



## Research Article

# Stably Properly Infinite and Purely Infinite Real $C^*$ -algebras

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## Abstract

This paper considers real  $C^*$ - and  $AW^*$ -algebras with quasitrace and explores their connection with the concept of stable finiteness, stable proper infiniteness and pure infiniteness. A review of existing results for complex algebras and their real analogues is presented, including the role of quasitrace in concept of finiteness and infiniteness. It is proved that a real  $C^*$ -algebra is stably properly infinite if and only if its complexification has the same property. The concept of quasitrace allows us to classify  $C^*$ -algebras as finite or infinite. If the quasitrace is trivial, such an algebra is called traceless or stably properly infinite. There are known results such as a connection between traceless and weakly purely infinite complex  $C^*$ -algebras and equivalence of all definitions of pure infiniteness under condition of real rank zero, so a natural question arises: how are traceless  $C^*$ -algebras related to infinite ones in the real case and if all definitions of pure infiniteness under condition of real rank zero coincide in real case? In this paper, we obtain the following results: a real analogue of the Cuntz-Blackadar-Handelman theorem, build the connection between weakly purely infinite and traceless real  $C^*$ -algebras through their enveloping complex  $C^*$ -algebras, and we give Stacey's theorem without the assumption simplicity through the class of stably properly infinite (traceless) real  $C^*$ -algebras. It would be reasonable to consider  $AW^*$ -algebras, since we consider real rank zero  $C^*$ -algebras as the class  $AW^*$ -algebras lies in the class of real rank zero  $C^*$ -algebras.

## Keywords

Real  $C^*$ -algebras, Proper Infiniteness, Pure Infiniteness, Quasitrace

## 1. Introduction

It is known that every stably finite  $C^*$ -algebra has a quasitrace [4], but the converse is true only for a simple  $C^*$ -algebra [5], that is, one without nontrivial two-sided ideals. Cuntz, Blackadar, and Handelman proved a more general result: a  $C^*$ -algebra has a quasitrace if its quotient algebra is stably finite [13], or, equivalently, if it is not stably properly infinite. The notion of 2-quasitrace allows one to classify  $C^*$ -algebras as stably finite or infinite. If the quasitrace is nontrivial, this allows one to conclude that the algebra is finite or stably finite. If the 2-quasitrace is trivial, such an algebra is called traceless

[14]. A natural question arises: how are traceless  $C^*$ -algebras related to infinite ones? There are four different definitions of infiniteness: locally pure, weakly pure, pure, and strongly pure infinite [6, 17, 18]. In particular, it was proved that a  $C^*$ -algebra is weakly purely infinite if and only if it is traceless [17]. In the paper [6], it was shown that under the condition of real rank zero the definitions of weak pure and pure infiniteness for  $C^*$ -algebras coincide. All the above-mentioned results relate to the complex case, so their adaptation to the real case is of particular interest. In the paper, Stacey proposed an analogue of the

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concept of pure infiniteness for real algebras. A similar result was obtained by Boersema [7]. However, in both cases the transfer of the properties of pure infiniteness from a real C\*-algebra to its complexification was carried out under the condition of simplicity of the latter. In the present paper, the following results will be obtained: a real analogue of the Cuntz-Blackadar-Handelman theorem, established the relation between real weakly pure infinite and traceless algebras via their enveloping algebras, considered Stacey's theorem without the assumption of simplicity of an algebra via the class of stably properly infinite (traceless) real C\*-algebras.

## 2. Purely Infinite Real C\*-algebras

### 2.1. Preliminaries

Let  $A$  be a Banach \*-algebra over a field  $R$ . An algebra  $A$  is called a C\*-algebra if  $\|x^*x\| = \|x\|^2$  for any  $x \in A$ . A real Banach \*-algebra  $R$  is called a real C\*-algebra if the norm on  $R$  can be extended to the complexification  $A = R + iR$  of  $R$  so that  $A$  is a (complex) C\*-algebra. We say that a C\*-algebra  $A$  is finite if for  $x \in A$ ,  $x^*x = \mathbb{1}$  implies  $xx^* = \mathbb{1}$ .  $A$  is called stably finite if  $M_n(A)$  is finite for all  $n \geq 1$ . Finiteness and stable finiteness in the real cases are defined similarly. Let  $A$  be a ring and  $S$  a non-empty subset of  $A$ , denote by  $R(S) = \{x \in A : sx = 0 \mid \forall s \in S\}$  and call  $R(S)$  the right annihilator of  $S$ . Similarly,  $L(S) = \{x \in A : xs = 0 \mid \forall s \in S\}$  denotes the left annihilator of  $S$ . A Baer \*-ring is a ring  $A$  such that for every non-empty subset  $S$  of  $A$ ,  $R(S) = gA$  for some projection  $g$ . An AW\*-algebra is a (real or complex) C\*-algebra that is a Baer \*-ring [1-3].

If  $S$  is an abelian semigroup, then there is a universal enveloping abelian group  $G(S)$ , called the Grothendieck group of  $S$ . There is a canonical homomorphism from  $S$  to  $G(S)$  that sends  $x$  to  $[(x + x, x)]$ . This homomorphism is injective if and only if  $S$  has a cancellation, i.e., one can omit the element  $z$  in the equality  $x_1 + y_2 + z = x_2 + y_1 + z$ . A strong identity in  $S$  is an element  $u > 0$  such that for any  $x \in G$  there is a positive integer  $n$  such that  $x \leq nu$ . Let  $S$  be an ordered abelian semigroup with strong identity  $u$ . If  $f$  is any monotone (i.e., order-preserving) homomorphism  $f: S \rightarrow \mathbb{R}$  such that  $f(u) = 1$ , we call  $f$  a (normalized) functional  $(S, u)$ . If, in addition,  $f(x) > 0$  for all  $x > 0$ , we call  $f$  a strict functional [9]. Recall that Grothendieck groups and states on them are inextricably linked with the condition of existence of a quasitrace in the complex and real cases. A C\*-subalgebra  $B$  of a C\*-algebra  $A$  is called hereditary if, for a positive element  $a$  of  $A$  and a positive element  $b$  of  $B$ , the inequality  $a \leq b$  implies that  $a$  belongs to  $B$ . Recall that a C\*-algebra  $A$  filter on a set  $X$  is a nonempty family  $\omega \subset 2^X$  satisfying three conditions:  $\emptyset \in \omega$ ; if  $A, B \in \omega$ , then  $A \cap B \in \omega$  (and hence any finite intersection of elements of  $\omega$  belongs to  $\omega$ ); If  $A \in \omega$  and  $A \subset B$ , then  $B \in \omega$ . In particular,  $X \in \omega$ . A filter  $\omega$  is called free if  $\cap \omega = \emptyset$ . A maximal (by inclusion)

filter is called an ultrafilter, that is, a filter  $\omega$  is an ultrafilter if for any filter  $\omega \subset \omega'$  we have  $\omega' = \omega$ .

**Definition 1.** [4, 11] Let  $A$  be a C\*-algebra, and the quasitrace on  $A$  is a function  $\tau: A \rightarrow \mathbb{C}$  that satisfies:

- 1)  $\tau(x^*x) = \tau(xx^*) \geq 0$  for all  $x \in A$ .
- 2)  $\tau(a + ib) = \tau(a) + i\tau(b)$  for  $a, b \in A_h$ .
- 3)  $\tau$  is linear on every abelian C\*-subalgebra  $B$  of  $A$ .

**Definition 2.** [12, 15] Let  $R$  be a real C\*-algebra with unity. A quasitrace  $\tau$  on  $R$  is a function  $\tau: R \rightarrow \mathbb{R}$  satisfying the following conditions:

- 1)  $\tau(x^*x) = \tau(xx^*) \geq 0$ ,  $x \in R$ ;
- 2)  $\tau(a + b) = \tau(a)$ , для  $a \in R_n, b \in R_k$ , i.e.,  $b^* = -b$ ;
- 3)  $\tau$  is linear on every abelian C\*-subalgebra  $B$  of  $R$ .

**Theorem 1.** A real C\*-algebra  $R$  is stably finite if  $A = R + iR$  is stably finite.

Recall that a C\*-algebra with a unity  $A$  is called properly infinite if it contains two mutually orthogonal projections, both equivalent with respect to  $\mathbb{1}$ . We say that  $A$  is stably properly infinite if  $M_n(A)$  is properly infinite for all  $n \geq 1$ .

**Theorem 2.** If a real C\*-algebra  $R$  is stably properly infinite, then  $A = R + iR$  is stably properly infinite.

Proofs in one direction is quite obvious, nevertheless converse proofs are quite difficult to obtain purely algebraically, so to simplify the proof, we apply the quasitrace. It is worth noting that in the properly infinite case, it is trivial. Accordingly, it is necessary to consider under what conditions the quasitrace is nontrivial in the real C\*-algebra.

### 2.2. Stably Not Properly Infinite Real C\*-algebras

Let  $R \subseteq B(H)$  be a separable real C\*-algebra with a unity, where  $H$  is a separable Hilbert space. Let  $M_\infty(R)$  be the \*-algebra consisting of all finite-dimensional square matrices over elements in  $R$ , i.e., we set  $M_\infty(R) = \bigcup_{n \in \mathbb{N}} M_n(R)$ . Consider the order  $\lesssim$  on  $M_\infty(R)$ , defined as  $a \lesssim b$ , if and only if there exists a sequence  $(v_n)_{n \in \mathbb{N}} \subset M_\infty(R)$  such that  $v_n^* b v_n \rightarrow a$  for  $n \rightarrow \infty$ . We set  $a \sim b$  if  $a \lesssim b$  and  $b \lesssim a$ , and we say that  $a$  is Cuntz equivalent to  $b$ , and we denote the equivalence class of  $a$  by  $\langle a \rangle$ . We set  $W(R) = M_\infty(R)_+ / \sim$  and equip it with the order  $\langle a \rangle \geq \langle b \rangle$  if  $a \lesssim b$ , and with the addition  $\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle$  such that  $W(R)$  is an ordered abelian semigroup. If we apply the same construction to the stabilization of  $R \otimes K(H)$  instead of the precompletion  $M_\infty(R)$ , we obtain the so-called full Cuntz semigroup, which appears explicitly as the quotient  $Cu(R) = (R \otimes K(H)) / \sim$  [14]. We formulate the definitions in terms of ordered abelian semigroups for the sake of generality. If  $S$  is an ordered abelian semigroup and  $x, y$  in  $S$ , we denote that  $x <_s y$  if there exists  $n \in \mathbb{N}$  such that  $(n+1)x \leq ny$ .

**Definition 3.** Let  $S$  be an ordered abelian semigroup, and let  $n \in \mathbb{N} \cup \{0\}$ . We say that  $S$  has  $n$ -comparison if for any  $x, y_0, \dots, y_n \in S$  satisfying  $x <_s y_j$  for all  $j = 0, \dots, n$ , we have  $x \leq y_0 + \dots + y_n$ .

Obviously,  $n$ -comparison implies  $m$ -comparison for  $n \leq m$ . Note that 0-comparison is equivalent to the property that  $(n+1)x \leq ny$  for some  $n \in \mathbb{N}$  implies  $x \leq y$ , which is also known as almost unperforation. If  $R$  is a real  $C^*$ -algebra, then  $W(R)$  is almost unperforated if and only if  $Cu(R)$  is almost unperforated [14], and it is easy to see that almost unperforation carries over to ideals and quotients.

It is worth noting that the definition of projection types in  $C^*$ -algebras is similar to that for  $AW^*$ -algebras.

**Proposition 1.** Let  $R$  be a real  $C^*$ -algebra and  $p \in R$  be a nonzero projection. The following conditions are equivalent:

- 1)  $p$  is properly infinite;
- 2)  $p \oplus p \precsim p$ ;

Proofs for the above assertions are similar to those in the complex case [6, 7].

**Definition 4.** Let  $S$  be an ordered abelian semigroup. A state on  $S$  is an order-preserving mapping  $f: S \rightarrow \mathbb{R}$ . The set of states on  $S$  is denoted by  $\Sigma(S)$ . If  $t \in S$ , then we denote by  $\Sigma(S, t)$  the set of states  $f \in \Sigma(S)$  satisfying  $f(t) = 1$ .

**Proposition 2.** [5] Let  $S$  be an ordered abelian semigroup with strong unity  $u$ , and let  $H$  be a subgroup of  $S$  that contains  $u$ . If  $f$  is any functional on  $(H, u)$ , then  $f$  extends to a functional on  $(G, u)$ .

It is worth noting that Definition 4 and Proposition 4 apply equally in the real and complex cases.

Let  $R$  be a real  $C^*$ -algebra with a unity. A state  $d$  on  $Cu(R)$   $d(\mathbb{I}_A) = 1$  is called a dimension function on  $R$ . Therefore, dimension functions are states on Cuntz semigroups with the obvious normalization assumption. For every positive element  $a \in M_\infty(R)_+$  and  $\epsilon > 0$ , we define the  $\epsilon$ -cutoff of  $a$  as  $(a - \epsilon)_+ = f_\epsilon(a)$ , where the right-hand side is determined by applying continuous functional calculus to the function  $f_\epsilon(t) = \max\{0, t - \epsilon\}$ . A dimension function  $d$  on  $R$  is lower semicontinuous if for all  $a \in M_\infty(A)_+$ ,  $d((a - \epsilon)_+) \rightarrow d(\langle a \rangle)$  for  $\epsilon \rightarrow 0$  [17]. Denote by  $DF(R)$  the dimension functions on  $R$ , and by  $LDF(R)$  the lower semicontinuous dimension functions on  $R$ . Given the quasitrace  $\tau$  on the real  $C^*$ -algebra  $R$ , one can define a lower semicontinuous dimension function  $d$  on  $R$  by  $d_\tau(\langle a \rangle) = \lim_{n \rightarrow \infty} (a^{1/n})$ . This correspondence is in fact an equivalence, as shown by Blackadar and Handelman [4] in the following theorem.

**Theorem 3.** Let  $A$  be a  $C^*$ -algebra. Then there is an affine bijection between the spaces  $QT(A)$  and  $LDF(A)$ .

Since the definitions of the quasitrace and the dimension function are given in the real case, it is reasonable to conjecture that the equivalence established by Blackadar and Handelman also applies in the real case. We now prove the real analog of the Cuntz-Blackadar-Handelman theorem (see [13] Theorem 2.3).

**Theorem 4.** Let  $R$  be a real  $C^*$ -algebra with a unity. If  $R$  is not stably properly infinite then  $R$  admits a quasitrace.

**Proof.** Suppose that  $R$  is not stably properly infinite and  $K_0(R)$  does not admit nonzero states. Let  $u = [1_R]_0$  be the equivalence class of the identity  $1_R$  in  $K_0(R)$ . First, we prove

the existence of  $k, l \in \mathbb{N}$  with  $k > l$  and  $ku \leq lu$ . Suppose that  $ku \leq lu$  implies  $k \leq l$ . Using Proposition 2, the function  $f: Z_u \rightarrow R$  by  $f(nu) = n$  for  $n \in \mathbb{Z}$  extends to a state on  $K_0(R)$ , which contradicts our assumption. Therefore, there exist some  $k, l \in \mathbb{N}$  with  $k > l$  and  $ku \leq lu$ .

Next, let  $e_n \in M_n(R)$  be the identity in  $M_n(R)$ . Since  $[e_n]_0 = nu$  for any  $n \in \mathbb{N}$ , it immediately follows that the inequality  $[e_k]_0 \leq [e_l]_0$  holds. We choose  $m \in \mathbb{N}$  such that  $e_k \oplus e_m \precsim e_l \oplus e_m$  and set  $n = l + m$  and  $d = k - l > 0$ . Then  $e_{n+d} \precsim e_n$ , and the iterative process will show that  $e_n$  is in fact properly infinite. Thus when  $R$  is not stably properly infinite,  $K_0(R)$  does admit nonzero states.

Based on the Theorem 2 and Theorem 4, we can assert the following corollary.

**Corollary 1.** Let  $R$  be a real  $C^*$ -algebra with a unity.  $R$  admits a quasitrace if and only if  $R$  is not stably properly infinite.

## 2.3. Purely Infinite Real $C^*$ -algebras

Stably properly infinite  $C^*$ -algebras are of particular interest because they simplify the consideration of certain types of purely infinite  $C^*$ -algebras. There are several formulations of purely infinite  $C^*$ -algebras. We will consider them all in order from weakest to strongest.

**Definition 5** [6]. A  $C^*$ -algebra  $A$  is called

- 1) *locally purely infinite* if and only if for every primitive ideal  $J$  of  $A$  and every element  $b \in A_+$  with  $\|b + J\| > 0$ , there exists a stable  $C^*$ -subalgebra  $D$  of the hereditary  $C^*$ -subalgebra generated by  $b$  such that  $D$  is not contained in  $J$ .
- 2) For a given strictly positive integer  $m$ , a  $C^*$ -algebra  $A$  is said to be  *$m$ -purely infinite* if and only if, for every pair of positive elements  $a, b$  in  $A$  such that  $b$  lies in the closed two-sided ideal  $A$  generated by  $a$ , and for every  $\epsilon > 0$ , there exists  $d_1, \dots, d_m$  in  $A$  such that  $\|b - (d_1^* a d_1 + \dots + d_m^* a d_m)\| < \epsilon$ , and there is no nonzero quotient algebra  $l^\infty(A)$  of dimension  $\leq m^2$ . We say that  $A$  is *weakly purely infinite* if  $A$  is  $m$ -purely infinite for some  $m \in \mathbb{N}$ .
- 3) *purely infinite* if and only if for every pair of positive elements  $a, b \in A_+ \setminus \{0\}$  such that  $b$  lies in the closed two-sided ideal  $\overline{\text{span}(AaA)}$  generated by  $a$ , and for every  $\epsilon > 0$ , there exists an element  $d \in A$  such that  $\|b - d^* a d\| < \epsilon$ , and there is no nonzero character on  $A$ .
- 4) *strongly purely infinite* if and only if for any  $a, b \in A_+ \setminus \{0\}$ ,  $\epsilon > 0$ , there exist elements  $s, t \in A$  such that  $\|a^2 - s^* a^2 s\| < \epsilon$ ,  $\|b^2 - t^* b^2 t\| < \epsilon$  and  $\|s^* a b t\| < \epsilon$ .

It should be noted that these definitions are given similarly in the real case under the regularity condition [7, 18]. However, it is rather difficult to connect the real and complex cases through these definitions, so we consider an analogue using

traceless  $C^*$ -algebras. Traceless  $C^*$ -algebras provide a different approach to considering various classes of infinite  $C^*$ -algebras.

**Proposition 5 [15].** Let  $A$  be a  $C^*$ -algebra with a unity. Then  $A$  does not admit a normalized 2-quasitrace if and only if  $M_n(A)$  is properly infinite for some  $n \in \mathbb{N}$ .

We say that  $A$  is *traceless* if every lower semicontinuous nonnegative 2-quasitrace on  $A_+$  is trivial, i.e., takes only the values 0 and  $+\infty$ .

However, traceless  $C^*$ -algebras are not always properly infinite or purely infinite, since there is an example of a finite  $C^*$ -algebra over which the matrix is properly infinite (see Proposition 5.6 in [17]). The following theorem establishes a connection between weakly purely infinite and traceless  $C^*$ -algebras.

**Theorem 5 [18].** Let  $A$  be a  $C^*$ -algebra, and  $\omega$  a free ultrafilter on  $\mathbb{N}$ .

Let  $A_\omega$  be the ultraproduct of  $A$ . Then the following statements are equivalent.

- 1)  $A_\omega$  is traceless, i.e., every lower semicontinuous 2-quasitrace on  $A_\omega$  is trivial;
- 2)  $A_\omega$  is weakly purely infinite;
- 3)  $A$  is weakly purely infinite;

In the following theorem, we give a partial real analogue of the above theorem.

**Theorem 6.** Let  $R$  be a real  $C^*$ -algebra. Then  $R$  is traceless if and only if  $R$  is weakly purely infinite.

*Proof.* Let  $R$  be traceless, then by Corollary 1  $A = R + iR$  is traceless, which implies by Theorem 3 that  $A = R + iR$  is weakly purely infinite. Since  $A = R + iR$  is weakly purely infinite, it follows that (see Proposition 2.2 in [18])  $R$  is weakly purely infinite. Conversely, let  $R$  be weakly purely infinite, but its complexification  $A = R + iR$  is not weakly purely infinite. Then  $A = R + iR$  is not traceless, and hence, by Theorem 4, it has a nontrivial quasitrace. A contradiction.

Thus, we have proved that a real stably properly infinite  $C^*$ -algebra is weakly purely infinite.

Of great interest is the condition under which all four definitions of pure infinity coincide. To this end, we consider a special class of  $C^*$ -algebras with real rank zero. The following theorem, obtained by E. Kirchberg, proves that all the above definitions of pure infiniteness coincide under the condition that the  $C^*$ -algebra has real rank zero.

**Theorem 7 [6].** Let  $A$  be a  $C^*$ -algebra of real rank zero. Then  $A$  is locally purely infinite if and only if  $A$  is strongly purely infinite.

This theorem yields an interesting corollary.

**Corollary 2.** A traceless  $C^*$ -algebra  $A$  with real rank zero is purely infinite as a  $C^*$ -algebra.

*Proof.* By Theorem 3,  $A$  is traceless if and only if it is weakly purely infinite. Consequently, if  $A$  has real rank zero and is weakly purely infinite, then by Theorem 5  $A$  is purely infinite.

This corollary relates stably properly infinite and purely infinite  $C^*$ -algebras.

*Remark.* The example of a traceless  $C^*$ -algebra mentioned in (see Proposition 5.6 in [17]) does not have real rank zero, (see Theorem 4.3 in [6]).

We are particularly interested in Stacey's result on real purely infinite  $C^*$ -algebras.

**Theorem 8 [18].** Let  $A = R + iR$  be a purely infinite  $C^*$ -algebra, then  $R$  is a purely infinite real  $C^*$ -algebra.

The theorem does not require  $A$  to be simple, but the converse result was obtained with  $A$  being simple [7]. We will consider the converse theorem without the simplicity condition. An obvious candidate is the real weakly purely infinite  $C^*$ -algebra from Theorem 4. We will also give the result under the condition that  $R$  has real rank zero, which implies that  $A = R + iR$  has real rank zero (see Proposition 2.2 in [19]).

**Theorem 9.** Let  $R$  be a weakly purely infinite  $C^*$ -algebra with real rank zero. Then  $A = R + iR$  is purely infinite.

*Proof.* By Corollary 1,  $R$  is stably properly infinite if and only if  $A = R + iR$  is stably properly infinite. Consequently, by Proposition 1,  $A = R + iR$  is traceless, and by Corollary 2,  $A = R + iR$  is pure infinite.

Thus, we have proved that  $A = R + iR$  is (weakly) pure infinite if and only if  $R$  is (weakly) purely infinite, using known and proper analogs of some theorems in the real case.

Since we consider real rank zero  $C^*$ -algebras, it is appropriate to consider  $AW^*$ -algebras [8, 10].

**Theorem 10.** Let  $A$  be a traceless  $C^*$ -algebra. Then, if  $A$  is an  $AW^*$ -algebra, then it is of type III  $AW^*$ -algebra.

*Proof.* Since  $A$  is an  $AW^*$ -algebra, Theorem 5 implies that weak pure infiniteness is equivalent to pure infiniteness. By the definition of purely infinite  $C^*$ -algebras,  $A$  does not contain nonzero finite projections. Therefore,  $A$  is an algebra of type III as an  $AW^*$ -algebra. The theorem is proved.

Furthermore, we give a corollary of the above theorem in the real case, based on Theorem 9 and (see Theorem 4.7.4 [1]).

**Corollary 3.** Let  $R$  be a traceless real  $C^*$ -algebra. Then, if  $R$  is a real  $AW^*$ -algebra, then it is of type III real  $AW^*$ -algebra.

### 3. Materials and Methods

To obtain the results, methods of operator algebras were used, as well as the method of extension of quasitrace from real  $C^*$ -algebra to its complexification.

### 4. Results

A real analogue of the Cuntz-Blackadar-Handelman theorem is obtained, investigated the connection between real weakly pure infinite and traceless algebras via their enveloping algebras, considered Stacey's theorem without the assumption of simplicity of an algebra via the class of stably properly infinite (traceless) real  $C^*$ -algebras.

## 5. Discussion

Four different definitions of infiniteness are considered: locally pure, weakly pure, pure, and strongly pure infinite. In particular, it was proved that a  $C^*$ -algebra is weakly purely infinite if and only if it is traceless. It was shown that under the condition of real rank zero the definitions of weak pure and pure infiniteness for  $C^*$ -algebras coincide. All the above-mentioned results related to the complex case, so their adaptation to the real case is of particular interest. Stacey and Boersema proposed an analogue of the concept of pure infiniteness for real algebras. However, in both cases the transfer of the properties of pure infiniteness from a real  $C^*$ -algebra to its complexification was carried out under the condition of simplicity of the latter.

## 6. Conclusions

Every stably finite  $C^*$ -algebra has a quasitrace, but the converse is true only for a simple  $C^*$ -algebra. Cuntz, Blackadar, and Handelman proved a more general result: a  $C^*$ -algebra has a quasitrace if its quotient algebra is stably finite, or, equivalently, if it is not stably properly infinite. The notion of quasitrace allows one to classify  $C^*$ -algebras as stably finite or infinite. If the quasitrace is nontrivial, this allows one to conclude that the algebra is finite or stably finite. If the 2-quasitrace is trivial, such an algebra is called traceless. A natural question arises: how are traceless  $C^*$ -algebras related to infinite ones? Since the class of real  $C^*$ -algebras is wider than complex counterparts, the results are new and do not have any analogues.

## Abbreviations

|          |                                                                               |
|----------|-------------------------------------------------------------------------------|
| $B(H)$   | Algebra of All Bounded Linear Operators Acting on a Complex Hilbert Space $H$ |
| $M_n(A)$ | Algebra of All $n \times n$ Matrices over $A$                                 |

## Author Contributions

**Kim Dmitriy:** Conceptualization, Supervision, Writing – original draft, Data curation, Investigation, Validation, Formal Analysis, Methodology, Writing – review & editing

## Conflicts of Interest

The author declares no conflicts of interest.

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## Research Field

**Kim Dmitriy:** Functional analysis, Theory of operator algebras, Theory of unbounded operators, Topology, K-theory