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Research Article

A Posteriori Error Estimates by FEM for Source Control Problems Governed by a System of Semi-Linear Convection-Diffusion Equations

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Abstract

In this paper, we consider the optimal source control problem of a system of 2-dimensional semi-linear steady convection-diffusion equations. The problem is modelized from temperature and consistency distribution in the gasification processes, so it is described by 2 non-linear elliptic partial differential equations with Dirichlet boundary condition. The problem is a optimal source control problem that controls the source term necessary to approximate the temperature to a proper target function. First, we derived the optimal condition. Based on setting the approximation problem of a given control problem in a first order polynomial finite element function space and deriving the optimality condition of the approximation problem, we evaluated a priori error between the optimal control, the optimal state, the conjugate state and its finite element approximation functions by using optimal condition of original and approximate problem. And we also evaluated the upper estimate of a posteriori error by finite element method (FEM). We proved the convergence to 0 of a posteriori error indicator (term of the right side of inequality) when division diameter converges to 0. For this, we acquired the lower bound estimation of a posteriori error and proved that a priori error and total variance error converges to 0 when division diameter converges to 0, so that we proved the convergence problem of a posteriori error indicator.

Keywords

System of Semi-Linear Convection-Diffusion Equations, Source Control, A Posteriori Error Estimates

1. Introduction

Differential equations and their control problems are frequently found in the technical processes accompanied by energy and in the fields of optimal design. In the preliminary literatures, they derived a priori and a posteriori error estimates for the non-linear differential equations by FEM, finite volume method (FVM) and spectral method [1-4, 7-9, 14, 16-18], and even now for optimal control problems [5, 6, 10-13, 15]. When solutions of differential equations are smooth, spectral method is useful for a posteriori error estimates, otherwise, FEM is preferable. One of the difficulties in estimating a posteriori errors by FEM is to estimate the sensitivity of a posteriori error indicator to h, mesh size, i.e, to prove the convergence of a posteriori error indicator to zero when h tends to zero. That's because, unlike by spec-

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tral method, by FEM, the error indicator η in estimating a posteriori error is represented by an infinite series of element-wise error indicators with respect to the total number of elements when h tends to zero.

$$\|y - y_h\|^2 \le \eta^2 = C\sum_k \|\eta_k\|_{0,k}^2 h_k^2 = \eta_\Omega^2 h^2,$$

$$\eta_\Omega^2 = \sum_k \|\eta_k\|_{0,k}^2$$

Therefore, upper bound estimate of a posteriori error makes sense only when error indicator converges to zero. In [13], for control problems of linear differential equations, convergence to zero for error indicators was discussed under the assumption that a sequence of approximate solutions converges to analytical solution in the sense of subsequences. And in [5], for semi-linear parabolic source control problems, we acquired a posteriori error estimates by spectral method but no convergence results of error indicator.

A priori and a posteriori error estimates for linear and semi-linear convection-diffusion equations were presented in [1, 2]. However, in case of a system of non-linear convection-diffusion equations, no error estimates and convergence results for not only equations but also their control problems might be presented at all.

The purpose of this paper is to discuss on a posteriori error estimates by FEM for source control problem governed by a system of semi-linear convection-diffusion equations and convergence of error indicator η to zero. For convergence of error indicator, we estimated lower bound of a posteriori error, a priori error and oscillation error by FEM, introduced some finite dimensional norm, and proved them by using Riesz's theorem in Hilbert space. This paper is organized as follows.

In section 1, we presented problems to solve and optimality conditions, in section 2, a priori error estimates, in section 3 and 4, upper and lower bound estimates for a posteriori error, in section 5, convergence of a posteriori error indicator.

Here are some notations.

$V := H_0^1(\Omega)$, $\quad H^0(\Omega) := L^2(\Omega)$,

$\|.\|_s$: norm in $H^s(\Omega)(s = 0, 1, 2)$

$(.,.)_s$: inner product in $H^s(\Omega)(s = 0, 1, 2)$

$\|.\|_{s,K}$, $\|.\|_{s,E}$: norm in $H^s(K), H^s(E)$ (where K is patch of element in Ω, E is edge of element in Ω, s = 0, 1, 2).

$(.,.)_{s,K}$, $(.,.)_{s,E}$: inner product in $H^s(K), H^s(E)$

$\|.\|_{0,\Gamma} = \|.\|_{L^2(\Gamma)}$: norm in $L^2(\Gamma)$

$(.,.)_{0,\Gamma}$: inner product in $L^2(\Gamma)$

$\|.\|_{0,p,K}$, $\|.\|_{0,p,E}$: norm in $L^p(K)$, $L^p(E)$

$(.,.)_{0,p,K}$, $(.,.)_{0,p,E}$: inner product in $L^p(K)$, $L^p(E)$

Assume that Ω is bounded convex polygonal domain in R^2 and Γ is boundary of Ω. The state model is as follows.

$$\begin{cases} -\Delta y + \beta(y,g) + a\nabla y = f, & \Omega \\ -\Delta g + \phi(y,g) + a\nabla g = u, & \Omega \\ y = 0,\ g = 0 & \Gamma = \partial\Omega \end{cases} \tag{1}$$

The weak solution of (1) satisfies the following equation.

$$\begin{cases} (\nabla y, \nabla \upsilon)_0 + (\beta(y,g),\upsilon)_0 + (a\nabla y,\upsilon)_0 = (f,\upsilon)_0, \forall \upsilon \in V \\ (\nabla g, \nabla \upsilon)_0 + (\phi(y,g),\upsilon)_0 + (a\nabla g,\upsilon)_0 = (u,\upsilon)_0, \forall \upsilon \in V \end{cases} \tag{2}$$

[Assumption 1] Functions $\beta(\cdot,\cdot)$, $\phi(\cdot,\cdot)$, $a(x)$ satisfy the following conditions.

$\beta(\cdot,\cdot),\ \phi(\cdot,\cdot) \in W_{loc}^{2,\infty}(R^2),\ a \in W^{1,\infty}(\Omega)^2, diva = 0,$

$\nabla\beta(\cdot,\cdot) > 0, \nabla\phi(\cdot,\cdot) > 0, \beta(s,t)\cdot s \ge 0, \phi(s,t)t \ge 0, s,t \in R^1$ Under this assumption, *y*, *g*, the solutions of (2), uniquely exist in $V \cap H^2(\Omega) \cap L^\infty(\Omega)$ [5]. Cost functional is

$$J(\mathrm{y};\mathrm{f},\mathrm{u}) := \frac{1}{2}\int_\Omega (\mathrm{y} - \mathrm{z}_d)^2 dx + \frac{\nu}{2}\int_\Omega f^2 dx + \frac{\nu}{2}\int_\Omega u^2 dx,$$

$\nu \ge s > 0, \nu \in R^1, z_d \in L^2(\Omega)$ - is known.

[Problem 1] $\inf_{(\mathrm{f},\mathrm{u}) \in \mathrm{U}_{ad}} J(\mathrm{y};\mathrm{f},\mathrm{u})$

where *y=y(f,u)* is the unique solution of (2) and $U_{ad} := U^2 = (L^2(\Omega))^2$.

[Theorem 1] (Optimality condition) Assume that $(\bar f, \bar u)$ is the solution of problem 1. Then there exists a function $\bar p_i \in V \cap L^\infty(\Omega)(i = 1,2)$ so that it satisfies the following system.

$$\begin{cases} (\bar p_1 + \nu \bar f, f)_0 = 0, & \forall f \in U := L^2(\Omega) \\ (\bar p_2 + \nu \bar u, u)_0 = 0, & \forall u \in U \end{cases}$$

$$\begin{cases} (\nabla \bar p_1, \nabla\upsilon)_0 + (\beta_y'(\bar y,\bar g)\cdot \bar p_1, \upsilon)_0 + (\phi_y'(\bar y,\bar g)\cdot \bar p_2, \upsilon)_0 - \\ \quad -(a\nabla \bar p_1, \upsilon)_0 = (\bar y - z_d, \upsilon)_0, \forall \upsilon \in V \\ (\nabla \bar p_2, \nabla\upsilon)_0 + (\beta_g'(\bar y,\bar g)\cdot \bar p_1, \upsilon)_0 + (\phi_g'(\bar y,\bar g)\cdot \bar p_2, \upsilon)_0 - \\ \quad -(a\nabla \bar p_2, \upsilon)_0 = 0, \forall \upsilon \in V \end{cases} \tag{3}$$

where $(\bar y, \bar g)$ stands for the unique weak solution of (2) when $\mathrm{f} = \bar f$, $\mathrm{u} = \bar u$.

(Proof) Let us identify $\dot y_f$, $\dot g_f$ with Gateaux differentiation to *f* at $(\bar f, \bar u)$ of y, g, the solution of (2).

$\dot y_f$, $\dot g_f$ satisfy the following equations, respectively.

$$(\nabla \dot y_f, \nabla\upsilon)_0 + (\beta_y'(\bar y,\bar g)\dot y_f, \upsilon)_0 + (\beta_g'(\bar y,\bar g)\dot g_f, \upsilon)_0 + \\ + (a\nabla \dot y_f, \upsilon)_0 = (f,\upsilon)_0, \quad \forall\upsilon \in V \tag{4}$$

$$(\nabla \dot{g}_f, \nabla \upsilon)_0 + (\phi'_y(\bar{y},\bar{g})\dot{y}_f, \upsilon)_0 + (\phi'_g(\bar{y},\bar{g})\dot{g}_f, \upsilon)_0 + \\ + (a\nabla \dot{g}_f, \upsilon)_0 = 0, \quad \forall \upsilon \in V \tag{5}$$

Let us identify $\dot{y}_u$, $\dot{g}_u$ with Gateaux differentiation to u at $(\bar{f},\bar{u})$ of y, g, the solution of (2). Then $\dot{y}_u$, $\dot{g}_u$ satisfy the following equations.

$$(\nabla \dot{y}_u, \nabla \upsilon)_0 + (\beta'_y(\bar{y},\bar{g})\dot{y}_u, \upsilon)_0 + (\beta'_g(\bar{y},\bar{g})\dot{g}_u, \upsilon)_0 + \\ + (a\nabla \dot{y}_u, \upsilon)_0 = 0, \quad \forall \upsilon \in V \tag{6}$$

$$(\nabla \dot{g}_u, \nabla \upsilon)_0 + (\phi'_y(\bar{y},\bar{g})\dot{y}_u, \upsilon)_0 + (\phi'_g(\bar{y},\bar{g})\dot{g}_u, \upsilon)_0 - \\ - (a\nabla \dot{g}_u, \upsilon)_0 = (u, \upsilon)_0, \quad \forall \upsilon \in V \tag{7}$$

Then, using Gateaux differentiation to f, u at $(\bar{f},\bar{u})$ of cost functional, the optimality condition is

$$(\bar{y}-z_d, \dot{y}_f)_0 + \nu(\bar{f}, f)_0 = 0, \ \forall f \in U \\ (\bar{y}-z_d, \dot{y}_u)_0 + \nu(\bar{u}, u)_0 = 0, \ \forall u \in U \tag{8}$$

Now let us consider $\bar{p}_i \in V(i = 1,2)$, the solution of the following conjugate system.

$$(\nabla \bar{p}_1, \nabla \upsilon)_0 + (\beta'_y(\bar{y},\bar{g}) \cdot \bar{p}_1, \upsilon)_0 + (\phi'_y(\bar{y},\bar{g}) \cdot \bar{p}_2, \upsilon)_0 + \\ - (a\nabla \bar{p}_1, \upsilon)_0 = (\bar{y}-z_d, \upsilon)_0, \ \forall \upsilon \in V \tag{9}$$

$$(\nabla \bar{p}_2, \nabla \upsilon)_0 + (\phi'_g(\bar{y},\bar{g}) \cdot \bar{p}_2, \upsilon)_0 + (\beta'_g(\bar{y},\bar{g}) \cdot \bar{p}_1, \upsilon)_0 - \\ - (a\nabla \bar{p}_2, \upsilon)_0 = 0, \ \forall \upsilon \in V \tag{10}$$

Let $\upsilon = p_1$ in (4) and $\upsilon = \dot{y}_f$ in (9), and by using assumption 1 and $(a\nabla \dot{y}_f, p_1)_0 = -(a\nabla p_1, \dot{y}_f)_0$, then we have

$$(\bar{y}-z_d, \dot{y}_f)_0 = (p_1, f)_0 + (\phi'_y p_2, \dot{y}_f)_0 - (\beta'_g p_1, \dot{g}_f)_0 \tag{11}$$

Let $\upsilon = p_2$ in (5) and $\upsilon = \dot{g}_f$ in (10), and by comparing 2 equalities, then we have the equality

$$(\phi'_y(\bar{y},\bar{g})p_2, \dot{g}_f)_0 = (\beta'_g(\bar{y},\bar{g})p_1, \dot{g}_f)_0, \\ \phi'_y(\bar{y},\bar{g})p_2 = \beta'_g(\bar{y},\bar{g})p_1 \tag{12}$$

From (11) and (12), we have

$$(\bar{y}-z_d, \dot{y}_f)_0 = (f, p_1)_0 \tag{13}$$

Let $\upsilon = p_1$ in (6) and $\upsilon = \dot{y}_u$ in (9), and by using assumption 1 and $(a\nabla \dot{y}_u, p_1)_0 = -(a\nabla p_1, \dot{y}_u)_0$, we have the equality

$$(\bar{y}-z_d, \dot{y}_u)_0 = (\phi'_y p_2, \dot{y}_u)_0 - (\beta'_g p_1, \dot{g}_u)_0 \tag{14}$$

Let $\upsilon = p_2$ in (7) and $\upsilon = \dot{g}_u$ in (10), and by comparing 2 equalities and using $(a\nabla \dot{g}_u, p_2)_0 = -(a\nabla p_2, \dot{g}_u)_0$, then we have

$$(u, p_2)_0 = (\phi'_y(\bar{y},\bar{g})p_2, \dot{y}_u)_0 - (\beta'_g(\bar{y},\bar{g})p_1, \dot{g}_u)_0 \tag{15}$$

Therefore, from (14) and (15), we have

$$(\bar{y}-z_d, \dot{y}_u)_0 = (u, p_2)_0 \tag{16}$$

and from (8), (13), (16), we have

$$\begin{cases} (\bar{p}_1 + \nu\bar{f}, f)_0 = 0, & \forall f \in U \\ (\bar{p}_2 + \nu\bar{u}, u)_0 = 0, & \forall u \in U \end{cases}$$

It is trivial that $\bar{p}_i \in H^2(\Omega)$, and we have $\bar{p}_i \in V \cap L^\infty(\Omega)(i=1,2)$ from the fact that embedding $H^2(\Omega) \subset C^0(\Omega) \subset L^\infty(\Omega)$ is continuous. Therefore, we can obtain the result of this theorem.

2. A Priori Error Estimate by FEM

Now we partition Ω into regular triangles $K_j(j = 1,2,\ldots,M)$, let us denote the diameter of triangle K_j by h_{K_j}, length of edge by h_{E_j}, $h := \max_{1\le j\le M} h_{K_j}$. Let T_h be the family of triangulation, E_h be the set of edges of T_h.

Let us denote finite element function space of $H_0^1(\Omega)$ as

$$V_h := \{v_h \in H_0^1(\Omega) \cap C(\bar{\Omega}) \mid v_h|_K \in P_r(K), \forall K \in T_h, v_h|_\Gamma = 0\}, \\ \bar{V}_h := \{v_h \in H^1(\Omega) \cap C(\bar{\Omega}) \mid v_h|_K \in P_1(K), \forall K \in T_h\}$$

where $P_1(K)$ is the linear polygonal space on element K, $U_h := U \cap \bar{V}_h$.

The state equation in the finite element function space is

$$\begin{cases} (\nabla y_h, \nabla \upsilon_h)_0 + (\beta(y_h, g_h), \upsilon_h)_0 + (a\nabla y_h, \upsilon_h)_0 = \\ \qquad = (f_h, \upsilon_h)_0, \forall \upsilon_h \in V_h \\ (\nabla g_h, \nabla \upsilon_h)_0 + (\phi(y_h, g_h), \upsilon_h)_0 + (a\nabla g_h, \upsilon_h)_0 = \\ \qquad = (u_h, \upsilon_h)_0, \forall \upsilon_h \in V_h \end{cases} \tag{17}$$

The finite element approximation of problem 1 is as follows.

[Problem 2]:

$$\min_{(f_h,u_h)\in U_h^2} J(y_h; f_h, u_h) := \frac{1}{2}\int_\Omega (y_h - z_d)^2 dx + \frac{\nu}{2}\int_\Omega f_h^2 dx + \frac{\nu}{2}\int_\Omega u_h^2 dx,$$

where $y_h \in V_h$ is the unique solution of (17).

[Lemma 1] When $((\bar{f}_h, \bar{u}_h), \bar{y}_h) \in U_h^2 \times V_h$ is the solution of problem 2, there exists a $(\bar{p}_{1h}, \bar{p}_{2h}) \in V_h^2$ so that it satisfies the following system of equations.

(*Optimality system*)

$$\begin{cases} (\bar{p}_{1h} + \nu \bar{f}_h, f_h)_0 = 0, & \forall f_h \in U_h \\ (\bar{p}_{2h} + \nu \bar{u}_h, u_h)_0 = 0, & \forall u_h \in U_h \end{cases}$$

$$\begin{cases} (\nabla \bar{p}_{1h}, \nabla \upsilon_h)_0 + (\beta_y'(\bar{y}_h, \bar{g}_h) \cdot \bar{p}_{1h}, \upsilon_h)_0 + \\ \quad + (\phi_y'(\bar{y}_h, \bar{g}_h) \cdot \bar{p}_{2h}, \upsilon_h)_0 - (a \nabla \bar{p}_{1h}, \upsilon_h)_0 = \\ \quad = (\bar{y}_h - z_d, \upsilon_h)_0, \quad \forall \upsilon_h \in V_h \\ (\nabla \bar{p}_{2h}, \nabla \upsilon_h)_0 + (\phi_g'(\bar{y}_h, \bar{g}_h) \cdot \bar{p}_{2h}, \upsilon_h)_0 + \\ \quad + (\beta_g'(\bar{y}_h, \bar{g}_h) \cdot \bar{p}_{1h}, \upsilon_h)_0 - (a \nabla \bar{p}_{2h}, \upsilon_h)_0 = \\ \quad = 0, \quad \forall \upsilon_h \in V_h \end{cases} \tag{18}$$

where $(\bar{y}_h, \bar{g}_h)$ is the unique solution of (17).

(Because the proof of this lemma is similar to theorem 1, we shall omit it here.)

[Lemma 2] Assume that condition $\|\beta_g'\|_\infty \cdot \|\phi_y'\|_\infty / \bar{C}^2 < 1$ holds true. Then the equation (18) has a unique solution, where $\bar{C}$ is the positive constant such that $(\nabla \upsilon, \nabla \upsilon)_0 + (a \nabla \upsilon, \upsilon)_0 \ge \bar{C} \|\upsilon\|_1^2$.

(Proof) To begin with, let us consider the following iteration scheme for existence of the solution.

$$(\nabla p_{1h}^{(m)}, \nabla \upsilon_h)_0 + (\beta_y' p_{1h}^{(m)}, \upsilon_h)_0 - (a \nabla p_{1h}^{(m)}, \upsilon_h)_0 = \\ = (y_h - z_d - \phi_y' p_{2h}^{(m-1)}, \upsilon_h)_0, \quad \forall \upsilon_h \in V_h$$

$$(\nabla p_{2h}^{(m)}, \nabla \upsilon_h)_0 + (\phi_g' p_{2h}^{(m)}, \upsilon_h)_0 - (a \nabla p_{2h}^{(m)}, \upsilon_h)_0 = \\ = (-\beta_g' p_{1h}^{(m)}, \upsilon_h)_0 \; 0, \quad \forall \upsilon_h \in V_h \tag{19}$$

By using assumption 1 and [1], the above iteration scheme has a unique solution.

Let us set $\upsilon_h = p_{ih}^{(m)} (\in V_h)$ in every equations above.

Reflecting $(a \nabla p_{ih}^{(m)}, p_{ih}^{(m)})_0 = 0, (diva = 0, i = 1, 2)$, from the above iteration scheme, we have

$$\left\| p_{1h}^{(m)} \right\|_1 \le (1/c_1) \|y_h - z_d\|_0 + (c_2 / c_1) \left\| p_{2h}^{(m-1)} \right\|_1$$

where $c_1 := \bar{C}, \; c_2 := \|\phi_y'\|_{0,\infty}$

Similarly, the following inequality holds

$$\left\| p_{2h}^{(m)} \right\|_1 \le (c_3 / c_1) \left\| p_{1h}^{(m)} \right\|_1 \le \\ \le (c_3 / \bar{C}^2) \|y_h - z_d\|_0 + (c_2 c_3 / \bar{C}^2) \left\| p_{2h}^{(m-1)} \right\|_1) \\ c_3 := \|\beta_g'\|_{0,\infty}$$

where c_i is constant independent of m. From the inequalities above, we have

$$\left\| p_{2h}^{(m)} \right\|_1 \le c_4 \left\| p_{2h}^{(m-1)} \right\|_1 + c_5 \|y_h - z_d\|_0, \\ (c_4 := c_2 c_3 / \bar{C}^2, \; c_5 := c_3 / \bar{C}^2)$$

$$\left\| p_{2h}^{(m)} \right\|_1 \le (c_4)^m \left\| p_{2h}^{(0)} \right\|_1 + (((c_4)^{m-1} + \dots + c_4) c_5) \|y_h - z_d\|_0$$

$$\text{If } c_4 < 1, \text{ then } \left\| p_{2h}^{(m)} \right\|_1 \le C_1, \; \left\| p_{1h}^{(m)} \right\|_1 \le C_2 \tag{20}$$

where $C_i (i = 1, 2)$ is constant independent of m.

Thus there exists an weak convergent subsequence, so that

$$p_{ih}^{(m)} \to \tilde{p}_{ih}, H^1(\Omega)(m \to \infty), \tilde{p}_{ih} \in V_h \; (i = 1, 2) \tag{21}$$

When *m* approaches to infinity under (21) in (19), it can be shown that $\tilde{p}_{ih} (i = 1, 2)$ is the solution of (18).

Now let us prove uniqueness. For 2 solutions $\hat{p}_{ih}, \; \hat{\hat{p}}_{ih} (i = 1, 2)$, let us notice $\bar{p}_{ih} = \hat{p}_{ih} - \hat{\hat{p}}_{ih} (i = 1, 2)$. In the first equation of (18), we fix $\hat{p}_{2h}$, subtract equations with each other and choose the trial function by $\upsilon_h = \hat{p}_{1h} - \hat{\hat{p}}_{1h}$, and then we have

$$\|\nabla \bar{p}_{1h}\|_0^2 + (\beta_y' |\bar{p}_{1h}|^2, 1)_0 - (a \nabla \bar{p}_{1h}, \bar{p}_{1h})_0 = 0$$

By applying Green's Formula to the third term of above equality, we have

$$(a \nabla \bar{p}_{1h}, \bar{p}_{1h})_0 = (a \cdot \frac{1}{2} \nabla (\bar{p}_{1h})^2, 1)_0 = \\ = -(diva, \frac{1}{2} (\bar{p}_{1h})^2)_0 + (a \cdot n \frac{1}{2} (\bar{p}_{1h})^2)_{0,\Gamma} = 0$$

and by recalling $\bar{p}_{1h}|_\Gamma = 0, \; diva = 0$ (assumption 1), we have $\bar{p}_{1h} = 0 \; (\Omega)$ from the above equality. i.e, $\hat{p}_{1h} = \hat{\hat{p}}_{1h}$.

Likewise, we can derive $\hat{p}_{2h} = \hat{\hat{p}}_{2h}$ for the second equation of (18), which completes the proof.

[Lemma 3] [3, 4] Let I_h be the orthogonal projector being $V \to V_h \subset H_0^1(\Omega)$. For $\forall \upsilon \in H^1(K), \; \forall K \in T_h, E \subset \partial K$,

the following estimates hold

1)
$\|v-I_h v\|_{0,K} \le Ch_k \|v\|_{1,\omega_K}$, $\|v-I_h v\|_{0,E} \le Ch_E^{1/2} \|v\|_{1,\omega_E}$

$\omega_K := \bigcup_{K\cap K' \ne \varnothing} K', \omega_E := \bigcup_{K\cap E\ne\varnothing} K$

2) $\|v\|_{0,E} \le C(h_K^{-1/2}\|v\|_{0,K} + h_K^{1/2}\|\nabla v\|_{0,K})$

3) $v \in L^\infty(K), \|v\|_{0,\infty,K} \le Ch_K^{-1}\|v\|_{0,K}, \|v\|_{1,K} \le Ch_K^{-1}\|v\|_{0,K}$

First, let us derive a priori error estimates. Let $p_i^h \in V_h (i=1,2)$ satisfy the following system.

$$\begin{cases} (\nabla p_1^h,\ \nabla \upsilon_h)_0 + (\beta_y'(y^h,g^h)\cdot p_1^h,\ \upsilon_h)_0 + \\ \quad +(\phi_y'(y^h,g^h)\cdot p_2^h,\ \upsilon_h)_0 + (a\nabla \overline{p}_1^h,\ \upsilon_h)_0 = \\ \quad =(y^h - z_d,\ \upsilon_h)_0, \forall \upsilon_h \in V_h,\quad \overline{p}_1^h \in V_h \\ (\nabla p_2^h,\ \nabla \upsilon_h)_0 + (\phi_g'(y^h,g^h)\cdot p_2^h,\ \upsilon_h)_0 + \\ \quad +(\beta_g'(y^h,g^h)\cdot p_1^h,\ \upsilon_h)_0 + (a\nabla p_2^h,\ \upsilon_h)_0 = 0, \\ \qquad \forall \upsilon_h \in V_h,\quad p_2^h \in V_h \end{cases} \tag{22}$$

where y^h, g^h is the solution of the system of equations.

$$\begin{cases} (\nabla y^h,\nabla \upsilon_h)_0 + (\beta(y^h,g^h),\upsilon_h)_0 + (a\nabla y^h,\upsilon_h)_0 = \\ \qquad = (f,\ \upsilon_h)_0,\ \forall \upsilon_h \in V_h,\ y^h \in V_h \\ (\nabla g^h,\nabla \upsilon_h)_0 + (\phi(y^h,g^h),\upsilon_h)_0 + (a\nabla g^h,\upsilon_h)_0 = \\ \qquad = (u,\upsilon_h)_0,\ \forall \upsilon_h \in V_h,\quad g^h \in V_h \end{cases}$$

[Lemma 4]

Let $(f,u,y,g,p_i), (f_h,u_h,y_h,g_h,p_{ih})(i=1,2)$ be the solutions of (3) and (18). Then the following inequality holds true.

$$\frac{\nu}{2}(\|f-f_h\|_0^2 + \|u-u_h\|_0^2) \le C\left\|p_1 - p_{1h}\right\|_0^2 + \left\|p_2 - p_{2h}\right\|_0^2 + Ch$$

(We denote the constants independent of h by C.)

This lemma can be proved easily from optimality condition.

[Theorem 2]

Let $(f,u,y,g,p_i), (f_h,u_h,y_h,g_h,p_{ih})(i=1,2)$ be the solution of (3), (18), respectively. And let us assume that $\|\Phi_y'\|_{0,\infty} < \overline{C}, \|\beta_g'\|_{0,\infty} < \overline{C}$. Then the following a priori error estimate holds.

$$\|f-f_h\|_0^2 + \|u-u_h\|_0^2 + \|y_h - y\|_1^2 + \|g_h - g\|_1^2 + \\ +\sum_{i=1}^{2}\left\|p_i - p_{ih}\right\|_1^2 \le C(\nu,\Omega)\cdot h$$

(We shall denote the constants independent of h by C in the proof.)

(Proof) First of all, let us estimate $\left\|p_1 - p_1^h\right\|_1$, where p_1^h is the solution of (22). Let us notice $\xi := I_h p_1 - p_1^h, \varsigma := p_1 - I_h p_1, \xi + \varsigma = p_1 - p_1^h$.

$$\begin{aligned} &\overline{C}\|\xi\|_1^2 \le (\nabla\xi,\nabla\xi)_0 + (\beta_y'(y^h,g^h)\xi,\xi)_0 + (a\nabla\xi,\xi)_0 = \\ &= (\nabla(I_h p_1),\nabla\xi)_0 + (\beta_y'(y^h,g^h)I_h p_1,\xi)_0 + \\ &+(a\nabla(I_h p_1),\xi)_0 - (\nabla p_1^h,\nabla\xi)_0 - (\beta_y'(y^h,g^h)p_1^h,\xi)_0 - \\ &-(\phi_y'(y^h,g^h)p_2^h,\xi)_0 - (a\nabla p_1^h,\xi)_0 - (\nabla p_1,\nabla\xi)_0 - \\ &-(\beta_y'(y,g)p_1,\xi)_0 - (\phi_y'(y,g)p_2,\xi)_0 - (a\nabla p_1,\xi)_0 + \\ &+(\nabla p_1,\nabla\xi)_0 + (\beta_y'(y,g)p_1,\xi)_0 + (\phi_y'(y,g)p_2,\xi)_0 + \\ &+(a\nabla p_1,\xi)_0 + (\phi_y'(y^h,g^h)p_2^h,\xi)_0 = -(\nabla\varsigma,\nabla\xi)_0 - \\ &-(\beta_y'(y,g)\varsigma,\xi)_0 - (a\nabla\varsigma,\xi)_0 + \\ &+(y-y^h,\xi)_0 + (\phi_y'(y^h,g^h)p_2^h - \phi_y'(y,g)p_2,\xi)_0 + \\ &+(\beta_y'(y^h,g^h)p_1^h - \beta_y'(y,g)p_1,\xi)_0 \end{aligned}$$

$$\begin{aligned} &\overline{C}\|\xi\|_1^2 \le C((\|\beta_y'\|_{0,\infty} + \|a\|_{0,\infty})\|\varsigma\|_1 + \left\|y-y^h\right\|_1)\|\xi\|_1 + \\ &+C\|\phi_y''\|_{0,\infty}(\left\|y-y^h\right\|_1 + \left\|g-g^h\right\|_1)\|\xi\|_1 + \\ &+\|\phi_y'\|_{0,\infty}\left\|p_2-p_2^h\right\|_1\|\xi\|_1 + C\|\beta_y''\|_{0,\infty}(\left\|y-y^h\right\|_1 + \left\|g-g^h\right\|_1)\|\xi\|_1 + \\ &+\|\beta_y'\|_{0,\infty}\left\|p_1-p_1^h\right\|_1\|\xi\|_1 \\ &\overline{C}\|\xi\|_1 \le \tilde{C}(\|\varsigma\|_1 + \left\|y-y^h\right\|_1 + \left\|g-g^h\right\|_1 + \left\|p_1-p_1^h\right\|_1 + \left\|p_2-p_2^h\right\|_1) \le \\ &\le Ch\|p_1\|_2 + \tilde{C}(\left\|y-y^h\right\|_1 + \left\|g-g^h\right\|_1 + \left\|p_1-p_1^h\right\|_1 + \left\|p_2-p_2^h\right\|_1) \le \\ &\le Ch + \tilde{C}(\left\|y-y^h\right\|_1 + \left\|g-g^h\right\|_1 + \left\|p_1-p_1^h\right\|_1 + \left\|p_2-p_2^h\right\|_1) \end{aligned}$$

$$\begin{aligned} \left\|p_1-p_1^h\right\|_1 &\le \|\xi\|_1 + \|\varsigma\|_1 \le Ch + \\ &+\tilde{C}(\left\|y-y^h\right\|_1 + \left\|g-g^h\right\|_1 + \left\|p_2-p_2^h\right\|_1) \end{aligned} \tag{23}$$

$$\tilde{C} := \frac{\hat{C}}{\overline{C} - \|\beta'\|_{0,\infty}},\quad \hat{C} := \max(1, \|\beta'\|_{0,\infty})$$

Second, let us estimate $\left\|y-y^h\right\|_1$.

This time, by letting $\xi := I_h y - y^h, \varsigma := y - I_h y, \xi + \varsigma = y - y^h$, and using lemma 3, assumption 1 and the above estimate, then we can write

$$\left\|y-y^h\right\|_1 \le \|\xi\|_1 + \|\varsigma\|_1 \le Ch + \|\beta_g'\|_{0,\infty}\left\|g-g^h\right\|_1$$

Third, let us estimate $\left\|g-g^h\right\|_1$.

This time, by letting $\xi := I_h g - g^h, \varsigma := g - I_h g, \xi + \varsigma = g - g^h$, and using lemma 3, assumption 1, then we can write

$$\|g - g^h\|_1 \le \|\xi\|_1 + \|\varsigma\|_1 \le Ch + \|\phi'_y\|_{0,\infty} \|y - y^h\|_1$$

And thus, we have

$$\|y - y^h\|_1 + \|g - g^h\|_1 \le Ch, \|p_1 - p_1^h\|_1 \le Ch \qquad (24)$$

Fourth, let us estimate $\|p_2 - p_2^h\|_1$.

Similarly to estimate $\|p_1 - p_1^h\|_1$, we have

$$\|p_2 - p_2^h\|_1 \le Ch + C_1(\|y - y^h\|_1 + \|g - g^h\|_1 + \|p_1 - p_1^h\|_1)$$

$$C_1 := \frac{\hat{C}}{\bar{C} - \|\beta'\|_{0,\infty}}, \quad \hat{C} := \max(1, \|\beta'\|_{0,\infty})$$

$$\|p_i - p_i^h\|_1 \le Ch + C(\Omega)(\|y - y^h\|_1 + \|g - g^h\|_1), i = 1, 2 \qquad (25)$$

Meanwhile, subtracting the equations which y^h, y_h satisfy, we have

$$(\nabla(y^h - y_h), \nabla \upsilon_h)_0 + (\beta(y^h, g^h) - \beta(y_h, g_h), \upsilon_h)_0 + \\ +(a\nabla(y^h - y_h), \upsilon_h)_0 = (f - f_h, \upsilon_h)_0$$

by letting a trial function by $v_h := y^h - y_h \in V_h$ and applying assumption 1, we have

$$\bar{C}\|y^h - y_h\|_1^2 \le C_\Omega \|f - f_h\|_0 + \\ +C_\Omega \|\beta'\|_{0,\infty} (\|y^h - y_h\|_0 + \|g^h - g_h\|_0)\|y^h - y_h\|_1$$

$$\bar{C}\|y^h - y_h\|_1 \le C_\Omega \|f - f_h\|_0 + C_2\|g^h - g_h\|_0 + \\ +C_\Omega^2 \|\beta'\|_{0,\infty} \|y^h - y_h\|_1$$

$$\|y^h - y_h\|_1 \le \hat{C}(\|f - f_h\|_0 + \|g^h - g_h\|_0)$$

Meanwhile, subtracting the equations which g^h, g_h satisfy, we have

$$(\nabla(g^h - g_h), \nabla \upsilon_h)_0 + (\phi(y^h, g^h) - \phi(y_h, g_h), \upsilon_h)_0 + \\ +(a\nabla(g^h - g_h), \upsilon_h)_0 = (u - u_h, \upsilon_h)_0$$

by letting a trial function by $v_h := g^h - g_h \in V_h$ and applying assumption 1, we have

$$\bar{C}\|g^h - g_h\|_1^2 \le (C_\Omega \|\phi'\|_{0,\infty} (\|y^h - y_h\|_0 + \|g^h - g_h\|_0) + \\ +\|u - u_h\|_0)\|g^h - g_h\|_1$$

$$\bar{C}\|g^h - g_h\|_1 \le C_\Omega^2 \|\phi'\|_{0,\infty} (\|y^h - y_h\|_1 + \|g^h - g_h\|_1) + \\ +C_\Omega \|u - u_h\|_0)$$

$$\|g^h - g_h\|_1 \le \hat{C}(\|y^h - y_h\|_0 + \|u - u_h\|_0),$$

$$\hat{C}_2 := \frac{C_2}{\bar{C} - C_\Omega^2 \|\phi'\|_{0,\infty}}, \quad C_2 = \max(C_\Omega, C_\Omega^2 \|\phi'\|_{0,\infty})$$

From the above results, we can write

$$\|y^h - y_h\|_1 \le C(\Omega)(\|f - f_h\|_0 + \|u - u_h\|_0), \\ C(\Omega) := C_3 / (1 - C_4^2) \qquad (26)$$

where

$C_4 := \max(C_1, \hat{C}_2)$,

$C_3 := C_1 \cdot \hat{C}_2, \ 1 = C_\Omega \ge \bar{C} > 0$

Due to the condition of this theorem, it holds $1 > \bar{C} - \max(\|\beta'\|_{0,\infty}, \|\phi'\|_{0,\infty}) > 0$.

By subtracting the equations which p_1^h, p_{1h} satisfy, we have

$$(\nabla(p_1^h - p_{1h}), \nabla \upsilon_h)_0 + (\beta'_y(y^h, g^h)p_1^h - \beta'_y(y_h, g_h)p_{1h}, \upsilon_h)_0 + \\ +(\phi'_y(y^h, g^h)p_2^h - \phi'_y(y_h, g_h)p_{2h}, \upsilon_h)_0 + (a\nabla(p_1^h - p_{1h}), \upsilon_h)_0 = \\ = (y^h - y_h, \upsilon_h)_0$$

by letting a trial function by $\upsilon_h = p_1^h - p_{1,h}$ under exploiting Lipschitz continuity of $\beta'(\cdot)$ and $\|\beta''\|_{0,\infty} \le C$, we have

$$\bar{C}\|p_1^h - p_{1h}\|_1^2 \le C_\Omega \|\beta'\|_{0,\infty} \|p_1^h - p_{1h}\|_1 + \\ +C_\Omega \|\phi'\|_{0,\infty} \|p_2^h - p_{2h}\|_1 + C_\Omega(\|\beta''\|_{0,\infty} + \|\phi''\|_{0,\infty}) \\ (\|y^h - y_h\|_1 + \|g^h - g_h\|_1)\|p_1^h - p_{1h}\|_1$$

$$\|p_1^h - p_{1h}\|_1 \le \hat{C}\|y^h - y_h\|_0 + \|g^h - g_h\|_0 + \|p_2^h - p_{2h}\|_1$$

Similarly to estimate $\|p_1^h - p_{1h}\|_1$, for p_2^h, p_{2h}, we have

$$\left\|p_2^h - p_{2h}\right\|_1 \le \hat{C}\left\|y^h - y_h\right\|_0 + \left\|g^h - g_h\right\|_0 + \left\|p_1^h - p_{1h}\right\|_1$$

$$\begin{aligned}&\left\|p_i^h - p_{ih}\right\|_1 \le C(\Omega)\left\|y^h - y_h\right\|_0 + \left\|g^h - g_h\right\|_0, i=1,2\\ &C(\Omega) := \hat{C}/(\bar{C}-\hat{C}), \hat{C} = \max(1, \|\phi'\|_\infty, \|\beta'\|_\infty)\end{aligned} \tag{27}$$

Meanwhile, the following inequalities hold.

$$\|y-y_h\|_1^2 \le \left\|y-y^h\right\|_1^2 + \left\|y^h - y_h\right\|_1^2,$$
$$\|g-g_h\|_1^2 \le \left\|g-g^h\right\|_1^2 + \left\|g^h - g_h\right\|_1^2,$$

$$\|p_i - p_{ih}\|_1^2 \le \left\|p_i - p_i^h\right\|_1^2 + \left\|p_i^h - p_{ih}\right\|_1^2 \ (i=1,2) \tag{28}$$

From lemma 4 and (23)~(28), we have

$$\begin{aligned}&\frac{\nu}{2}(\|u-u_h\|_0^2 + \|f-f_h\|_0^2) \le \|p_1 - p_{1h}\|_1^2 + \|p_2 - p_{2h}\|_1^2 + Ch \le\\ &\le C(\Omega)^2(\|u-u_h\|_0^2 + \|f-f_h\|_0^2) + Ch\end{aligned}$$

Now let us choose ν being $\nu/2 - C(\Omega)^2 > 0$.

$$\begin{aligned}&(\frac{\nu}{2} - C(\Omega)^2)(\|u-u_h\|_0^2 + \|f-f_h\|_0^2) \le Ch\\ &\|u-u_h\|_0^2 + \|f-f_h\|_0^2 \le C(\nu,\Omega)h\end{aligned} \tag{29}$$

By summing up (23)~(29), we can complete the proof.

[Lemma 5] For the solution of (23), we have the following result.

$$\|y_h\|_{0,\infty} \le c_1\|y_h\|_{1,4} \le C, \quad \|g_h\|_{0,\infty} \le c_2\|g_h\|_{1,4} \le C$$

where C is constant independent of h.

(Proof) From theorem 2, for y_h, g_h, the solution of (22), we have the following a priori error estimate.

$$\|y-y_h\|_1 \le Ch^{1/2}, \quad \|g-g_h\|_1 \le Ch^{1/2} \tag{30}$$

Let us denote the interpolation of y by $y^i \in V_h$. Because of $y \in H^{3/2}(\Omega) \cap W^{1,4}(\Omega)$, we can get

$$\left\|y^i\right\|_{1,q} \le C\|y\|_{1,q}, \forall q \in (1,\infty), \left\|y-y^i\right\|_1 \le Ch^{1/2}\|y\|_{H^{3/2}(\Omega)} \tag{31}$$

and by applying triangle inequality, using (30) and (31), we have

$$\begin{aligned}&\|y_h\|_{0,\infty} \le \|y_h\|_{1,4} \le \left\|y_h - y^i\right\|_{1,4} + \left\|y^i\right\|_{1,4} \le\\ &\le C_1 h^{-1/2}\left\|y_h - y^i\right\|_1 + C_2\|y\|_{1,4} \le\\ &\le C_1 h^{-1/2}(\|y_h - y\|_1 + \left\|y - y^i\right\|_1) + C_2\|y\|_{1,4} \le \tilde{C}\end{aligned}$$

Hence the relation $\|y_h\|_{0,\infty} \le c_1\|y_h\|_{1,4}$ holds true. We can deduce the second result of lemma in a similar way. So we can complete the proof.

3. A Posteriori Error Estimate by FEM

Now let us assume that $p_i^h \in V (i=1,2)$ satisfy the following system.

$$\begin{cases}(\nabla p_1^h, \nabla\upsilon)_0 + (\beta_y'(y^h, g^h)p_1^h, \upsilon)_0 + (\phi_y'(y^h, g^h)p_2^h, \upsilon)_0 -\\ \qquad -(a\nabla p_1^h, \upsilon)_0 = (y^h - z_d, \upsilon)_0, \ \forall\upsilon \in V\\ (\nabla p_2^h, \nabla\upsilon)_0 + (\phi_g'(y^h, g^h)p_2^h, \upsilon)_0 + (\beta_g'(y^h, g^h)p_1^h, \upsilon)_0 -\\ \qquad -(a\nabla p_2^h, \upsilon)_0 = 0, \quad \forall\upsilon \in V\\ (\nabla y^h, \nabla\upsilon)_0 + (\beta(y^h, g^h), \upsilon)_0 + (a\nabla y^h, \upsilon)_0 = (f_h, \upsilon)_0,\\ \qquad \forall\upsilon \in V\\ (\nabla g^h, \nabla\upsilon)_0 + (\phi(y^h, g^h), \upsilon)_0 + (a\nabla g^h, \upsilon)_0 = (u_h, \upsilon)_0,\\ \qquad \forall\upsilon \in V\end{cases} \tag{32}$$

We can assume that y^h, g^h, $p_i^h (i=1,2)$, the solutions of (32), exist uniquely in $H_0^1(\Omega) \cap \mathrm{H}^2(\Omega)$.

[Assumption 2] Functional J is strictly convex near (f,u) and there exists a constant C such that

$$(J'(\tilde{\nu}) - J'(\tilde{w}), \tilde{\nu} - \tilde{w}) \ge C(\|f-w_1\|_0^2 + \|u-w_2\|_0^2) \tag{33}$$

where $w := (w_1, w_2)$ is an arbitrary point in the neighborhood of $\nu := (f,u)$.

We shall assume that assumption 1, 2 are valid in the subsequent theorems.

[Lemma 6]

Let (f,u,y,g,p_i), $(f_h,u_h,y_h,g_h,p_{ih})(i=1,2)$ be the solution of (3), (17), respectively. Then we have

$$\begin{aligned}&\nu(\|f-f_h\|_0^2 + \|u-u_h\|_0^2) \le\\ &\le C(\left\|p_1^h - p_{1h}\right\|_0^2 + \left\|p_2^h - p_{2h}\right\|_0^2 + \eta_9^2 + \eta_{10}^2)\\ &\eta_9^2 := \|\nu f_h + p_{1h}\|_0^2, \ \eta_{10}^2 := \|\nu u_h + p_{2h}\|_0^2\end{aligned}$$

where $p_i^h (i=1,2)$ are the functions defined in (32), *C>0* is the constant independent of h.

(Proof) Starting from (33), we obtain

$$\begin{aligned}
&\nu(\|f-f_h\|_0^2+\|u-u_h\|_0^2)\le\\
&\le C(\nu)(\tilde{J}'(f,u)-\tilde{J}'(f_h,u_h),(f-f_h,u-u_h))_0=\\
&=C(\nu)((\nu f+p_1,f-f_h)_0+(\nu u+p_2,u-u_h)_0+\\
&\quad+(\nu f_h+p_1^h,f_h-f)_0+(\nu u_h+p_2^h,u_h-u)_0)\le\\
&\le C(\nu)((\nu f_h+p_1^h,f_h-f)_0+(\nu u_h+p_2^h,u_h-u)_0)
\end{aligned}$$

Notice that we used optimality condition here.

From the above inequality, we have

$$\begin{aligned}
&\nu(\|f-f_h\|_0^2+\|u-u_h\|_0^2)\le\\
&\le C(\nu)((p_1^h-p_{1h},f_h-f)_0+(p_2^h-p_{2h},u_h-u)_0+\\
&\quad+(\nu f_h+p_{1h},f_h-f)_0+(\nu u_h+p_{2h},u_h-u)_0)\le\\
&\le C(\nu,\delta)\left\|p_1^h-p_{1h}\right\|_0^2+\delta\|f-f_h\|_0^2+\\
&\quad+C(\nu,\delta)\left\|p_2^h-p_{2h}\right\|_0^2+\delta\|u-u_h\|_0^2+\eta_9^2+\eta_{10}^2
\end{aligned}$$

which completes the proof.

[Lemma 7]

Let $(f_h,u_h,y_h,g_h,p_{ih}),(f_h,u_h,y^h,g^h,p_i^h)(i=1,2)$ be the solution of (18), (32), respectively. And assume that $0<\bar{C}-\max(\left\|\beta_g'\right\|_{0,\infty},\left\|\phi_y'\right\|_{0,\infty})\le 1$.

Then the following inequality holds

$$\left\|y_h-y^h\right\|_1^2+\left\|g_h-g^h\right\|_1^2+\sum_{i=1}^{2}\left\|p_i^h-p_{ih}\right\|_1^2\le C\sum_{i=1}^{10}\eta_i^2$$

where

$$\eta_1^2:=\sum_{K\in T_h}h_K^2\left\|\gamma_{1,K}\right\|_{0,K}^2,$$
$$\gamma_{1,K}:=y_h-z_d+\Delta p_{1h}-\beta_y'(y_h,g_h)p_{1h}-\phi_y'(y_h,g_h)p_{2h}+a\nabla p_{1h}$$

$$\eta_2^2:=\sum_{E\in E_h}h_E\left\|\gamma_{1E}\right\|_{0,\mathrm{E}}^2,$$
$$\gamma_{1E}:=\begin{cases}\left[\dfrac{\partial p_{1h}}{\partial n}\right]_E=(\nabla p_{1h}-\nabla\bar{p}_{1h})\cdot n_K, & E\cap\Gamma=\varnothing\\[2ex] \left.\dfrac{\partial p_{1h}}{\partial n}\right|_E, & E\cap\Gamma\ne\varnothing\end{cases}$$

where $\left[\dfrac{\partial p_{1h}}{\partial n}\right]_E$ is the jump of directional derivative on the common boundary E of two elements $K,\bar{K}$.

$$\eta_3^2:=\sum_{K\in T_h}h_K^2\left\|\gamma_{2,K}\right\|_{0,K}^2,$$
$$\gamma_{2,K}:=-\beta_g'(y_h,g_h)p_{1h}+\Delta p_{2h}-\phi_g'(y_h,g_h)p_{2h}+a\nabla p_{2h}$$

$$\eta_4^2:=\sum_{E\in E_h}h_E\left\|\gamma_{2E}\right\|_{0,\mathrm{E}}^2,$$
$$\gamma_{2E}:=\begin{cases}\left[\dfrac{\partial p_{2h}}{\partial n}\right]_E=(\nabla p_{2h}-\nabla\bar{p}_{2h})\cdot n_K, & E\cap\Gamma=\varnothing\\[2ex] \left.\dfrac{\partial p_{2h}}{\partial n}\right|_E, & E\cap\Gamma\ne\varnothing\end{cases}$$

where $\left[\dfrac{\partial p_{2h}}{\partial n}\right]_E$ is the jump of directional derivative on the common boundary E of two elements $K,\bar{K}$.

$$\eta_5^2:=\sum_{K\in T_h}h_K^2\left\|r_{1,K}\right\|_{0,K}^2,$$
$$r_{1,K}:=f_h+\Delta y_h-\beta(y_h,g_h)-a\nabla y_h$$

$$\eta_6^2:=\sum_{E\in E_h}h_E\left\|\gamma_{1E}\right\|_{0,\mathrm{E}}^2,$$
$$\gamma_{1E}:=\begin{cases}\left[\dfrac{\partial y_h}{\partial n}\right]_E=(\nabla y_h-\nabla\bar{y}_h)\cdot n_K, & E\cap\Gamma=\varnothing\\[2ex] \left.\dfrac{\partial y_h}{\partial n}\right|_E, & E\cap\Gamma\ne\varnothing\end{cases}$$

where $\left[\dfrac{\partial y_h}{\partial n}\right]_E$ is the jump of directional derivative on the common boundary E of two elements $K,\bar{K}$.

$$\eta_7^2:=\sum_{K\in T_h}h_K^2\left\|r_{2,K}\right\|_{0,K}^2,$$
$$r_{2,K}:=u_h+\Delta g_h-\phi(y_h,g_h)-a\nabla g_h$$

$$\eta_8^2:=\sum_{E\in E_h}h_E\left\|\gamma_{2E}\right\|_{0,\mathrm{E}}^2,$$
$$\gamma_{2E}:=\begin{cases}\left[\dfrac{\partial g_h}{\partial n}\right]_E=(\nabla g_h-\nabla\bar{g}_h)\cdot n_K, & E\cap\Gamma=\varnothing\\[2ex] \left.\dfrac{\partial g_h}{\partial n}\right|_E, & E\cap\Gamma\ne\varnothing\end{cases}$$

where $\left[\dfrac{\partial g_h}{\partial n}\right]_E$ is the jump of directional derivative on the common boundary E of 2 elements $K,\bar{K}$.

(Proof) *Step 1*:

Let $\xi := p_1^h - p_{1h}, \xi_I := I_h\xi \in V_h \subset H_0^1(\Omega)$.

First, for bilinear form $B(\upsilon,\upsilon) := (\nabla\upsilon,\nabla\upsilon)_0 + (a\nabla\upsilon,\upsilon)_0$, let us prove that there is a constant $\bar{C} > 0$ independent of v such that $B(\upsilon,\upsilon) \ge \bar{C}\|\upsilon\|_1^2$.

Relation

$$B(\upsilon,\upsilon) := \|\nabla\upsilon\|_0^2 + (a\nabla\upsilon,\upsilon)_0 = \|\nabla\upsilon\|_0^2 + \frac{1}{2}(a\nabla\upsilon^2,1)_0, \quad \forall\upsilon \in V = H_0^1(\Omega)$$

$$(a\nabla\upsilon^2,1)_0 = (div(a\upsilon^2),1)_0 - (diva,\upsilon^2)_0 = = (a\upsilon^2,n)_{0,\Gamma} - (diva,\upsilon^2)_0 = -(diva,\upsilon^2)_0$$

holds true, so we can deduce the following inequality.

$$B(\upsilon,\upsilon) := \|\nabla\upsilon\|_0^2 + \frac{1}{2}(a\nabla\upsilon^2,1)_0 = \|\nabla\upsilon\|_0^2 - \frac{1}{2}(diva,\upsilon^2)_0 = \ge \bar{C}\|\upsilon\|_1^2, \quad \forall\upsilon \in V = H_0^1(\Omega)$$

where $\bar{C} = c_1$ is the constant in the norm equality

$$c_1\|\upsilon\|_1 \le \|\nabla\upsilon\|_0 \le c_2\|\upsilon\|_1, \ \forall\upsilon \in H_0^1(\Omega).$$

By exploiting the assumption 1 on a, $\beta'(\cdot)$, and the inequality

$$(\nabla\xi_h,\nabla\xi_h)_0 + (a\nabla\xi_h,\xi_h)_0 \ge \bar{C}\|\xi_h\|_1^2$$

we can deduce the following inequality.

$$\begin{aligned}
\bar{C}\|\xi\|_1^2 &\le (\nabla\xi,\nabla\xi)_0 + (\beta_y'(y^h,g^h)\xi,\xi)_0 + (a\nabla\xi,\xi)_0 \le \\
&\le (\nabla p_1^h,\nabla\xi)_0 + (\beta_y'(y^h,g^h)p_1^h,\xi)_0 - (\nabla p_{1h},\nabla\xi)_0 - \\
&\quad - (\beta_y'(y^h,g^h)p_{1h},\xi)_0 - (a\nabla p_1^h,\xi)_0 + (a\nabla p_{1h},\xi)_0 = \\
&= (y^h - y_h,\xi)_0 + (\phi_y'(y^h,g^h)p_2^h,\xi)_0 - (\phi_y'(y_h,g_h)p_{2h},\xi)_0 - \\
&\quad - (\nabla p_{1h},\nabla\xi)_0 - (\beta_y'(y_h,g_h)p_{1h},\xi)_0 - (y_h - z_d,\xi)_0 + \\
&\quad + ((\beta_y'(y_h,g_h) - \beta_y'(y^h,g^h))p_{1h},\xi)_0 - \\
&\quad - (a\nabla p_{1h},\xi)_0 + (\phi_y'(y_h,g_h)p_{2h},\xi)_0 = \\
&= (y_h - z_d + a\nabla p_{1h},\xi)_0 + (y^h - y_h,\xi)_0 - (\nabla p_{1h},\nabla\xi)_0 - \\
&\quad - (\beta_y'(y_h,g_h)p_{1h} + \phi_y'(y_h,g_h)p_{2h},\xi)_0 + \\
&\quad + ((\beta_y'(y_h,g_h) - \beta_y'(y^h,g^h))p_{1h},\xi)_0 + (\phi_y'(y^h,g^h)p_2^h - \\
&\quad - (\phi_y'(y_h,g_h)p_{2h},\xi)_0 = \\
&= \sum_{K\in T_h}(\gamma_{1K},\xi)_{0,K} - \sum_{E\in E_h}(\gamma_{1E},\xi)_{0,E} + \\
&\quad + ((\beta_y'(y_h,g_h) - \beta_y'(y^h,g^h))p_{1h},\xi)_0 + \\
&\quad + (\phi_y'(y^h,g^h)p_2^h - (\phi_y'(y_h,g_h)p_{2h},\xi)_0
\end{aligned}$$

By adding this to

$$(\nabla p_{1h},\nabla\xi_I)_0 + (\beta_y'(y_h,g_h)p_{1h} + \phi_y'(y_h,g_h)p_{2h},\xi_I)_0 - - (a\nabla p_{1h},\xi_I)_0 - (y_h - z_d,\xi_I)_0 = 0$$

and applying lemma 3, then we can obtain the following estimation.

$$\begin{aligned}
\bar{C}\|\xi\|_1^2 &\le \sum_{K\in T_h} C\|\gamma_{1K}\|_{0,K} h_K \|\xi\|_{1,K} + \\
&\quad + \sum_{E\in E_h} C\|\gamma_{1E}\|_{0,E} h_E^{1/2}\|\xi\|_{1,K} + \\
&\quad + C(\|y^h - y_h\|_0 + \|g^h - g_h\|_0)\|\xi\|_1
\end{aligned}$$

$$\begin{aligned}
&\le \sum_{K\in T_h} C\|\gamma_{1K}\|_{0,K}^2 h_K^2 + \sum_{E\in E_h} C\|\gamma_{1E}\|_{0,E}^2 h_E + \\
&\quad + C(\|y^h - y_h\|_1^2 + \|g^h - g_h\|_1^2) + \delta\|\xi\|_1^2 \le \\
&\le \eta_1^2 + \eta_2^2 + C(\|y^h - y_h\|_1^2 + \|g^h - g_h\|_1^2)
\end{aligned}$$

$$\|p_1^h - p_{1h}\|_1 \le C(\|y_h - y^h\|_1^2 + \|g_h - g^h\|_1^2) + \eta_1^2 + \eta_2^2 \quad (34)$$

Step 2:

Let $\xi := p_2^h - p_{2h}$, $\xi_I := I_h\xi \in V_h \subset H_0^1(\Omega)$. By using the assumption 1 on a, $\phi'(\cdot)$, then we have

$$\begin{aligned}
\bar{C}\|\xi\|_1^2 &\le (\nabla\xi,\nabla\xi)_0 + (\beta_g'(y^h,g^h)\xi,\xi)_0 + (a\nabla\xi,\xi)_0 = \\
&= (\nabla p_2^h,\nabla\xi)_0 + (\beta_g'(y^h,g^h)p_2^h,\xi)_0 - \\
&\quad - (\nabla p_{2h},\nabla\xi)_0 - (\beta_g'(y^h,g^h)p_{2h},\xi)_0 - \\
&\quad - (a\nabla p_2^h,\xi)_0 + (a\nabla p_{2h},\xi)_0 = \\
&= -((\phi_g'(y^h,g^h) - \phi_g'(y_h,g_h))p_1^h,\xi)_0 - \\
&\quad - (\beta_g'(y^h,g^h)p_{2h} - \beta_g'(y_h,g_h)p_{2h},\xi)_0 + \\
&\quad + (-a\nabla p_{2h},\xi)_0 - (\beta_g'(y_h,g_h)p_{2h},\xi)_0 - \\
&\quad - (\nabla p_{2h},\nabla\xi)_0 - (\phi_g'(y_h,g_h)p_{1h},\xi)_0
\end{aligned}$$

By adding this to

$$(\nabla p_{2h},\nabla\xi_I)_0 + (\phi_g'(y_h,g_h)p_{2h},\xi_I)_0 - (a\nabla p_{2h},\xi_I)_0 + + (\beta_g'(y_h,g_h)p_{1h},\xi_I)_0 = 0$$

then we have the following estimation.

$$
\begin{aligned}
\bar{C}\|\xi\|_1^2 &\le \sum_{K\in T_h}(\gamma_{2K},\xi-\xi_I)_{0,K}+\sum_{E\in E_h}(\gamma_{2E},\xi-\xi_I)_{0,E}+\\
&+((\phi_g'(y_h,g_h)-\phi_g'(y^h,g^h))p_{1h},\xi)_0+\\
&+\left\|y^h-y_h\right\|_0(\|p_{2h}\|_{0,4}+\|p_{1h}\|_{0,4})\|\xi\|_{0,4}+\\
&+\left\|g^h-g_h\right\|_0(\|p_{2h}\|_{0,4}+\|p_{1h}\|_{0,4})\|\xi\|_{0,4}\le\\
&\le\sum_{K\in T_h}\|\gamma_{2K}\|_{0,K}^2Ch_K^2+\sum_{E\in E_h}\|\gamma_{2E}\|_{0,E}^2Ch_K+\\
&+C(\left\|y^h-y_h\right\|_0+\left\|g^h-g_h\right\|_0)+\delta\|\xi\|_1^2\le\\
&\le\eta_3^2+\eta_4^2+C(\left\|y^h-y_h\right\|_1^2+\left\|g^h-g_h\right\|_1^2)+\\
&+\delta\|\xi\|_1^2\frac{n!}{r!(n-r)!}
\end{aligned}
$$

$$
\begin{aligned}
\left\|p_2^h-p_{2h}\right\|_1^2 &\le C(\eta_3^2+\eta_4^2)+\\
&+C(\left\|y_h-y^h\right\|_1^2+C\left\|g_h-g^h\right\|_1^2)
\end{aligned}
\tag{35}
$$

Step 3:

Let $\xi:=\xi_1+\xi_2,\xi_1:=y^h-y_h,\xi_2:=g^h-g_h,$ $\xi_{1I}:=I_h\xi_1\in V_h$, $\xi_{2I}:=I_h\xi_2\in V_h,\xi\in V$

By using the assumption on β,ϕ , we can obtain

$$
\begin{aligned}
&\bar{C}(\|\nabla\xi_1\|_0^2+\|\nabla\xi_2\|_0^2)\le\\
&\le(\nabla(y^h-y_h),\nabla\xi_1)_0+(\nabla(g^h-g_h),\nabla\xi_2)_0+\\
&+(\beta(y_h,g^h),\xi_1)_0-(\beta(y^h,g^h),\xi_1)_0+\\
&+(a\nabla(y^h-y_h),\xi_1)_0+
\end{aligned}
$$

$$
\begin{aligned}
&+(\phi(y^h,g_h)-\phi(y^h,g^h),g^h-g_h)_0+\\
&+(a\nabla(g^h-g_h),\xi_2)_0=(\nabla y^h,\nabla\xi_1)_0-(\nabla y_h,\nabla\xi_1)_0+\\
&+(\nabla g^h,\nabla\xi_2)_0-(\nabla g_h,\nabla\xi_2)_0+
\end{aligned}
$$

$$
\begin{aligned}
&+(\beta(y^h,g^h),\xi_1)_0+(\beta(y_h,g^h)-\beta(y_h,g_h),\xi_1)_0+\\
&+(\beta(y_h,g_h),\xi_1)_0+(\phi(y^h,g^h),\xi_2)_0-(\phi(y^h,g_h),\xi_2)_0+\\
&+(a\nabla y^h,\xi_1)_0-(a\nabla y_h,\xi_1)_0+(a\nabla g^h,\xi_2)_0-\\
&-(a\nabla g_h,\xi_2)_0-(\phi(y_h,g_h),\xi_2)_0+(\phi(y_h,g_h),\xi_2)_0=\\
&=(f_h,\xi_1)_0-(u_h,\xi_2)_0-(\nabla y_h,\nabla\xi_1)_0-\\
&-(\beta(y_h,g_h)+a\nabla y_h,\xi_1)_0-(\nabla g_h,\nabla\xi_2)_0-\\
&-(\phi(y_h,g_h),\xi_2)_0-a\nabla g_h,\xi_2)_0+\\
&+(\beta(y_h,g^h)-\beta(y_h,g_h),\xi_1)_0+(\phi(y_h,g_h)-\phi(y^h,g_h),\xi_2)_0
\end{aligned}
$$

By adding this to

$$
\begin{aligned}
&(\nabla y_h,\nabla\xi_{1I})_0+(\beta(y_h,g_h),\xi_{1I})_0+\\
&\qquad+(a\nabla y_h,\xi_{1I})_0-(f_h,\xi_{1I})_0=0,\\
&(\nabla g_h,\nabla\xi_{2I})_0+(\phi(y_h,g_h),\xi_{2I})_0+\\
&\qquad+(a\nabla g_h,\xi_{2I})_0-(u_h,\xi_{2I})_0=0
\end{aligned}
$$

then we have

$$
\begin{aligned}
&=\sum_{K\in T_h}(f_h+\Delta y_h-\beta(y_h,g_h)-a\nabla y_h,\xi_1-\xi_{1I})_{0,K}-\\
&-\sum_{E\in E_h}(\frac{\partial y_h}{\partial n},\xi-\xi_I)_{0,\partial K}-\\
&-\sum_{K\in T_h}(u_h+\Delta g_h-\phi(y_h,g_h)-a\nabla g_h,\xi_2-\xi_{2I})_{0,K}-\\
&-\sum_{E\in E_h}(\frac{\partial g_h}{\partial n},\xi-\xi_I)_{0,\partial K}+\\
&+(\|\beta_g'\|_{0,\infty}+\|\phi_y'\|_{0,\infty})(\left\|y^h-y_h\right\|_0^2+\left\|g^h-g_h\right\|_0^2)+\\
&+\delta\|\xi_2\|_1^2\le\\
&\le\sum_{K\in T_h}\|r_{1,K}\|_{0,K}^2Ch_K^2+\sum_{K\in T_h}\|r_{2,K}\|_{0,K}^2Ch_K^2+\\
&+\sum_{E\in E_h}\|r_{1,E}\|_{0,E}^2Ch_K+C_\Omega^2\|\beta_g'\|_{0,\infty}\left\|y^h-y_h\right\|_1^2+\\
&+C_\Omega^2\|\phi_y'\|_{0,\infty}\left\|g^h-g_h\right\|_1^2)+\sum_{E\in E_h}\|r_{2,E}\|_{0,E}^2Ch_K+\\
&+\delta\|\xi_1\|_1^2+\delta\|\xi_2\|_1^2
\end{aligned}
$$

where C_Ω is the equivalence constant between the space $H_0^1(\Omega),\ H^1(\Omega)$.

From the assumption 1, using $0<\bar{C}-C_\Omega^2\max(\|\beta_g'\|_{0,\infty},\|\phi_y'\|_{0,\infty})\le 1$, we can deduce

$$
\left\|y^h-y_h\right\|_1^2+\left\|g^h-g_h\right\|_1^2\le C\sum_{i=5}^{8}\eta_i^2 \tag{36}
$$

By summing up (34)~(36), we can complete the proof.

[Theorem 3]

Let $(f,u,y,g,p_i),(f_h,u_h,y_h,g_h,p_{ih})(i=1,2)$ be the solution of (3) and (18), respectively. If the condition $1>\bar{C}-\max(\|\beta'\|_{0,\infty},\|\phi'\|_{0,\infty})>0$ is valid, then we can obtain the following upper bound estimate for a posteriori error

$$\|f-f_h\|_0^2+\|u-u_h\|_0^2+\|y_h-y\|_1^2+$$
$$+\|g_h-g\|_1^2+\sum_{i=1}^{2}\|p_i-p_{ih}\|_1^2\le C\sum_{i=1}^{10}\eta_i^2$$

(Proof) From lemma 6 and lemma 7, we have

$$\nu(\|f-f_h\|_0^2+\|u-u_h\|_0^2)\le$$
$$\le C(\|p_{1h}-p_1^h\|_1^2+\|p_{2h}-p_2^h\|_1^2)\le C\sum_{i=1}^{10}\eta_i^2 \tag{37}$$

Meanwhile, the following inequalities hold.

$$\begin{aligned}&\|y-y_h\|_1^2\le\|y-y^h\|_1^2+\|y^h-y_h\|_1^2\\&\|g-g_h\|_1^2\le\|g-g^h\|_1^2+\|g^h-g_h\|_1^2\end{aligned} \tag{38}$$

$$\|p_i-p_{ih}\|_1^2\le\|p_i-p_i^h\|_1^2+\|p_i^h-p_{ih}\|_1^2 \tag{39}$$

By subtracting the equation satisfying $y,\ y^h$, and choosing a trial function as $\upsilon=y-y^h$, and using assumption 1, then we have

$$\bar{C}\|\nabla(y-y^h)\|_0\le C_\Omega(\|\beta(y,g)-\beta(y^h,g^h)\|_0+\|f-f_h\|_0)$$
$$\|\nabla(y-y^h)\|_0\le C'_\Omega\|\nabla\beta'\|_{0,\infty}(\|y-y^h\|_0+\|g-g^h\|_0)+$$
$$+C_\Omega\|f-f_h\|_0)$$

This time, by subtracting the equation satisfying $g,\ g^h$, and choosing a trial function as $\upsilon=g-g^h$, using assumption 1 and the monotonity of $\phi(\cdot,\cdot)$, then we have

$$\|g-g^h\|_1\le\bar{C}_\Omega\|u-u_h\|_0$$

$$\bar{C}\|\nabla(y-y^h)\|_0\le C_\Omega^2\|\beta'\|_{0,\infty}\|\nabla(y-y^h)\|_0+$$
$$+C_\Omega(\|g-g^h\|_1+\|f-f_h\|_0)\le$$
$$\le C_\Omega^2\|\beta'\|_{0,\infty}\|\nabla(y-y^h)\|_0+$$
$$+C_\Omega(\|u-u_h\|_0+\|f-f_h\|_0)$$

From assumption 1 and above inequalities, we can deduce the following.

$$\|\nabla(y-y^h)\|_0\le\frac{C_\Omega}{\bar{C}-C_\beta}(\|g-g^h\|_0+\|f-f_h\|_0)$$

Denoting the equivalence constant of norm of $H^1(\Omega),\ H_0^1(\Omega)$ by C_1, and taking into account $\|\beta'\|_{0,\infty}\le C_\beta$ and assumption 1, then it follows that

$$\|y-y^h\|_1\le\frac{C_\Omega}{C_1(\bar{C}-C_\beta)}(\|g-g^h\|_0+\|f-f_h\|_0)\le$$
$$\le\frac{C_\Omega}{C_1(\bar{C}-C_\beta)}(C_1^{-1}\|u-u_h\|_0+\|f-f_h\|_0)\le$$
$$\le\tilde{C}(\|u-u_h\|_0+\|f-f_h\|_0)$$

$$\|y-y^h\|_1\le\tilde{C}(\|u-u_h\|_0+\|f-f_h\|_0)$$

where $\tilde{C}:=\dfrac{C_\Omega C_2}{C_1(\bar{C}-C_\beta)},\ C_2:=\max(C_1,1)$.

Similarly, subtracting the equation satisfying $p_1,\ p_1^h$, choosing a trial function as $\upsilon=p_1-p_1^h$, taking into account assumption 1, and then we have

$$\|\nabla(p_1-p_1^h)\|_0^2+(\beta'_y(y,g)p_1-\beta'_y(y^h,g^h)p_1^h,p_1-p_1^h)_0+$$
$$+(\phi'_y(y,g)p_2-\phi'_y(y^h,g^h)p_2^h,p_1-p_1^h)_0+$$
$$+(a\nabla(p_1-p_1^h),p_1-p_1^h)_0=(y-y^h,p_1-p_1^h)_0$$
$$\bar{C}\|p_1-p_1^h\|_1\le C_\Omega(\|\beta'_y(y,g)p_1-\beta'_y(y^h,g^h)p_1^h\|_0+$$
$$+\|\phi'_y(y,g)p_2-\phi'_y(y^h,g^h)p_2^h\|_0+\|y-y^h\|_0)$$

$$\le C_\beta\|p_1-p_1^h\|_1+C_\phi\|p_2-p_2^h\|_1+\tilde{C}(\|y-y^h\|_1+\|g-g^h\|_1),$$
$$(\tilde{C}:=\max(\|p_1^h\|_0,\|p_2^h\|_0)$$

$$\|p_1-p_1^h\|_1\le C(\|f-f_h\|_0+\|u-u_h\|_0+\|p_2-p_2^h\|_1)$$

Similarly, subtracting the equation satisfying $p_2,\ p_2^h$, choosing a trial function as $\upsilon=p_2-p_2^h$, taking into account assumption 1 and above estimations, and then we have

$$\|\nabla(p_2-p_2^h)\|_0^2+(\beta'_g(y,g)p_2-\beta'_g(y^h,g^h)p_2^h,p_2-p_2^h)_0+$$
$$+(\phi'_g(y,g)p_1-\phi'_g(y^h,g^h)p_1^h,p_2-p_2^h)_0+$$
$$+(a\nabla(p_2-p_2^h),p_2-p_2^h)_0=$$

$$= (y - y^h, p_2 - p_2^h)_0$$

$$\bar{C}\left\|\nabla(p_2 - p_2^h)\right\|_0 \le C_\Omega(\left\|\beta_g'(y,g)p_2 - \beta_g'(y^h,g^h)p_2^h\right\|_0 + $$
$$+\left\|\phi_g'(y,g)p_1 - \phi_g'(y^h,g^h)p_1^h\right\|_0 + \left\|y - y^h\right\|_0)$$

$$\le C_\beta\left\|\nabla(p_2 - p_2^h)\right\|_0 + C_\phi\left\|\nabla(p_1 - p_1^h)\right\|_0 +$$
$$+\tilde{C}(\left\|y - y^h\right\|_1 + \left\|g - g^h\right\|_1 \;(\tilde{C} := C_\Omega \max(\left\|p_2^h\right\|_0, \left\|p_1^h\right\|_0)$$

$$\left\|p_2 - p_2^h\right\|_1 \le C(\left\|f - f_h\right\|_0 + \left\|u - u_h\right\|_0 + \left\|p_4 - p_4^h\right\|_1),$$
$$(C := \frac{\tilde{C}}{\bar{C} - C_\beta})$$

$$\tilde{C} := \frac{\hat{C}}{\bar{C} - \hat{C}},\; \hat{C} := \max(C_1, C_2),\; C_i := \left\|p_{ih}\right\|_{0,\infty}$$

Due to assumption 1, all the above conditions are satisfied when $1 > \bar{C} - \max(\left\|\beta'\right\|_{0,\infty}, \left\|\phi'\right\|_{0,\infty}) > 0$. Thus there holds

$$\begin{aligned}\left\|p_1 - p_1^h\right\|_1 &\le \hat{C}(\left\|f - f_h\right\|_0 + \left\|u - u_h\right\|_0)\\ \left\|p_2 - p_2^h\right\|_1 &\le \hat{C}(\left\|f - f_h\right\|_0 + \left\|u - u_h\right\|_0)\end{aligned} \tag{40}$$

and for all v such that $v > \hat{C}$, by summing up (37)~(40), we can get the desired result. This completes the proof.

4. Lower Bound Estimate for a Posteriori Error by FEM

Let us introduce the following function using barycentric coordinate $\lambda_{K_1}, \lambda_{K_2}, \lambda_{K_3}$ [3]

$$b_K := \begin{cases} 27\lambda_{K_1}\lambda_{K_2}\lambda_{K_3}, & K \\ 0, & \Omega \setminus K \end{cases}$$

And let us introduce the following function depending on the end points (number 1, 2) of edge $E \in E_h$, which is the common edge of element K, K'.

$$b_K := \begin{cases} 4\lambda_{K_i}\lambda_{K_j}, & K \\ 4\lambda_{K_i}\lambda_{K_m}, & K' \;,\; (\omega_E := K \cup K') \\ 0, & (\omega_E)^c \end{cases}$$

[Lemma 8] [3] For every element $K \in T_h$ and edge $E \in E_h$, b_K, b_E have the following characteristics.

$$\sup b_K \subset K, b_K \in [0,1], \max_{x \in K} b_K = 1,$$

$$\int_K b_K dx = \frac{9}{20}|K| \le Ch_K^2, \left\|\nabla b_K\right\|_{0,K} \le Ch_K^{-1} \cdot \left\|b_K\right\|_{0,K}$$

$$\sup b_E \subset \omega_E, b_E \in [0,1], \max_{x \in \omega_E} b_E = 1,$$

$$\int_{\omega_E} b_E dx = \frac{1}{3}|\omega_E| \le Ch_E^2, \left\|\nabla b_E\right\|_{0,\omega_E} \le Ch_E^{-1} \cdot \left\|b_E\right\|_{0,\omega_E}$$

[Theorem 4]

Let (f,u,y,g,p_i), $(f_h,u_h,y_h,g_h,p_{ih})(i = 1,2,3,4)$ be the solution of (3), (18). Then we have the following lower bound estimate for a posteriori error by FEM.

$$\sum_{i=1}^{10}\eta_i^2 \le C(\left\|f - f_h\right\|_0^2 + \left\|u - u_h\right\|_0^2 + \left\|y_h - y\right\|_1^2 +$$
$$+\left\|g_h - g\right\|_1^2 + \sum_{i=1}^{2}\left\|p_i - p_{ih}\right\|_1^2) + osc + Ch$$

where C is independent of h and two of Cs are not the same.

$$osc := \sum_{i=1}^{2}(osc(\gamma_{iK}) + osc(\gamma_{iE})) + \sum_{i=1}^{2}(osc(r_{iK}) + osc(r_{iE})) +$$
$$+ osc(z_h) + osc(z_{1h})$$

$$osc(\gamma_{iK}) := \sum_{K \in T_h} h_K^2 \left\|\gamma_{iK} - \bar{\gamma}_{iK}\right\|_{0,K}^2,$$
$$osc(\gamma_{iE}) := \sum_{E \in T_h} h_E^2 \left\|\gamma_{iE} - \bar{\gamma}_{iE}\right\|_{0,E}^2 \;(i = 1,2)$$

$$osc(r_{iK}) := \sum_{K \in T_h} h_K^2 \left\|r_{iK} - \bar{r}_{iK}\right\|_{0,K}^2,$$
$$osc(r_{iE}) := \sum_{E \in T_h} h_E^2 \left\|r_{iE} - \bar{r}_{iE}\right\|_{0,E}^2 \;(i = 1,2)$$

$$osc(z_h) := \sum_{K \in T_h} \left\|z_h - \bar{z}_h\right\|_{0,K}^2,$$
$$osc(z_{1h}) := \sum_{K \in T_h} \left\|z_{1h} - \bar{z}_{1h}\right\|_{0,K}^2$$
$$z_h := u_h + p_{4h},\quad z_{1h} := f_h + p_{1h}$$

(Proof) · Estimation of η_1^2:

Let us consider the average $\bar{\gamma}_{1K} := \frac{1}{|K|}\int_K \gamma_{1K} dx$.

$$\left\|\gamma_{1K}\right\|_{0,K}^2 \le \left\|\bar{\gamma}_{1K}\right\|_{0,K}^2 + \left\|\gamma_{1K} - \bar{\gamma}_{1K}\right\|_{0,K}^2,$$

$$
\begin{aligned}
&\|\bar{\gamma}_{1K}\|_{0,K}^2 \sim (\bar{\gamma}_{1K}, b_K\bar{\gamma}_{1K})_{0,K} = \\
&= (\gamma_{1K}, b_K\bar{\gamma}_{1K})_{0,K} - (\gamma_{1K}-\bar{\gamma}_{1K}, b_K\bar{\gamma}_{1K})_{0,K} = \\
&= (y_h - z_d - \beta_y'(y_h,g_h)p_{1h} - \phi_y'(y_h,g_h)p_{2h} + \\
&\quad + a\nabla p_{1h} + \Delta p_{1h}, b_K\bar{\gamma}_{1K})_{0,K} + I_4 = \\
&= (y_h - z_d - \beta_y'(y_h,g_h)p_{1h} - \phi_y'(y_h,g_h)p_{2h} + \\
&\quad + a\nabla p_{1h} + \Delta p_{1h}, b_K\bar{\gamma}_{1K})_{0,K} +
\end{aligned}
$$

$$
\begin{aligned}
&+(-(y-z_d)+\beta_y'(y,g)p_1 - \phi_y'(y,g)p_2 - \\
&\quad - a\nabla p_1 - \Delta p_1, b_K\bar{\gamma}_{1K})_{0,K} + I_4 = (y_h - y, b_K\bar{\gamma}_{1K})_{0,K} + \\
&+(\beta_y'(y,g)p_1 - \beta_y'(y_h,g_h)p_{1h}, b_K\bar{\gamma}_{1K})_{0,K} +
\end{aligned}
$$

$$
\begin{aligned}
&+(\phi_y'(y,g)p_2 - \phi_y'(y_h,g_h)p_{2h} + \\
&+\Delta(p_{1h}-p_1) + a\nabla(p_{1h}-p_1), b_K\bar{\gamma}_{1K})_{0,K} + \\
&+ I_4 = I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

where $I_4 := (\gamma_{1K} - \bar{\gamma}_{1K}, b_K\bar{\gamma}_{1K})_{0,K}$.

By exploiting lemma 3 and lemma 8, we have

$$
\begin{aligned}
I_1 &\le \|y_h - y\|_{0,K} \|b_K\bar{\gamma}_{1K}\|_{0,K} \le \\
&\le \|y_h - y\|_{1,K} \|b_K\|_{0,\infty,K} \|\bar{\gamma}_{1K}\|_{0,K} \le \\
&\le C\|y_h - y\|_{1,K} h_K^{-1} \|b_K\|_{0,K} \|\bar{\gamma}_{1K}\|_{0,K} \le \\
&\le Ch_K^{-1}\|y_h - y\|_{1,K} \|\bar{\gamma}_{1K}\|_{0,K}
\end{aligned}
$$

Taking into account $\|\beta'(\cdot)\|_{0,\infty} \le C$, we can obtain the following estimates.

$$
\begin{aligned}
I_2 &\le C\left\|p_1 - p_1^h\right\|_{0,K} \|b_K\bar{\gamma}_{1K}\|_{0,K} + \\
&\quad + \left\|\beta_y'(y,g) - \beta_y'(y_h,g_h)\right\|_{0,4,K} \|p_{1h}\|_{0,4,K} \|b_K\bar{\gamma}_{1K}\|_{0,K} + \\
&\quad + \left\|\phi_y'(y,g) - \phi_y'(y_h,g_h)\right\|_{0,4,K} \|p_{3h}\|_{0,4,K} \|b_K\bar{\gamma}_{1K}\|_{0,K} \le \\
&\le Ch_K^{-1}\left\|p_1 - p_1^h\right\|_{0,K} \|\bar{\gamma}_{1K}\|_{0,K} + \\
&\quad + Ch_K^{-1}(\|y - y_h\|_{0,K} + \|g - g_h\|_{0,K})\|\bar{\gamma}_{1K}\|_{0,K} \le \\
&\le Ch_K^{-1}(\left\|p_1 - p_1^h\right\|_{1,K} + \|y_h - y\|_{1,K} + \|g - g_h\|_{1,K})\|\bar{\gamma}_{1K}\|_{0,K}
\end{aligned}
$$

$$
\begin{aligned}
I_3 &\le \|\Delta(p_{1h}-p_1)\|_{0,K} \|b_K\bar{\gamma}_{1K}\|_{0,K} + \\
&\quad + \|a\nabla(p_{1h}-p_1)\|_{0,K} \|b_K\bar{\gamma}_{1K}\|_{0,K} \le \\
&\le Ch_K^{-1}\|p_{1h} - p_1\|_{1,K} \|\bar{\gamma}_{1K}\|_{0,K}
\end{aligned}
$$

$$
\begin{aligned}
&\|\bar{\gamma}_{1K}\|_{0,K} \le Ch_K^{-1}(\|y_h - y\|_{1,K} + \|p_{1h} - p_1\|_{1,K} + \|g_h - g\|_{1,K}) \\
&h_K^2\|\gamma_{1K}\|_{0,K}^2 \le C(\|y_h - y\|_{1,K}^2 + \|p_{1h} - p_1\|_{1,K}^2 + \|g_h - g\|_{1,K}^2) \\
&\quad + h_K^2\|\gamma_{1K} - \bar{\gamma}_{1K}\|_{0,K}^2
\end{aligned}
$$

$$
\begin{aligned}
\eta_1^2 &:= \sum_{K\in T_h} h_K^2\|\gamma_{1K}\|_{0,K}^2 \le \\
&\le C(\|y_h - y\|_1^2 + \|p_{1h} - p_1\|_1^2 + \|g_h - g\|_1^2) + osc(\gamma_{1K}),
\end{aligned}
$$

Estimation of η_2^2:

Let us consider the average $\bar{\gamma}_{1E} := \frac{1}{|E|}\int_E \gamma_{1E}\,dl$.

$$
\|\gamma_{1E}\|_{0,E}^2 \le \|\bar{\gamma}_{1E}\|_{0,E}^2 + \|\gamma_{1E} - \bar{\gamma}_{1E}\|_{0,E}^2
$$

$$
\begin{aligned}
&\|\bar{\gamma}_{1E}\|_{0,E}^2 \sim (\bar{\gamma}_{1E}, b_E\bar{\gamma}_{1E})_{0,E} = \\
&= (\gamma_{1E}, b_E\bar{\gamma}_{1E})_{0,E} - (\gamma_{1E} - \bar{\gamma}_{1E}, b_E\bar{\gamma}_{1E})_{0,E} = \\
&= (\nabla p_{1h}, \nabla(b_E\bar{\gamma}_{1E})_{0,\omega_E} + (\Delta p_{1h}, b_E\bar{\gamma}_{1E})_{0,\omega_E} = \\
&= (\nabla(p_{1h} - p_1), \nabla(b_E\bar{\gamma}_{1E}))_{0,\omega_E} + \\
&\quad + (y - z_d - \beta_y'(y,g)p_1 - \phi_y'(y,g)p_2 + \\
&\quad\quad + a\nabla p_1, b_E\bar{\gamma}_{1E})_{0,\omega_E} + I_4 +
\end{aligned}
$$

$$
\begin{aligned}
&+(\beta_y'(y_h,g_h)p_{1h} + \phi_y'(y_h,g_h)p_{2h} + a\nabla p_{1h} - \\
&\quad -(y_h - z_d), b_E\bar{\gamma}_{1E})_{0,\omega_E} + (\Delta p_{1h} - \beta_y'(y_h,g_h)p_{1h} - \\
&\quad -\phi_y'(y_h,g_h)p_{2h} + a\nabla p_{1h} + y_h - z_d, b_E\bar{\gamma}_{1E})_{0,\omega_E} =
\end{aligned}
$$

$$
\begin{aligned}
&= (\nabla(p_{1h} - p_1), \nabla(b_E\bar{\gamma}_{1E}))_{0,\omega_E} + \\
&\quad + (\phi_y'(y_h,g_h)p_{2h} - \phi_y'(y,g)p_2 + y - y_h, b_E\bar{\gamma}_{1E})_{0,\omega_E} \\
&\quad + (\beta_y'(y_h,g_h)p_{1h} - \beta_y'(y,g)p_1 + y - y_h, b_E\bar{\gamma}_{1E})_{0,\omega_E} + \\
&\quad + (\gamma_{1K}, b_E\bar{\gamma}_{1E})_{0,\omega_E} + I_4 = I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

where $I_4 := (\gamma_{1E} - \bar{\gamma}_{1E}, b_E\bar{\gamma}_{1E})_{0,E}$.

Using lemma 3 and lemma 8, we also have the following estimate.

$$
\begin{aligned}
I_1 &\le Ch_K^{-1}\|p_1 - p_{1h}\|_{1,\omega_E} \|b_E\bar{\gamma}_{1E}\|_{0,\omega_E} \le \\
&\le Ch_K^{-1}\|p - p_{1h}\|_{1,\omega_E} \|b_E\|_{0,\omega_E} \bar{\gamma}_{1E} \le \\
&\le Ch_K^{-1/2}\|p_1 - p_{1h}\|_{1,\omega_E} \|\bar{\gamma}_{1E}\|_{0,E}
\end{aligned}
$$

Here $\|b_E\|_{0,\omega_E} \le Ch_K$ is taken into account. Likewise, considering the Lipschitz continuity of $\beta'(\cdot,\cdot)$, $\|\beta''(\cdot)\|_{0,\infty} \le C$, we can deduce the following estimates.

$$
\begin{aligned}
I_2 &\le Ch_K^{-1/2}(\|p_1 - p_{1h}\|_{1,\omega_E} + \\
&\quad + \|y - y_h\|_{1,\omega_E} + \|g - g_h\|_{1,\omega_E})\|\bar{\gamma}_{1E}\|_{0,E}
\end{aligned}
$$

$$I_3 \le \|\gamma_{1K}\|_{0,\omega_E} \|b_E\bar{\gamma}_{1E}\|_{0,\omega_E} \le$$

$$\le \|b_E\|_{0,\omega_E} \|\gamma_{1K}\|_{0,\omega_E} \bar{\gamma}_{1E} \le Ch_K^{1/2} \|\gamma_{1K}\|_{0,\omega_E} \|\bar{\gamma}_{1E}\|_{0,E}$$

$$h_K^{1/2} \|\bar{\gamma}_{1E}\|_{0,E} \le C(\|p_1 - p_{1h}\|_{1,\omega_E} + \|y - y_h\|_{1,\omega_E} + \|g - g_h\|_{1,\omega_E}) + Ch_K \|\gamma_{1K}\|_{0,\omega_E}$$

$$\eta_2^2 = \sum_{E\in E_h} h_E \|\gamma_{1E}\|_{0,E}^2 \le \le C(\|p_1 - p_{1h}\|_1^2 + \|y - y_h\|_1^2 + \|g - g_h\|_1^2) + C\sum_{K\in T_h} h_K^2 \|\gamma_{1K}\|_{0,K}^2 + osc(\gamma_{1E}) = C(\|p_1 - p_{1h}\|_1^2 + \|y - y_h\|_1^2 + \|g - g_h\|_1^2) + \eta_1^2 + osc(\gamma_{1E})$$

Similarly to the estimations of η_1^2, η_2^2, we can get the followings.

$$\eta_3^2 \le \sum_{K\in T_h} h_K^2 \|\gamma_{2K}\|_{0,K}^2 \le \le C(\|y_h - y\|_1^2 + \|g_h - g\|_1^2 + \|p_{1h} - p_1\|_1^2 + \|p_{2h} - p_2\|_1^2 + \|g_h - g\|_1^2) + osc(\gamma_{2K})$$

$$\eta_4^2 = \sum_{E\in E_h} h_E \|\gamma_{2E}\|_{0,E}^2 \le \le C(\|p_1 - p_{1h}\|_1^2 + \|p_2 - p_{2h}\|_1^2 + \|y - y_h\|_1^2 + \|g - g_h\|_1^2) + C\sum_{K\in T_h} h_K^2 \|\gamma_{2K}\|_{0,K}^2 + \sum_{E\in E_h} h_E \|\gamma_{2E} - \bar{\gamma}_{2E}\|_{0,E}^2 = C(\|p_1 - p_{1h}\|_1^2 + \|p_2 - p_{2h}\|_1^2 + \|y - y_h\|_1^2 + \|g - g_h\|_1^2) + \eta_3^2 + osc(\gamma_{2E})$$

$$\eta_5^2 = \sum_{K\in T_h} h_K^2 \|r_{1K}\|_{0,K}^2 \le \le C(\|y - y_h\|_1^2 + \|g - g_h\|_1^2 + \|f - f_h\|_0^2) + \sum_{K\in T_h} h_K^2 \|r_{1K} - \bar{r}_{1K}\|_{0,K}^2 = C(\|y - y_h\|_1^2 + \|g - g_h\|_1^2 + \|f - f_h\|_0^2) + osc(r_{1K})$$

$$\eta_6^2 = \sum_{E\in E_h} h_K \|r_{1E}\|_{0,E}^2 \le \le C(\|y_h - y\|_1^2 + \|g_h - g\|_1^2 + \|f - f_h\|_0^2) + C\sum_{K\in T_h} h_K^2 \|r_{1K}\|_{0,K}^2 + \sum_{E\in E_h} h_K \|r_{1E} - \bar{r}_{1E}\|_{0,E}^2 \le \le C(\|y_h - y\|_1^2 + \|g_h - g\|_1^2 + \|f - f_h\|_0^2) + \eta_5^2 + osc(r_{1E}).$$

$$\eta_7^2 = \sum_{K\in T_h} h_K^2 \|r_{2K}\|_{0,K}^2 \le \le C(\|u - u_h\|_0^2 + \|g - g_h\|_1^2) + \sum_{K\in T_h} h_K^2 \|r_{2K} - \bar{r}_{2K}\|_{0,K}^2 = C(\|u - u_h\|_0^2 + \|g - g_h\|_1^2) + osc(r_{2K}),$$

$$\eta_8^2 = \sum_{E\in E_h} h_K \|r_{2E}\|_{0,E}^2 \le \le C(\|g_h - g\|_1^2 + \|u - u_h\|_0^2) + C\sum_{K\in T_h} h_K^2 \|r_{2K}\|_{0,K}^2 + \sum_{E\in E_h} h_K \|r_{2E} - \bar{r}_{2E}\|_{0,E}^2 \le \le C(\|g_h - g\|_1^2 + \|u - u_h\|_0^2) + \eta_7^2 + osc(r_{2E}).$$

$$\eta_9^2 \le \|z_h\|_0^2 \le C(\|u - u_h\|_0^2 + \|p_2 - p_{2h}\|_1^2) + Ch(\|u\|_0^2 + \|p_2\|_0^2) + \sum_{K\in T_h} \|z_h - \bar{z}_h\|_{0,K}^2 =$$

$$= C(\|u - u_h\|_0^2 + \|p_2 - p_{2h}\|_1^2) + Ch + osc(z_h)$$

$$\eta_{10}^2 \le \|f_h + p_{1h}\|_0^2 \le C(\|f - f_h\|_0^2 + \|p_1 - p_{1h}\|_1^2) + Ch(\|f\|_1^2 + \|p_1\|_1^2) + osc(z_{1h})$$

By summing up $\eta_1^2 \sim \eta_{10}^2$, we can derive the result of this theorem. This completes the proof.

5. Convergence of a Posteriori Error Indicator

[Lemma 9] For osc, oscillation error (Theorem 4), $osc \to 0 \ \ (h \to 0)$

(We shall denote the constants independent of h by the same character C in the proof.)

(Proof) *Step 1*: Let us consider the following inequality.

$$\sum_{K\in T_h} \|\gamma_{2K}\|_{0,K}^2 + \sum_{E\in E_h} \|\gamma_{2E}\|_{0,E}^2 \le C$$

(C is independent of h). (41)

where r_{2K}, r_{2E} are the quantities defined in lemma 7 being

$$r_{2K} := \Delta g_h - \phi(y_h, g_h) + a\nabla g_h$$

$$r_{2E} = \begin{cases} \left[\dfrac{\partial g_h}{\partial n} \right]_E = (\nabla g_h - \nabla \overline{g}_h) \cdot n_K, \ E \cap \Gamma = \varnothing \\ \left. \dfrac{\partial g_h}{\partial n} \right|_E, \ E \cap \Gamma = \varnothing \end{cases}$$

Due to coercivity of objective function J with repect to control u_h, f_h, it follows that u_h, f_h is bounded in $L^2(\Omega)$ independently of h, i.e. $\|u_h\|_0 = \|u_h\|_{L^2(\Omega)} \le C, \| f_h\|_0 = \| f_h\|_{L^2(\Omega)} \le C$ (C is constant independent of h).

Letting $v_h = g_h$ in the variational equation

$$\begin{aligned} &(\nabla g_h, \nabla \upsilon_h)_0 + (\phi(y_h, g_h), \upsilon_h)_0 + \\ &+ (a\nabla g_h, \upsilon_h)_0 = (u_h, \upsilon_h)_0, \forall \upsilon_h \in V_h \end{aligned} \tag{42}$$

satisfied by $g_h \in V_h$, and by using assumption 1, lemma 5 and $(\nabla g_h, \nabla g_h)_0 + (a\nabla g_h, g_h)_0 \ge \overline{C}\|g_h\|_1^2$, then it follows

$$\|g_h\|_0 \le \|g_h\|_1 \le C\|u_h\|_0 + \|\phi(y_h, g_h)\|_{0,\infty} \le \widehat{C}$$

From (42), we have

$$\begin{aligned} &\sum_{K\in T_h} (\Delta g_h, \upsilon_h)_{0,K} + \sum_{E\in E_h} (-r_{2E}, \upsilon_h)_{0,E} \le \\ &\le (\|\phi(y_h, g_h)\|_{0,\infty} + \|u_h\|_0)\|\upsilon_h\|_0 \le C\|\upsilon_h\|_0 \end{aligned}$$

Let us introduce norm in V_h by $\|v_h\|_h := (\sum_{K\in T_h} \|v_h\|_{0,K}^2 + \sum_{E\in E_h} \|v_h\|_{0,E}^2)^{1/2}$. Because all finite dimensional norms are equivalent, it follows

$$C_1\|v_h\|_h \le \|v_h\|_0 \le C_2\|v_h\|_h$$

Let us consider the functional

$$F(\mathrm{g}_h)(v_h) := \sum_{K\in T_h} (\Delta \mathrm{g}_h, v_h)_{0,K} + \sum_{E\in E_h} (-\mathrm{r}_{2E}, v_h)_{0,\mathrm{E}}$$

in v_h.

Considering the fact $|F(\mathrm{y}_h)(\mathrm{v}_h)| \le C\|v_h\|_0 \le \tilde{C}\|v_h\|_h$, $F(\mathrm{y}_h)$ is bounded linear functional in V_h, and from Riesz's representation theorem, there is a unique $a_h \in V_h$ such that $F(\mathrm{y}_h)(v_h) = (a_h, v_h)_h$, and it follows

$$\|F(\mathrm{y}_h)\| = \|a_h\|_h$$

$$\|a_h\|_h = (\sum_{K\in T_h} \|\Delta g_h\|_{0,K}^2 + \sum_{E\in E_h} \|r_{2E}\|_{0,\mathrm{E}}^2) \le \tilde{C}$$

Recalling assumption 1 and lemma 5, $\|\phi(y_h, g_h)\|_{0,\infty} \le C$ (C is independent of h), so we have the following result.

$$\sum_{K\in T_h} \|r_{2K}\|_{0,K}^2 := \sum_{K\in T_h} \|\Delta g_h - \phi(y_h, g_h) - a\nabla g_h + u_h\|_{0,K}^2 \le$$

$$\le 2\sum_{K\in T_h} \|\Delta g_h\|_{0,K}^2 + 2\|\phi(y_h, g_h) + a\nabla g_h - u_h\|_0^2 \le C$$

Step 2:

$$\sum_{K\in T_h} \|\gamma_{1K}\|_{0,K}^2 + \sum_{E\in E_h} \|\gamma_1\|_{0,E}^2 \le C \tag{43}$$

$$\sum_{K\in T_h} \|\gamma_{1K}\|_{0,K}^2 + \sum_{E\in E_h} \|\gamma_{1E}\|_{0,E}^2 \le C \tag{44}$$

$$\sum_{K\in T_h} \|\gamma_{2K}\|_{0,K}^2 + \sum_{E\in E_h} \|\gamma_{2E}\|_{0,E}^2 \le C \tag{45}$$

where C is constant independent of h.

By summing up (43)~(45), and using the definition of osc, we can obtain the result of this theorem. This completes the proof.

Now we can state the convergence of a posteriori error indicator η.

[Theorem 5]

Let (f, u, y, g, p_i), $(f_h, u_h, y_h, g_h, p_{ih})(i = 1,2)$ be the solution of (3), (18), respectively. We have the following convergence results when $\mathrm{h} \to 0$.

1) $\|u - u_h\|_0 \to 0, \ \|f - f_h\|_0 \to 0, \ \|y - y_h\|_1 \to 0$

2) $\|g - g_h\|_1 \to 0, \ \|p_i - p_{1h}\|_1 \to 0 \ \ (i = 1,2)$

3) $\eta^2 := \sum_{i=1}^{10} \eta_i^2 \to 0$

(Proof) We can prove 1), 2) from theorem 2, and then 3) from lemma 9, theorem 2 and theorem 4.

6. Conclusion

In this paper, for the optimal source control problem of a system of 2-dimensional semi-linear steady convection-diffusion equations, we acquired the optimality condition. And we estimated the a priori and a posteriori error and proved the convergence to 0 of a posteriori error indicator when division diameter converges to 0.

Abbreviations

FEM	Finite Element Method
FVM	Finite Volume Method

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Conflicts of Interest

The authors declare no conflicts of interest.

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