(1-b_{kj})},\ i,j,k\in N. \tag{12}$$

Definition 10 is equivalent to Definition 9.

Theorem 1. Let $R=(r_{ij})_{n\times n}$ $(r_{ij}=[r_{ij}^{-},r_{ij}^{+}])$ be the IFPR. Then Eq. (12) satisfies the additive reciprocity between the related preferences, i.e. $r_{ij}=1-r_{ji}$.

Proof. For $i,j,k\in N$, $i\neq j\neq k$, no reciprocals appear in Eq. (12) for any $b_{ij}\in[r_{ij}^{-},r_{ij}^{+}]$. For $i,j,k\in N$, $i=j=k$, Eq. (12) is represented as: $\forall$ $b_{ii}=0.5$, $\exists$ $b_{ik}=b_{ii}$, $\exists$ $b_{kj}=b_{jj}$: $0.5=\frac{0.5\times0.5}{0.5\times0.5+(1-0.5)(1-0.5)}$, which satisfies the additive reciprocity. If $i,j,k\in N$, $i\neq j=k$, $\forall$ $b_{ij}\in[r_{ij}^{-},r_{ij}^{+}]$, $\exists$ $b_{ik}^{*}\in[r_{ik}^{-},r_{ik}^{+}]$ $\wedge$ $b_{kj}=b_{jj}$: $b_{ij}=\frac{0.5b_{ik}^{*}}{0.5b_{ik}^{*}+0.5(1-b_{ik}^{*})}=b_{ik}^{*}$, which implies that $b_{ij}=b_{ik}^{*}$ and, therefore, the reciprocity does not appear. Finally, if $i,j,k\in N$, $i=j\neq k$, $\forall$ $b_{ij}=b_{ii}=0.5$, $\exists$ $b_{ik}\in[r_{ik}^{-},r_{ik}^{+}]$ $\wedge$ $\exists$ $b_{kj}\in[r_{kj}^{-},r_{kj}^{+}]$:

$$0.5=\frac{b_{ik}b_{ki}}{b_{ik}b_{ki}+(1-b_{ik})(1-b_{ki})}\Leftrightarrow b_{ik}=1-b_{ki},$$

which means that the additive reciprocity between the related preferences is satisfied. As a result, Theorem 1 is proved.

If a set of objects is provided, pairs of objects are selected at random. This random action should not influence the multiplicative consistency for the IFPR. This random action is modeled by a permutation of objects. Accordingly, if the multiplicative consistency has rationally defined, the multiplicative consistency for the IFPR should not change under any permutation of objects.

Theorem 2. Eq. (12) is invariant under any permutation of objects.

Proof. If any permutation of objects, σ acts to the set of objects, IFPR $R=(r_{ij})_{n\times n}$ is converted into $R^{\sigma}=(r_{\sigma(i)\sigma(j)})_{n\times n}$. Then any $i,j,k\in N$, $i\neq j\neq k$ are moved to $\sigma(i),\ \sigma(j),\ \sigma(k)\in N$, $\sigma(i)\neq\sigma(j)\neq\sigma(k)$ because of bijective property of the permutation. Let's represent $\sigma(i)=i_1$, $\sigma(j)=j_1$, $\sigma(k)=k_1$, then expression $i_1,\ j_1,\ k_1\in N$ holds. Then, for $b_{i_1j_1}\in r_{i_1j_1}=[r_{i_1j_1}^{-},r_{i_1j_1}^{+}]$, there exist some $b_{i_1k_1}\in[r_{i_1k_1}^{-},r_{i_1k_1}^{+}]$, $b_{k_1j_1}\in[r_{k_1j_1}^{-},r_{k_1j_1}^{-}]$:

$$b_{i_1j_1}=\frac{b_{i_1k_1}b_{k_1j_1}}{b_{i_1k_1}b_{k_1j_1}+(1-b_{i_1k_1})(1-b_{k_1j_1})}. \tag{13}$$

So $$b_{\sigma(i)\sigma(j)}=\frac{b_{\sigma(i)\sigma(k)}b_{\sigma(k)\sigma(j)}}{b_{\sigma(i)\sigma(k)}b_{\sigma(k)\sigma(j)}+(1-b_{\sigma(i)\sigma(k)})(1-b_{\sigma(k)\sigma(j)})},\ i,j,k\in N. \tag{14}$$

Eq. (14) means that $R^{\sigma}=(r_{\sigma(i)\sigma(j)})_{n\times n}$ is multiplicatively consistent for any permutation. As a result, Theorem 2 is proved.

Without loss of generality, we restrict the indices to $i,j,k\in N,\quad i\le k\le j$.

Theorem 3. Eq. (12) holds if and only if for any $b_{ij}\in[r_{ij}^{-},r_{ij}^{+}]$, there exist some $b_{(i+t)(i+t+1)}\in[r_{(i+t)(i+t+1)}^{-},r_{(i+t)(i+t+1)}^{+}]$, $t=0,1,\cdots,j-i-1$ such that

$$b_{ij}=\frac{\prod_{t=0}^{j-i-1}b_{(i+t)(i+t+1)}}{\prod_{t=0}^{j-i-1}b_{(i+t)(i+t+1)}+\prod_{t=0}^{j-i-1}(1-b_{(i+t)(i+t+1)})},\quad i,j\in N,\quad i\le j. \tag{15}$$

Proof. Suppose that Eq. (12) hold. Let's denote $s=j-i$. If $s=2$, then $j=i+2$. By Eq. (12), we have:

$$b_{i(i+2)}=\frac{b_{i(i+1)}b_{(i+1)(i+2)}}{b_{i(i+1)}b_{(i+1)(i+2)}+(1-b_{i(i+1)})(1-b_{(i+1)(i+2)})}. \tag{16}$$

Therefore, Eq. (15) holds. Suppose that Eq. (15) holds for $s=n\ (j=i+n)$. Then, we have:

$$b_{i(i+n)}=\frac{\prod_{t=0}^{n-1}b_{(i+t)(i+t+1)}}{\prod_{t=0}^{n-1}b_{(i+t)(i+t+1)}+\prod_{t=0}^{n-1}(1-b_{(i+t)(i+t+1)})}. \tag{17}$$

We prove that it holds for $s=n+1(j=i+n+1)$. Let

$$A=\prod_{t=0}^{n-1}b_{(i+t)(i+t+1)}+\prod_{t=0}^{n-1}(1-b_{(i+t)(i+t+1)}). \tag{18}$$

Using Eq. (18) at Eq. (12), we have:

$$b_{i(i+n+1)}=\frac{b_{i(i+n)}b_{(i+n)(i+n+1)}}{b_{i(i+n)}b_{(i+n)(i+n+1)}+(1-b_{i(i+n)})(1-b_{(i+n)(i+n+1)})}=\frac{\dfrac{\prod_{t=0}^{n-1}b_{(i+t)(i+t+1)}}{A}b_{(i+n)(i+n+1)}}{\dfrac{\prod_{t=0}^{n-1}b_{(i+t)(i+t+1)}}{A}b_{(i+n)(i+n+1)}+\left(1-\dfrac{\prod_{t=0}^{n-1}b_{(i+t)(i+t+1)}}{A}\right)(1-b_{(i+n)(i+n+1)})}$$

$$=\frac{\dfrac{\prod_{t=0}^{n}b_{(i+t)(i+t+1)}}{A}}{\dfrac{\prod_{t=0}^{n}b_{(i+t)(i+t+1)}}{A}+\dfrac{\prod_{t=0}^{n}(1-b_{(i+t)(i+t+1)})}{A}}=\frac{\prod_{t=0}^{n}b_{(i+t)(i+t+1)}}{\prod_{t=0}^{n}b_{(i+t)(i+t+1)}+\prod_{t=0}^{n}(1-b_{(i+t)(i+t+1)})}.$$

Accordingly, Eq. (15) holds.

Conversely, suppose that Eq. (15) holds.

Let

$$A_{ik}=\prod_{t=0}^{k-i-1}b_{(i+t)(i+t+1)}+\prod_{t=0}^{k-i-1}(1-b_{(i+t)(i+t+1)}),\quad A_{kj}=\prod_{t=0}^{j-k-1}b_{(i+t)(i+t+1)}+\prod_{t=0}^{j-k-1}(1-b_{(i+t)(i+t+1)}).\tag{19}$$

Then, we have:

$$\frac{b_{ik}b_{kj}}{b_{ik}b_{kj}+(1-b_{ik})(1-b_{kj})}=\frac{\dfrac{\prod_{t=0}^{k-i-1}b_{(i+t)(i+t+1)}}{A_{ik}}\dfrac{\prod_{t=0}^{j-k-1}b_{(j+t)(j+t+1)}}{A_{kj}}}{\dfrac{\prod_{t=0}^{k-i-1}b_{(i+t)(i+t+1)}}{A_{ik}}\dfrac{\prod_{t=0}^{j-k-1}b_{(j+t)(j+t+1)}}{A_{kj}}+\left(1-\dfrac{\prod_{t=0}^{k-i-1}b_{(i+t)(i+t+1)}}{A_{ik}}\right)\left(1-\dfrac{\prod_{t=0}^{j-k-1}b_{(j+t)(j+t+1)}}{A_{kj}}\right)}$$

$$=\frac{\prod_{t=0}^{j-i-1}b_{(i+t)(i+t+1)}}{\prod_{t=0}^{j-i-1}b_{(i+t)(i+t+1)}+\prod_{t=0}^{j-i-1}(1-b_{(i+t)(i+t+1)})}=b_{ij}.$$

As a result, Theorem 3 is proved.

Remark 1. Theorem 3 is correct and can reduce a large amount of computations in judging and improving the multiplicative consistency for the IFPR because of using off-diagonal elements $b_{(i+t)(i+t+1)}$, $t=1,2,\cdots,j-i-1$ instead of using $b_{i(i+t)}b_{(i+t)j}$, $t=1,2,\cdots,j-i-1$.

4. A Method to Improve the Multiplicative Consistency of IFPR

In this section, we derive evaluative expressions to judge the multiplicative consistency for IFPRs using only off-diagonal elements. In addition, two-stage linear models that calculate multiplicatively consistent IFPR minimizing indeterminacy are constructed. Based on these models, an iterative algorithm to improve the multiplicative consistency for IFPRs is developed.

The following theorem is an important tool to judge the multiplicative consistency for IFPRs.

Theorem 4. Let $R=(r_{ij})_{n\times n}$ $(r_{ij}=[r_{ij}^-,r_{ij}^+])$ be an IFPR. R is multiplicatively consistent according to Definition 10 if and only if the following expressions are true:

$$r_{ij}^-\ge\frac{\prod_{t=0}^{j-i-1}r_{(i+t)(i+t+1)}^-}{\prod_{t=0}^{j-i-1}r_{(i+t)(i+t+1)}^-+\prod_{t=0}^{j-i-1}(1-r_{(i+t)(i+t+1)}^-)},\quad r_{ij}^+\le\frac{\prod_{t=0}^{j-i-1}r_{(i+t)(i+t+1)}^+}{\prod_{t=0}^{j-i-1}r_{(i+t)(i+t+1)}^++\prod_{t=0}^{j-i-1}(1-r_{(i+t)(i+t+1)}^+)},\ i,j\in N,\ i\le j.\tag{20}$$

Proof. If R is multiplicatively consistent, then for any $r_{ij}^-\in[r_{ij}^-,r_{ij}^+]$, $i,j\in N$, there are some $b_{(i+t)(i+t+1)}\in[r_{(i+t)(i+t+1)}^-,r_{(i+t)(i+t+1)}^+]$, $t=0,1,\cdots,j-i-1$ such that

$$r_{ij}^-=\frac{\prod_{t=0}^{j-i-1}b_{(i+t)(i+t+1)}}{\prod_{t=0}^{j-i-1}b_{(i+t)(i+t+1)}+\prod_{t=0}^{j-i-1}(1-b_{(i+t)(i+t+1)})}.$$

Since function $f(x)=\dfrac{x}{x+a}$, $a>0$ is monotonically increasing and $b_{(i+t)(i+t+1)}\ge r_{(i+t)(i+t+1)}^-$, $t=0,1,\cdots,j-i-1$, we have:

$$r_{ij}^- = \frac{\prod_{t=0}^{j-i-1} b_{(i+t)(i+t+1)}}{\prod_{t=0}^{j-i-1} b_{(i+t)(i+t+1)} + \prod_{t=0}^{j-i-1} (1-b_{(i+t)(i+t+1)})} \geq \frac{\prod_{t=0}^{j-i-1} b_{(i+t)(i+t+1)}}{\prod_{t=0}^{j-i-1} b_{(i+t)(i+t+1)} + \prod_{t=0}^{j-i-1} (1-r_{(i+t)(i+t+1)}^-)}$$

$$\geq \frac{\prod_{t=0}^{j-i-1} r_{(i+t)(i+t+1)}^-}{\prod_{t=0}^{j-i-1} r_{(i+t)(i+t+1)}^- + \prod_{t=0}^{j-i-1} (1-r_{(i+t)(i+t+1)}^-)}, \quad i,j \in N, \quad i \leq j.$$

In addition, for any $r_{ij}^+ \in [r_{ij}^-, r_{ij}^+]$, $i,j \in N,\ i \leq j$, there are some $b_{(i+t)(i+t+1)} \in [r_{(i+t)(i+t+1)}^-, r_{(i+t)(i+t+1)}^+]$, $t = 0,1,\cdots,j-i-1$ such that

$$r_{ij}^+ = \frac{\prod_{t=0}^{j-i-1} b_{(i+t)(i+t+1)}}{\prod_{t=0}^{j-i-1} b_{(i+t)(i+t+1)} + \prod_{t=0}^{j-i-1} (1-b_{(i+t)(i+t+1)})}.$$

Since function $f(x) = \frac{x}{x+a}$, $a > 0$ is monotonically increasing and $b_{(i+t)(i+t+1)} \leq r_{(i+t)(i+t+1)}^+$, $t = 0,1,\cdots,j-i-1$, we have:

$$r_{ij}^+ = \frac{\prod_{t=0}^{j-i-1} b_{(i+t)(i+t+1)}}{\prod_{t=0}^{j-i-1} b_{(i+t)(i+t+1)} + \prod_{t=0}^{j-i-1} (1-b_{(i+t)(i+t+1)})} \leq \frac{\prod_{t=0}^{j-i-1} b_{(i+t)(i+t+1)}}{\prod_{t=0}^{j-i-1} b_{(i+t)(i+t+1)} + \prod_{t=0}^{j-i-1} (1-r_{(i+t)(i+t+1)}^+)} \leq \frac{\prod_{t=0}^{j-i-1} r_{(i+t)(i+t+1)}^+}{\prod_{t=0}^{j-i-1} r_{(i+t)(i+t+1)}^+ + \prod_{t=0}^{j-i-1} (1-r_{(i+t)(i+t+1)}^+)},$$
$$i,j \in N, \quad i \leq j.$$

As a result, Theorem 4 is proved.

Based on Eq. (20), we construct a model to judge the multiplicative consistency of IFPR $R = (r_{ij})_{n\times n}$ $(r_{ij} = [r_{ij}^-, r_{ij}^+])$.

Model 2. $\alpha^* = \min \sum_{\substack{i,j\in N \\ i<j}} (\alpha_{ij}^- + \alpha_{ij}^+)$

$$s.t.\begin{cases} r_{ij}^- + \alpha_{ij}^- \geq \dfrac{\prod_{t=0}^{j-i-1} r_{(i+t)(i+t+1)}^-}{\prod_{t=0}^{j-i-1} r_{(i+t)(i+t+1)}^- + \prod_{t=0}^{j-i-1} (1-r_{(i+t)(i+t+1)}^-)}, \ i,j \in N,\ i \leq j \\ r_{ij}^+ - \alpha_{ij}^+ \leq \dfrac{\prod_{t=0}^{j-i-1} r_{(i+t)(i+t+1)}^+}{\prod_{t=0}^{j-i-1} r_{(i+t)(i+t+1)}^+ + \prod_{t=0}^{j-i-1} (1-r_{(i+t)(i+t+1)}^+)}, \ i,j \in N,\ i \leq j \\ \alpha_{ij}^-,\ \alpha_{ij}^+ \geq 0,\ i,j \in N,\ i \leq j \end{cases}. \tag{21}$$

If the optimal value of Model 2 is $\alpha^* = 0$, then IFPR $R = (r_{ij})_{n\times n}$ is multiplicatively consistent. Otherwise, IFPR

$R=(r_{ij})_{n\times n}$ is multiplicatively inconsistent.

Let Eq. (20) be convert into equivalent expressions:

$$\frac{r_{ij}^-}{1-r_{ij}^-}\ge\prod_{t=0}^{j-i-1}\frac{r_{(i+t)(i+t+1)}^-}{1-r_{(i+t)(i+t+1)}^-},\ \frac{r_{ij}^+}{1-r_{ij}^+}\le\prod_{t=0}^{j-i-1}\frac{r_{(i+t)(i+t+1)}^+}{1-r_{(i+t)(i+t+1)}^+},\quad i,j\in N,\ i\le j. \tag{22}$$

Let's introduce the following expressions:

$$\log\frac{r_{ij}^-}{1-r_{ij}^-}=d_{ij}^-,\ \log\frac{r_{ij}^+}{1-r_{ij}^+}=d_{ij}^+,\ i,j\in N,\ i\le j. \tag{23}$$

If IFPR $R=(r_{ij})_{n\times n}$ is multiplicatively inconsistent, the values $\log\frac{r_{ij}^-}{1-r_{ij}^-}=d_{ij}^-$ and $\log\frac{r_{ij}^+}{1-r_{ij}^+}=d_{ij}^+$ should be changed into $\log\frac{\bar r_{ij}^-}{1-\bar r_{ij}^-}=d_{ij}^-+\varepsilon_{ij}^+-\varepsilon_{ij}^-$ and $\log\frac{\bar r_{ij}^+}{1-\bar r_{ij}^+}=d_{ij}^++\eta_{ij}^+-\eta_{ij}^-$, where $\varepsilon_{ij}^-,\ \varepsilon_{ij}^+,\ \eta_{ij}^-,\ \eta_{ij}^-\ge 0$, $i,j\in N$, $i\le j$. Then, interval judgments $[r_{ij}^-,r_{ij}^+]$, $i,j\in N$, $i\le j$ are also changed. Based on it, we construct two-stage optimal models to calculate multiplicatively consistent IFPR $R=(r_{ij})_{n\times n}$ that keeps the initial information as much as possible and minimizes indeterminacy (the sum of widths of all interval membership degrees). Among them, the first optimal model is to calculate multiplicatively consistent IFPR keeping the initial information as much as possible. The second optimal model is to minimize indeterminacy while preserving the first objective value. Based on the above ideas, the corresponding optimal model is constructed by minimizing the deviation between R and $\bar R$.

Model 3. $J=\min\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\varepsilon_{ij}^-+\varepsilon_{ij}^++\eta_{ij}^-+\eta_{ij}^+)$

$$s.t.\begin{cases} d_{ij}^-+\varepsilon_{ij}^+-\varepsilon_{ij}^-\ge\sum_{t=0}^{j-i-1}(d_{(i+t)(i+t+1)}^-+\varepsilon_{(i+t)(i+t+1)}^+-\varepsilon_{(i+t)(i+t+1)}^-),\ i,j\in N,\ i\le j\\ d_{ij}^++\eta_{ij}^+-\eta_{ij}^-\le\sum_{t=0}^{j-i-1}(d_{(i+t)(i+t+1)}^++\eta_{(i+t)(i+t+1)}^+-\eta_{(i+t)(i+t+1)}^-),\ i,j\in N,\ i\le j\\ d_{ij}^-+\varepsilon_{ij}^+-\varepsilon_{ij}^-\le d_{ij}^++\eta_{ij}^+-\eta_{ij}^-,\ i,j\in N,\ i\le j\\ (d_{ij}^-+\varepsilon_{ij}^+-\varepsilon_{ij}^-)+(d_{ji}^++\eta_{ji}^+-\eta_{ji}^-)=0,\ i,j\in N,\ i\le j\\ (d_{ij}^++\eta_{ij}^+-\eta_{ij}^-)+(d_{ji}^-+\varepsilon_{ji}^+-\varepsilon_{ji}^-)=0,\ i,j\in N,\ i\le j\\ \varepsilon_{ij}^+,\ \varepsilon_{ij}^-,\ \eta_{ij}^+,\ \eta_{ij}^-\ge 0,\ i,j\in N,\ i\le j\end{cases}. \tag{24}$$

In Eq. (24), the first two constraints ensure $\bar R$ to be multiplicatively consistent, and the third constraint shows that $\bar r_{ij}^-\le\bar r_{ij}^+$, $i,j\in N$, $i\le j$. Thus,

$$\frac{r_{ij}^-}{1-r_{ij}^-}\le\frac{r_{ij}^+}{1-r_{ij}^+}\to d_{ij}^-=\log\frac{r_{ij}^-}{1-r_{ij}^-}\le d_{ij}^+=\log\frac{r_{ij}^+}{1-r_{ij}^+}\to d_{ij}^-+\varepsilon_{ij}^+-\varepsilon_{ij}^-\le d_{ij}^++\eta_{ij}^+-\eta_{ij}^-.$$

In revised interval judgments $[\bar r_{ij}^-,\bar r_{ij}^+]$, $i,j\in N$, $i\le j$ the lower limit $\bar r_{ij}^-$ is represented by $\log\frac{\bar r_{ij}^-}{1-\bar r_{ij}^-}=d_{ij}^-+\varepsilon_{ij}^+-\varepsilon_{ij}^-$, and therefore $\bar r_{ij}^-=\frac{2^{d_{ij}^-+\varepsilon_{ij}^+-\varepsilon_{ij}^-}}{1+2^{d_{ij}^-+\varepsilon_{ij}^+-\varepsilon_{ij}^-}}$. In addition, the upper limit $\bar r_{ji}^+$ is represented by $\log\frac{\bar r_{ji}^+}{1-\bar r_{ji}^+}=d_{ji}^++\eta_{ji}^+-\eta_{ji}^-$, and therefore $\bar r_{ji}^+=\frac{2^{d_{ji}^++\eta_{ji}^+-\eta_{ji}^-}}{1+2^{d_{ji}^++\eta_{ji}^+-\eta_{ji}^-}}$. From Definition 2, the equations $\bar r_{ij}^-+\bar r_{ji}^+=1$, $i,j\in N$, $i\le j$ should hold. Then, we have:

$$\begin{aligned}\bar r_{ij}^-+\bar r_{ji}^+&=\frac{2^{d_{ij}^-+\varepsilon_{ij}^+-\varepsilon_{ij}^-}}{1+2^{d_{ij}^-+\varepsilon_{ij}^+-\varepsilon_{ij}^-}}+\frac{2^{d_{ji}^++\eta_{ji}^+-\eta_{ji}^-}}{1+2^{d_{ji}^++\eta_{ji}^+-\eta_{ji}^-}}=1\\ &\Leftrightarrow\frac{2^{d_{ij}^-+\varepsilon_{ij}^+-\varepsilon_{ij}^-}}{1+2^{d_{ij}^-+\varepsilon_{ij}^+-\varepsilon_{ij}^-}}=\frac{1}{1+2^{d_{ji}^++\eta_{ji}^+-\eta_{ji}^-}}\\ &\Leftrightarrow 2^{d_{ij}^-+\varepsilon_{ij}^+-\varepsilon_{ij}^-}\times(1+2^{d_{ji}^++\eta_{ji}^+-\eta_{ji}^-})=1+2^{d_{ij}^-+\varepsilon_{ij}^+-\varepsilon_{ij}^-}\\ &\Leftrightarrow 2^{d_{ij}^-+\varepsilon_{ij}^+-\varepsilon_{ij}^-}\times 2^{d_{ji}^++\eta_{ji}^+-\eta_{ji}^-}=1\\ &\Leftrightarrow 2^{d_{ij}^-+\varepsilon_{ij}^+-\varepsilon_{ij}^-+d_{ji}^++\eta_{ji}^+-\eta_{ji}^-}=1\\ &\Leftrightarrow d_{ij}^-+\varepsilon_{ij}^+-\varepsilon_{ij}^-+d_{ji}^++\eta_{ji}^+-\eta_{ji}^-=0.\end{aligned}$$

Accordingly. the fourth constraint holds. In the same way, the fifth constraint follows from equation $\bar{r}_{ij}^{+}+\bar{r}_{ji}^{-}=1$.

Remark 2. The objective function and the constraint conditions in Model 3 are all bounded linear and the feasible region of Model 3 is nonempty (such as r_{ij}^{-}, r_{ij}^{+}, $i,j\in N$, $i\le j$ are equal to 0.5). According to Weierstrass theorem [1], Model 3 has at least one optimal solution.

Stage 2 is to choose the IFPR with the minimum indeterminacy among the solutions of Model 3.

Model 4.

$$I^{*}=\min\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\frac{2^{d_{ij}^{+}+\eta_{ij}^{+}-\eta_{ij}^{-}}}{1+2^{d_{ij}^{+}+\eta_{ij}^{+}-\eta_{ij}^{-}}}-\frac{2^{d_{ij}^{-}+\varepsilon_{ij}^{+}-\varepsilon_{ij}^{-}}}{1+2^{d_{ij}^{-}+\varepsilon_{ij}^{+}-\varepsilon_{ij}^{-}}}\right)\ s.t\begin{cases}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\varepsilon_{ij}^{-}+\varepsilon_{ij}^{+}+\eta_{ij}^{-}+\eta_{ij}^{+}\right)=J^{*}\\ \text{Eq. (24)}\end{cases}. \tag{25}$$

Objective function of Model 4 is obviously equivalent to the following function because function $g(\alpha)=\dfrac{2^{\alpha}}{1+2^{\alpha}}$ is increasing and $d_{ij}^{+}+\eta_{ij}^{+}-\eta_{ij}^{-}\ge d_{ij}^{-}+\varepsilon_{ij}^{+}-\varepsilon_{ij}^{-}$:

$$T^{*}=\min\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(d_{ij}^{+}+\eta_{ij}^{+}-\eta_{ij}^{-}-d_{ij}^{-}-\varepsilon_{ij}^{+}+\varepsilon_{ij}^{-}\right).$$

Then, Model 4 is expressed as.

Model 5. $T^{*}=\min\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(d_{ij}^{+}+\eta_{ij}^{+}-\eta_{ij}^{-}-d_{ij}^{-}-\varepsilon_{ij}^{+}+\varepsilon_{ij}^{-}\right)$

$$s.t\begin{cases}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\varepsilon_{ij}^{-}+\varepsilon_{ij}^{+}+\eta_{ij}^{-}+\eta_{ij}^{+}\right)=J^{*}\\ \text{Eq. (24)}\end{cases}.$$

By solving Model 3 and Model 5, we obtain the multiplicative consistent IFPR $\bar{R}=(\bar{r}_{ij})_{n\times n}$ $(\bar{r}_{ij}=[\bar{r}_{ij}^{-},\bar{r}_{ij}^{+}])$ that keeps the initial information as much as possible and minimizes indeterminacy.

In Model 3, if the optimal objective value is $J^{*}=0$, then IFPR $R=(r_{ij})_{n\times n}$ is multiplicatively consistent. Otherwise, the IFPR is multiplicatively inconsistent.

Definition 11. Let $R=(r_{ij})_{n\times n}$ be an IFPR and $U_{\bar{R}_n}$ be the set of $n\times n$-type multiplicative consistent IFPRs corresponding to the IFPR R according to Definition 10, then the consistency index (CI) of R is defined as the deviation between R and $U_{\bar{R}_n}$

$$CI(R)=D(R,U_{\bar{R}}), \tag{26}$$

where $D(R,U_{\bar{R}})=\min\limits_{\bar{R}\in U_{\bar{R}}}d(R,\bar{R})$ and

$$d(R,\bar{R})=\frac{1}{n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\left|r_{ij}^{-}-\bar{r}_{ij}^{-}\right|+\left|r_{ij}^{+}-\bar{r}_{ij}^{+}\right|\right). \tag{27}$$

Based on above results, we construct an algorithm to calculate an acceptable consistent IFPR $\tilde{R}=(\tilde{r}_{ij})_{n\times n}$ corresponding to R.

Algorithm 1.

Input. IFPR $R=(r_{ij})_{n\times n}$, prescribed threshold $CI_{*}\ge 0$ and the given maximum number of iterations $z_{\max}\ge 1$.

Output. An acceptable consistent IFPR $\tilde{R}=(\tilde{r}_{ij})_{n\times n}$ associated with $R=(r_{ij})_{n\times n}$.

Step 1. Let $z=1$, $R^{(z)}=R$.

Step 2. Calculate the multiplicative consistent IFPR $\bar{R}$ corresponding to $R^{(1)}$ by Model 3 and Model 5. Therefore, $\bar{R}$ is the multiplicative consistent IFPR minimizing the sum of widths of all interval membership degrees.

Step 3. Calculate consistency index $CI(R^{(z)})$ using Eq. (26). If $CI(R^{(z)})\le CI_{*}$ or $z\ge z_{\max}$ go to Step 6. Otherwise go to next.

Step 4. Calculate a deviation matrix $D^{(z)}=(d_{ij}^{(z)})_{n\times n}$ $(d_{ij}^{(z)}=(d_{ij}^{(z)-},d_{ij}^{(z)+}))$ between $R^{(z)}$ and $\bar{R}$:

$$d_{ij}^{(z)-}=\left|r_{ij}^{(z)-}-\bar{r}_{ij}^{-}\right|,\ d_{ij}^{(z)+}=\left|r_{ij}^{(z)+}-\bar{r}_{ij}^{+}\right|,\ i,j\in N,\ i\le j.$$

Let's introduce the following set:

$$C^{(z)}=\{(i,j)^{(z)}\,\big|\,\min\{d_{ij}^{(z)-},d_{ij}^{(z)+}\}\ge CI_{*},\ i,j\in N,\ i<j\}.$$

Step 5. Based on $R^{(z)}$, $\bar{R}$ and $C^{(z)}$, construct the new IFPR $R^{(z+1)}$, where

$$\begin{cases} r_{ij}^{(z+1)-} = (1-z\alpha)r_{ij}^{(z)-} + z\alpha\bar{r}_{ij}^{-} \\ r_{ij}^{(z+1)+} = (1-z\alpha)r_{ij}^{(z)+} + z\alpha\bar{r}_{ij}^{+} \\ r_{ji}^{(z+1)-} = (1-z\alpha)r_{ji}^{(z)-} + z\alpha\bar{r}_{ji}^{-} \\ r_{ji}^{(z+1)+} = (1-z\alpha)r_{ji}^{(z)+} + z\alpha\bar{r}_{ji}^{+} \\ \quad 0<\alpha<1,\ (i,j)\in C^{(z)} \end{cases}, \tag{28}$$

$$r_{ij}^{(z+1)-} = r_{ij}^{(z)-},\ r_{ij}^{(z+1)+} = r_{ij}^{(z)+},\ r_{ji}^{(z+1)-} = r_{ji}^{(z)-}, r_{ji}^{(z+1)+} = r_{ji}^{(z)+},\ (i,j)\notin C^{(z)}. \tag{29}$$

Let $z = z+1$, and go to Step 3.

Step 6. Let $\tilde{R} = R^{(z)}$ and output $\tilde{R}$.

Theorem 5. The consistency index of Algorithm 1 converges to zero.

Proof. $\tilde{R}$ constructed by Algorithm 1 is an IFPR:

$$\begin{aligned}\mu_{ij}^{(z+1)-} + \mu_{ji}^{(z+1)+} &= ((1-z\alpha)\mu_{ij}^{(z)-} + z\alpha\bar{\mu}_{ij}^{-}) + (1-z\alpha)\mu_{ji}^{(z)+} + z\alpha\bar{\mu}_{ji}^{+} \\ &= (1-z\alpha)(\mu_{ij}^{(z)-} + \mu_{ji}^{(z)+}) + z\alpha(\bar{\mu}_{ij}^{(z)-} + \bar{\mu}_{ji}^{(z)+}) = 1 - z\alpha + z\alpha = 1.\end{aligned}$$

In Algorithm 1, as iteration number z is increased to $\frac{1}{\alpha}$, then $1-z\alpha$ converses to zero and $z\alpha$ converses to 1. Then, we have in Step 5 of Algorithm 4.1 because of Eq. (28):

$$\mu_{ij}^{(z+1)-} \to \bar{\mu}_{ij}^{-},\ \mu_{ij}^{(z+1)+} \to \bar{\mu}_{ij}^{+},\ \mu_{ji}^{(z+1)-} \to \bar{\mu}_{ji}^{-},\ \mu_{ji}^{(z+1)+} \to \bar{\mu}_{ji}^{+}.$$

As a result, the inequality $CI(R^{(z_*)}) \le CI_*$ holds for some integer $z_* > 0$, and therefore, Theorem 5 is proved.

Remark 3. In order to keep the initial information as much as possible, we only revise the most inconsistent elements in the IFPR.

Example 1 [19]. Consider IFPR $R = (r_{ij})_{3\times 3}$ on three alternatives $X = \{x_1, x_2, x_3\}$:

$$R = \begin{pmatrix} [0.5,0.5] & [\frac{4}{7},\frac{7}{11}] & [\frac{1}{5},\frac{21}{22}] \\ [\frac{4}{11},\frac{3}{7}] & [0.5,0.5] & [\frac{3}{5},\frac{3}{5}] \\ [\frac{1}{22},\frac{4}{5}] & [\frac{2}{5},\frac{2}{5}] & [0.5,0.5] \end{pmatrix}.$$

According to Model 2, IFPR $R = (r_{ij})_{3\times 3}$ multiplicatively inconsistent. Therefore, we calculate ε_{ij}^{+}, ε_{ij}^{-}, η_{ij}^{+}, η_{ij}^{-} by solving Model 3 and 5. Then we obtain $\varepsilon_{12}^{-} = 1$, $\varepsilon_{13}^{+} = 0.1$, $\varepsilon_{23}^{-} = 1$, $\eta_{12}^{+} = 0.2$, $\eta_{13}^{-} = 1$, $\eta_{23}^{+} = 0.9$ and the other variables c_{ij}^{+}, c_{ij}^{-}, d_{ij}^{+}, d_{ij}^{-} are equal to zero. So, the optimal value of Mode 3 and 5 is $J^* = 4.2$ and the multiplicative consistent IFPR $\bar{R}_1$ is as follows:

$$\bar{R}_1 = \begin{pmatrix} [0.5000,0.5000] & [0.3291,0.6813] & [0.2165,0.8853] \\ [0.3187,0.6709] & [0.5000,0.5000] & [0.3556,0.7868] \\ [0.1147,0.7835] & [0.2132,0.6444] & [0.5000,0.5000] \end{pmatrix}$$

The indeterminacy (the sum of widths of all interval membership degrees) of IFPR $\bar{R}_1$ is 2.9044.

Next, we calculate acceptably consistent IFPR $\tilde{R}$ by using Algorithm 1.

Step 3. Consistency index of IFPR R is $CI(R) = 0.134 > CI_* = 0.1$.

Step 4. Deviation matrix $D^{(1)} = (d_{ij}^{(1)})_{n\times n}$ between R and $\bar{R}_1$ is constructed:

$$D^{(1)} = \begin{pmatrix} (0,0) & (0.2423,0.0449) & (0.0165,0.0692) \\ & (0,0) & (0.2444,0.1681) \\ & & (0,0) \end{pmatrix}.$$

$$C^{(1)} = \{(i,j)^{(1)} \mid \min\{d_{ij}^{(1)-}, d_{ij}^{(1)+}\} > CI_*,\ i,j\in N,\ \ i<j\} = (2,3).$$

Step 5. Denoting $\alpha = 0.6$, we obtain:

$$r_{23}^{-(2)} = 0.45336,\ r_{23}^{+(2)} = 0.71208,\ r_{32}^{-(2)} = 0.28792,\ r_{32}^{+(2)} = 0.54664.$$

Then IFPR $R^{(2)}$ is as follows:

$$R^{(2)} = \begin{pmatrix} [0.5,0.5] & [\frac{4}{7},\frac{7}{11}] & [\frac{1}{5},\frac{21}{22}] \\ [\frac{4}{11},\frac{3}{7}] & [0.5,0.5] & [0.45336,0.71208] \\ [\frac{1}{22},\frac{4}{5}] & [0.28792,0.54664] & [0.5,0.5] \end{pmatrix}.$$

Step 3. Consistency index of IFPR $R^{(2)}$ is $CI(R^{(2)}) = 0.0909 < CI_* = 0.1$.

Step 6. Let $\tilde{R} = R^{(2)}$ and output $\tilde{R}$

On the other hand, we calculate c_{ij}^+, c_{ij}^-, d_{ij}^+, d_{ij}^- by solving Model 1 of [19] in the same IFPR $R=(r_{ij})_{3\times 3}$. Then, we obtain $c_{12}^- = c_{21}^- = d_{12}^+ = d_{21}^- = 2.0794$ and the other variables c_{ij}^+, c_{ij}^-, d_{ij}^+, d_{ij}^- are equal to zero. So, the optimal value of Mode 1 is $F^* = 8.3176$ and the multiplicative consistent IFPR $\bar{R}_2$ is as follows:

$$\bar{R}_2 = \begin{pmatrix} [0.5, 0.5] & [\frac{1}{7}, \frac{14}{15}] & [\frac{1}{5}, \frac{21}{22}] \\ [\frac{1}{15}, \frac{6}{7}] & [0.5, 0.5] & [\frac{3}{5}, \frac{3}{5}] \\ [\frac{1}{22}, \frac{4}{5}] & [\frac{2}{5}, \frac{2}{5}] & [0.5, 0.5] \end{pmatrix}.$$

The indeterminacy (the sum of widths of all interval membership degrees) of IFPR $\bar{R}_2$ is 3.09.

The results of our method and the method of [19] are summarized in Table 1.

Table 1. *The resuts by different methods.*

Method	The optimal value of models	Indeterminacy (sum of interval membership degrees)	Acceptable consistent IFPR
[16]	F*=8.3176	3.09	×
Our paper	J*=4.2	2.0794	O

Table 1 clearly shows that our method is superior to the method of [19]. In particular, our method needs a few calculations because it only calculates the element of second row and thid column of the IFPR R using off-diagonal elements.

5. Conclusion

In the course making researches on IFPRs, we have proposed a method that improves the multiplicative consistency using only off-diagonal elements of an IFPR and minimizes indeterminacy. This method keeps the initial information as much as possible. The first, we have formulated a multiplicative consistency for the IFPR and proved that satisfies the additive reciprocity and is invariant under any permutation of objects. The second, the equations equivalent to the multiplicative consistency as well as using only off-diagonal elements of the IFPR have been derived. Based on the equations, linear models to calculate the multiplicative consistent IFPR minimizing indeterminacy by using only off-diagonal elements of the IFPR have been constructed. The third, we have proposed an algorithm to improve the multiplicative consistency of IFPR and proved that the consistency index of the algorithm converges to zero. This algorithm only revises the most inconsistent elements in the IFPR to keep the initial information as much as possible. The proposed method can reduce a large amount of calculations and is correct in judging and improving the multiplicative consistency of IFPR in comparison with previous methods because of using only off-diagonal elements of IFPR.

In further, we are going to research a method for determining expert' weights taking account of their power and skill and comprehensive aggregation operators in group decision making with IFPRs.

Abbreviations

IFPR Interval Fuzzy Preference Relation

Conflicts of Interest

The authors declare no conflicts of interest.

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