Engineering Mathematics
2026; 10(1): 1-13
http://www.sciencepublishinggroup.com/j/engmath
doi: 10.11648/j.engmath.20261001.11

Existence and Uniqueness of a Weak Solution of a Time Fractional Reaction-Diffusion System of Fitzhugh-Nagumo Type

Yong-Dok Han*, **Kang-Song Yun**, **Chol-Gwang Kim**

Faculty of Mathematics, University of Sciences, Pyongyang, DPR Korea

Email address:
han631008@star-co.net.kp (Yong-Dok Han), yun991213@star-co.net.kp (Kang-Song Yun), kim010609@star-co.net.kp (Chol-Gwang Kim)
*Corresponding author

To cite this article:
Yong-Dok Han, Kang-Song Yun, Chol-Gwang Kim. (2026). Existence and Uniqueness of a Weak Solution of a Time Fractional Reaction-Diffusion System of Fitzhugh-Nagumo Type. *Engineering Mathematics, 10*(1), 1-13.
https://doi.org/10.11648/j.engmath.20261001.11

Received: 28 January 2025; **Accepted:** 12 August 2025; **Published:** 23 January 2026

Abstract: In the mathematical theory of nerve impulse propagation, the Fitzhugh-Nagumo Reaction-Diffusion System has attracted a great deal of attention. The Fitzhugh-Nagumo Reaction-Diffusion System provides a prototype for chemical and other nerve conduction and biological systems. In this paper, we define two types of weak solutions of the time fractional Fitzhugh-Nagumo Reaction-Diffusion System, namely (1) -weak solutions and (2) -weak solutions, and demonstrate the existence and uniqueness of these weak solutions. First, we have obtained a generalization of [1, Lemma 1] in Lemma 2.1 and using Lemma 2.1 and Galerkin's approximation sequence, we have found the existence of (1)-weak solutions and (2)-weak solutions. We also obtained a generalization of the result of [10, Lemma 6] to Hilbert spaces in Lemma 2.2, and using this result we proved the uniqueness of the (2)-weak solution. Lemma 2.1 and Lemma 2.2 of this paper are results that can be effectively used to show the existence and uniqueness of weak solutions of time fractional partial differential equations. And the existence and uniqueness results of the weak solution of the time fractional Fitzhugh-Nagumo Reaction-Diffusion System can be used in the numerical solutions of this reaction-diffusion system. Also, we can be used in the optimal control problems described in this system.

Keywords: Time-Fractional Reaction-Diffusion System, Existence, Uniqueness

1. Introduction

Fitzhugh-Nagumo Reaction-Diffusion System(FNRDS) is an equation describing the neuron signal propagation, which have studied in [2, 8, 9, 11, 14].

In [2], numerical calculations algorithm for fractional order FNRDS(neglecting diffusion term) and examples of numerical calculations for different fractional orders were studied.

The stability of a solution about the initial value problem of first order FNRDS(neglecting diffusion term) was studied in [8].

In [9], Spectrum and stability of travelling pulses in a coupled FNRDS were studied.

In [11], the numerical calculation algorithm of first order FNRDS (neglecting diffusion term) and the solution behavior by numerical calculation were studied.

In [14], the bifurcation of initial boundary value problem for first order FNRDS in R^1 was studied.

The existence and uniqueness of a weak solution of a time fractional linear diffusion equation were studied in [3], and the existence and uniqueness of a weak solution for a time fractional Navier-Stokes system were studied in [15].

In this paper, we consider weak solution of the following initial boundary value problem of the time fractional FNRDS in the bounded domain $\Omega \subset R^n (n = 1, 2, 3)$. Here, we assume that the boundary Γ is smooth enough:

$$\frac{\partial^\alpha y_1}{\partial t^\alpha} - \Delta y_1 + a_1 y_1^3 - a_2 y_1 + a_3 y_2 = f_1(t, x) \text{ in } Q = (0, T) \times \Omega, \quad (1)$$

$$\frac{\partial^\alpha y_2}{\partial t^\alpha} + b_1 y_2 - b_2 y_1 = f_2(t, x) \text{ in } Q = (0, T) \times \Omega, \quad (2)$$

$$\frac{\partial y_1}{\partial \nu}=0,\ \frac{\partial y_2}{\partial \nu}=0 \text{ on } \Sigma=(0,T)\times\Gamma, \tag{3}$$

$$y_1(0,x)=u_1(x),\ \ y_2(0,x)=u_2(x) \text{ in } \Omega, \tag{4}$$

where ν is a outward unit normal on Γ, $\frac{\partial y_i}{\partial \nu}=\nabla y_i\cdot\nu\ (i=1,2)$ is a normal derivative on Γ, $\frac{\partial^\alpha y_i}{\partial t^\alpha}$ (or ${}_0^cD_t^\alpha y_i$) is the Caputo Fractional Derivative(CFD) of order $\alpha\in(\frac{1}{2},1)$. Also, we assume that $f_1,f_2\in L^2(0,T;L^2(\Omega))$ and $u_1,u_2\in L^2(\Omega)$.

We organized this paper as follows.

First, we present some preliminaries concepts and definitions that will be used in subsequent sections in section 2.

Next, we demonstrate the existence of the weak solution of the time fractional FNRDS in section 3. Finally, we prove the uniqueness of the weak solutions in section 4. The main result is as follows in this paper:

Definition 1.1.

$$y_1\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))\cap C_w(0,T;L^2(\Omega)),$$

$$y_2\in L^\infty(0,T;L^2(\Omega))\cap C_w(0,T;L^2(\Omega))$$

$$(\frac{\partial^\alpha y_1}{\partial t^\alpha}\in L^2(0,T;H^1(\Omega)^*)+L^{\frac{4}{3}}(Q),$$

$$\frac{\partial^\alpha y_2}{\partial t^\alpha}\in L^2(0,T;L^2(\Omega)))$$

are called (1)-weak solutions of the initial boundary value problem (1)-(4), if for any $z_1\in H^1(\Omega),\ z_2\in L^2(\Omega)$ and almost all $t\in(0,T)$, the equalities

$$\left\langle\frac{\partial^\alpha y_1}{\partial t^\alpha},z_1\right\rangle+\int_\Omega\left(\nabla y_1\nabla z_1+a_1y_1^3z_1-a_2y_1z_1+a_3y_2z_1\right)dx=\int_\Omega f_1z_1dx, \tag{5}$$

$$\int_\Omega\left(\frac{\partial^\alpha y_2}{\partial t^\alpha}z_2+b_1y_2z_2-b_2y_1z_2\right)dx=\int_\Omega f_2z_2dx, \tag{6}$$

$$y_1(0,x)=u_1(x),\ y_2(0,x)=u_2(x)\quad (\text{in }\ L^2(\Omega)) \tag{7}$$

are satisfied, where $H^1(\Omega)^*$ is a adjoint space of $H^1(\Omega)$, $\langle\cdot,\cdot\rangle$ is a duality pairing of $H^1(\Omega)^*$, $H^1(\Omega)$.

Definition 1.2.

$$y_1\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))\cap C_w(0,T;L^2(\Omega)),$$

$$y_2\in L^\infty(0,T;L^2(\Omega))\cap C_w(0,T;L^2(\Omega))$$

$$\left(\frac{\partial^\alpha y_1}{\partial t^\alpha}\in L^2(0,T;L^2(\Omega)),\ \frac{\partial^\alpha y_2}{\partial t^\alpha}\in L^2(0,T;L^2(\Omega))\right)$$

are called (2)-weak solutions of the initial-boundary value problem (1)-(4), if for any $z_1\in H^1(\Omega),\ z_2\in L^2(\Omega)$ and almost all $t\in(0,T)$, the equalities

$$\int_\Omega\left(\frac{\partial^\alpha y_1}{\partial t^\alpha}z_1+\nabla y_1\nabla z_1+a_1y_1^3z_1-a_2y_1z_1+a_3y_2z_1\right)dx=\int_\Omega f_1z_1dx, \tag{8}$$

$$\int_\Omega\left(\frac{\partial^\alpha y_2}{\partial t^\alpha}z_2+b_1y_2z_2-b_2y_1z_2\right)dx=\int_\Omega f_2z_2dx, \tag{9}$$

$$y_1(0,x)=u_1(x),\ y_2(0,x)=u_2(x)\quad (\text{in }\ L^2(\Omega)) \tag{10}$$

are satisfied.

Theorem 1.1. There exist (1)-weak solutions of the problem (1)-(4).

Theorem 1.2. If $u_1\in H^2(\Omega)$ and $\frac{\partial u_1}{\partial\nu}|_\Gamma=0$, there exist (2)-weak solutions of (1)-(4).

Theorem 1.3. The (2)-weak solution $y=(y_1,y_2)$ of (1)-(4) is unique.

2. Preliminaries

In this section, we provide some preliminaries concepts and definitions for fractional integral and derivative and prove Lemma 2.1 and Lemma 2.2.

Suppose that X is a Banach space. Let $\alpha\in(0,1]$, $\nu:[0,T]\to X$. We define the left, right Riemann-Liouville(RL) integrals of ν :

$${}_0I_t^\alpha v(t)=\int_0^t g_\alpha(t-s)v(s)ds,$$

$${}_tI_T^\alpha v(t)=\int_t^T g_\alpha(s-t)v(s)ds,\ \ 0<t<T$$

(see [4]), where g_α denotes the RL kernel

$$g_\alpha(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)},\ \ t>0.$$

Further, ${}_0^cD_t^\alpha v(t)$, ${}_0D_t^\alpha v(t)$ and ${}_tD_T^\alpha v(t)$ stand the left CFD, left and right RL fractional derivative operators of order α, respectively; that is

$${}_0^cD_t^\alpha v(t)=\frac{d}{dt}\left(\int_0^t g_{1-\alpha}(t-s)(v(s)-v(0))ds\right),\ \ t>0,$$

$${}_0D_t^\alpha v(t)=\frac{d}{dt}\left(\int_0^t g_{1-\alpha}(t-s)v(s)ds\right),\ \ t>0,$$

$${}_tD_T^\alpha v(t)=-\frac{d}{dt}\left(\int_t^T g_{1-\alpha}(s-t)v(s)ds\right),\ \ t<T.$$

(see [4]).

Generally, for $u:[0,\infty)\times R^n\to R^1$, the left CFD with respect to time of the function u can be written as

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha}=\frac{\partial}{\partial t}\int_0^t g_{1-\alpha}(t-s)\left(u(s,x)-u(0,x)\right)ds,\quad t>0.$$

Let $X_0,\ X,\ X_1$ be Hilbert spaces and $X_0\hookrightarrow X\hookrightarrow X_1$ be continuous.
Suppose that $v:R\to X_1$, $\hat{v}$ denotes Fourier transform(FT)

$$F[v](\tau)=\hat{v}(\tau)=\int_{-\infty}^{+\infty}e^{-i\tau t}v(t)dt.$$

and R is a set of the real numbers. We have

$$F\left[{}_{-\infty}D_t^\alpha v\right](\tau)=(i\tau)^\alpha\hat{v}(\tau)$$

(see [4]), where

$${}_{-\infty}D_t^\alpha v(t)=\frac{d}{dt}\left(\int_{-\infty}^t g_{1-\alpha}(t-s)v(s)ds\right),\quad t>0.$$

For given $\gamma\in(0,1)$, we introduce a following space.

$$\mathcal{W}^\gamma(R\,;X_0,X_1)=$$
$$\left\{v\in L^2(R\,;X_0):\ (1+|\tau|^\gamma)\hat{v}(\tau)\in L^2(R\,;X_1)\right\}.$$

Obviously, it is a Hilbert space for the norm

$$\|v\|_{\mathcal{W}^\gamma}=\left(\|v\|^2_{L^2(R;X_0)}+\||\tau|^\gamma\hat{v}(\tau)\|^2_{L^2(R;X_1)}\right)^{\frac{1}{2}}$$

(see [5, Eq.(5.18)]). For any $J\subset R$, we define the space $\mathcal{W}^\gamma_J$ as:

$$\mathcal{W}^\gamma_J(R\,;X_0,X_1)=\left\{u:v|_J=u,\ v\in\mathcal{W}^\gamma(R\,;X_0,X_1)\right\},$$
$$\mathcal{W}^\gamma(0,T;X_0,X_1)=\mathcal{W}^\gamma_{[0,T]}(R\,;X_0,X_1).$$

Let

$${}_0C^\infty(0,T)=\left\{v\in C^\infty(0,T):\ \text{support }v\subset(0,T]\right\},$$

$$H_l^\gamma(0,T;X_0,X_1)=$$
$$\left\{v\in L^2(0,T;X_0):\ \exists v_k\in\ {}_0C^\infty(0,T;X_0),\right.$$
$$\left.\|v_k-v\|_{H_l^\gamma(0,T;X_0,X_1)}\to0\ (k\to\infty)\right\},$$

$$\|v\|_{H_l^\gamma(0,T;X_0,X_1)}=\left(\|v\|^2_{L^2(0,T;X_0)}+\|{}_0D_t^\gamma v(t)\|^2_{L^2(0,T;X_1)}\right)^{\frac{1}{2}}$$

and

$$H_l^\gamma(R\,;X_0,X_1)=\left\{v\in L^2(R\,;X_0):\exists v_k\in\ C_0^\infty(R\,;X_0),\right.$$
$$\left.\|v_k-v\|_{H_l^\gamma(R\,;X_0,X_1)}\to0\ (k\to\infty)\right\},$$

$$\|v\|_{H_l^\gamma(R;X_0,X_1)}=\left(\|v\|^2_{L^2(R;X_0)}+\|{}_{-\infty}D_t^\gamma v(t)\|^2_{L^2(R;X_1)}\right)^{\frac{1}{2}}$$

By Plancherel theorem, $\mathcal{W}^\gamma(R\,;X_0,X_1)=H_l^\gamma(R\,;X_0,X_1)$ holds and the norms are equivalent.

Lemma 2.1. Suppose that H is the real Hilbert space and $\alpha\in(\frac{1}{2},1)$, $u\in L^2(0,T;H)$, ${}_0^cD_t^\alpha u\in L^2(0,T;H)$ hold. Then

$${}_0I_t^\alpha\left({}_0^cD_t^\alpha u(t),u(t)\right)\geq\frac{1}{2}\left(\|u(t)\|^2-\|u(0)\|^2\right)\quad(t\in(0,T)),$$

here $\|\cdot\|$ is the norm of H and $(\cdot,\cdot)$ is the scalar product of H.

Proof: Putting

$$v(t)=u(t)-u(0),\ g(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{v(\tau)}{(t-\tau)^\alpha}d\tau,$$

we have

$$g\in L^2(0,T;H),\ \frac{d}{dt}g(t)=\ {}_0D_t^\alpha v(t)\in L^2(0,T;H)$$

by the condition for $u(t)$.

Therefore, by [6, Theorem 2.2 of Chapter I], there exist

$$\tilde{g}\in L^2(R\,;H),\ \frac{d\tilde{g}}{dt}\in L^2(R\,;H)$$

such that $\tilde{g}|_{[0,T]}=g,\ \tilde{g}(t)=0\ (t\leq0)$.

So, there exists $\{\tilde{g}_k\}\subset C_0^\infty(R\,;H)$ such that

$$\tilde{g}_k(t)=0\ (t\leq0),\ \|\tilde{g}_k-\tilde{g}\|_{L^2(R\,;H)}\to0,$$
$$\left\|\frac{d}{dt}\tilde{g}_k-\frac{d}{dt}\tilde{g}\right\|_{L^2(R\,;H)}\to0\ \ (k\to+\infty)$$

by [6, Theorem 2.1 of Chapter I].

Now if we put $\tilde{v}=\ {}_{-\infty}D_t^{1-\alpha}\tilde{g},\ \tilde{v}_k=\ {}_{-\infty}D_t^{1-\alpha}\tilde{g}_k$, we find that

$$\tilde{v}=\ {}_{-\infty}I_t^\alpha\left(\frac{d\tilde{g}}{dt}\right),\ \tilde{v}_k=\ {}_{-\infty}I_t^\alpha\left(\frac{d\tilde{g}_k}{dt}\right),$$
$$\tilde{v}|_{[0,T]}=v,\ \tilde{v}=0,\ \tilde{v}_k=0\ (t\leq0),$$
$${}_{-\infty}D_t^\alpha\tilde{v}=\frac{d}{dt}\tilde{g},\ {}_{-\infty}D_t^\alpha\tilde{v}_k=\frac{d}{dt}\tilde{g}_k,$$
$$\|\tilde{v}_k-\tilde{v}\|_{L^2(R\,;H)}\to0,$$
$$\|{}_{-\infty}D_t^\alpha\tilde{v}_k-\ {}_{-\infty}D_t^\alpha\tilde{v}\|_{L^2(R\,;H)}\to0\ \ (k\to+\infty),$$

namely,

$$\|\tilde{v}_k-\tilde{v}\|_{H_l^\alpha(R\,;H,H)}=\|\tilde{v}_k-\tilde{v}\|_{\mathcal{W}^\alpha(R\,;H,H)}\to0\ \ (k\to+\infty).$$

Therefore, putting $\tilde{v}_k|_{[0,T]}=v_k,\ u_k(t)=v_k(t)+u(0)$, we obtain

$$\|u_k-u\|_{L^2(0,T;H)}\to0,\ \ \|{}_0^cD_t^\alpha u_k-\ {}_0^cD_t^\alpha u\|_{L^2(0,T;H)}\to0\quad(k\to+\infty).\tag{11}$$

By [4, Theorem 2.1], we obtain

$$\frac{d}{dt}\left(\int_0^t\frac{1}{\Gamma(1-\alpha)}\cdot\frac{u_k(\tau)-u_k(0)}{(t-\tau)^\alpha}d\tau\right)=$$

$$\int_0^t \frac{1}{\Gamma(1-\alpha)} \cdot \frac{u_k'(\tau)}{(t-\tau)^\alpha} d\tau$$

and $u_k(t) \in C^1([0,T];H)$. Now let us prove

$$\left({}_0^cD_t^\alpha u_k(t), u_k(t)\right) \geq \frac{1}{2}\, {}_0^cD_t^\alpha \left(\|u_k(t)\|^2\right).$$

Similarly to [1, Lemma 1],

$$\begin{aligned}
&\left({}_0^cD_t^\alpha u_k(t), u_k(t)\right) - \frac{1}{2}\, {}_0^cD_t^\alpha \left(u_k(t), u_k(t)\right) = \\
&= \frac{1}{\Gamma(1-\alpha)} \left(\int_0^t \frac{u_k'(\tau)}{(t-\tau)^\alpha} d\tau, u_k(t)\right) - \\
&\frac{1}{2}\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{2(u_k'(\tau), u_k(\tau))}{(t-\tau)^\alpha} d\tau = \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{(u_k'(\tau), u_k(t)-u_k(\tau))}{(t-\tau)^\alpha} d\tau = \\
&\frac{1}{\Gamma(1-\alpha)} \left(\int_0^t \frac{u_k'(\tau)}{(t-\tau)^\alpha} d\tau, \int_\tau^t u_k'(\eta) d\eta\right) = \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t d\tau \int_\tau^t \frac{(u_k'(\tau), u_k'(\eta))}{(t-\tau)^\alpha} d\eta = \\
&\frac{1}{\Gamma(1-\alpha)} \int_0^t d\eta \int_0^\eta \frac{(u_k'(\tau), u_k'(\eta))}{(t-\tau)^\alpha} d\tau = \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\eta)^\alpha d\eta \int_0^\eta \frac{(u_k'(\tau), u_k'(\eta))}{(t-\eta)^\alpha (t-\tau)^\alpha} d\tau = \\
&= \frac{1}{2\Gamma(1-\alpha)} \int_0^t (t-\eta)^\alpha \\
&\frac{\partial}{\partial \eta} \left(\int_0^\eta \frac{u_k'(\tau)}{(t-\tau)^\alpha} d\tau, \int_0^\eta \frac{u_k'(\tau)}{(t-\tau)^\alpha} d\tau\right) d\eta = \\
&= \frac{\alpha}{2\Gamma(1-\alpha)} \int_0^t (t-\eta)^{(\alpha-1)} \\
&\left(\int_0^\eta \frac{u_k'(\tau)}{(t-\tau)^\alpha} d\tau, \int_0^\eta \frac{u_k'(\tau)}{(t-\tau)^\alpha} d\tau\right) d\eta \geq 0
\end{aligned}$$

Thus, we have

$${}_0I_t^\alpha \left({}_0^cD_t^\alpha u_k(t), u_k(t)\right) \geq \frac{1}{2}\left(\|u_k(t)\|^2 - \|u_k(0)\|^2\right). \quad (12)$$

Because $\mathcal{W}^\alpha(0,T;H,H) \hookrightarrow C(0,T;H)$ is continuous,

$$\frac{1}{2}\left(\|u_k(t)\|^2 - \|u_k(0)\|^2\right) \to \frac{1}{2}\left(\|u(t)\|^2 - \|u(0)\|^2\right) \quad (k \to +\infty) \qquad (13)$$

holds. Also, since

$$\begin{aligned}
&|{}_0I_t^\alpha \left({}_0^cD_t^\alpha u_k(t), u_k(t)\right) - {}_0I_t^\alpha \left({}_0^cD_t^\alpha u(t), u(t)\right)| = \\
&= \|\frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} [({}_0^cD_\tau^\alpha u_k(\tau), u_k(\tau)) - \\
&({}_0^cD_\tau^\alpha u(\tau), u(\tau))] d\tau\| \leq \\
&\leq \frac{1}{\Gamma(\alpha)} [\left(\int_0^t \|{}_0^cD_\tau^\alpha u_k(\tau)\|^2 d\tau\right)^{1/2} \cdot \\
&\left(\int_0^t \frac{\|u_k(\tau)-u(\tau)\|^2}{(t-\tau)^{2-2\alpha}} d\tau\right)^{1/2} + \\
&+ \left(\int_0^t \|{}_0^cD_\tau^\alpha (u_k(\tau)-u(\tau))\|^2 d\tau\right)^{1/2} \cdot \\
&\left(\int_0^t \frac{\|u(\tau)\|^2}{(t-\tau)^{2-2\alpha}} d\tau\right)^{1/2}] \leq \\
&\leq c_1 [\left(\|u_k - u\|^2_{C([0,T];H)} \int_0^t \frac{1}{(t-\tau)^{2-2\alpha}} d\tau\right)^{1/2} + \\
&+ \left(\|u\|^2_{C([0,T;H])} \cdot \int_0^t \|{}_0^cD_\tau^\alpha (u_k(\tau)-u(\tau))\|^2 d\tau\right)^{1/2} \cdot \\
&\left(\int_0^t \frac{1}{(t-\tau)^{2-2\alpha}} d\tau\right)^{1/2}],
\end{aligned}$$

$$\int_0^t \frac{1}{(t-\tau)^{2-2\alpha}} d\tau = \frac{t^{2\alpha-1}}{2\alpha-1} \quad (\alpha \in (1/2,1)),$$

we have from (11) that

$${}_0I_t^\alpha \left({}_0^cD_t^\alpha u_k(t), u_k(t)\right) \to {}_0I_t^\alpha \left({}_0^cD_t^\alpha u(t), u(t)\right)$$

$$(k \to +\infty,\ t \in (0,T))$$

holds. Thus from the above result and (11), (12), (13), this lemma is proved.

Lemma 2.2. Suppose that H is the real Hilbert space, $\alpha \in (\frac{1}{2}, 1)$, $u \in L^2(0,T;H)$, ${}_0D_t^\alpha u \in L^2(0,T;H)$, $u(0) = 0$ hold. Then

$$\int_0^T \left({}_0D_t^\alpha u(t), u(t)\right) dt = \int_0^T \left({}_0D_t^{\alpha/2} u(t),\ {}_tD_T^{\alpha/2} u(t)\right) dt \geq 0$$

is satisfied. Here $(\cdot,\cdot)$ is the scalar product of H.

Proof: Putting

$$g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u(\tau)}{(t-\tau)^\alpha} d\tau,$$

we have

$$g \in L^2(0,T;H),\ \frac{d}{dt} g(t) = {}_0D_t^\alpha u(t) \in L^2(0,T;H)$$

by the condition for $u(t)$.

Then, similarly to Lemma 2.1, there is $\{\tilde{u}_k\} \subset C_0^\infty(R\,;H)$ such that

$$\tilde{u}_k|_{[0,T]} = u_k,\ \tilde{u}|_{[0,T]} = u,$$

$$\|\tilde{u}_k - \tilde{u}\|_{L^2(R\,;H)} \to 0,\ \|\ {}_{-\infty}D_t^\alpha \left(\tilde{u}_k - \tilde{u}\right)\|_{L^2(R\,;H)} \to 0 \quad (k \to +\infty)$$

Hence,

$$\left|\int_0^T ({}_0D_t^\alpha u_k(t), u_k(t))\,dt - \int_0^T ({}_0D_t^\alpha u(t), u(t))\,dt\right| \le$$
$$\le \|{}_0D_t^\alpha (u_k - u)\|_{L^2(0,T;H)} \|u\|_{L^2(0,T;H)} +$$
$$+ \|{}_0D_t^\alpha u_k\|_{L^2(0,T;H)} \|u_k - u\|_{L^2(0,T;H)} \to 0$$
$$(k \to +\infty)$$

holds. By [7, Lemma 2.5], the norm

$$\|u\|_{H_l^{\alpha/2}(0,T;H,H)},\ \|u\|_{H_r^{\alpha/2}(0,T;H,H)}$$

and $\|u\|_{\mathcal{W}^{\alpha/2}(0,T;H,H)}$ are equivalent, where

$$\|u\|_{H_r^{\alpha/2}(0,T;H,H)} = \left(\|u\|^2_{L^2(0,T;H)} + \|{}_tD_T^{\alpha/2} u\|^2_{L^2(0,T;H)}\right)^{\frac{1}{2}}.$$

Thus, since $\mathcal{W}^\alpha(0,T;H,H) \hookrightarrow \mathcal{W}^{\alpha/2}(0,T;H,H)$ is continuous,

$$\left|\int_0^T \left({}_0D_t^{\alpha/2} u_k(t),\ {}_tD_T^{\alpha/2} u_k(t)\right) dt - \right.$$
$$\left.\int_0^T \left({}_0D_t^{\alpha/2} u(t),\ {}_tD_T^{\alpha/2} u(t)\right) dt\right| \le$$
$$\le \left\|{}_0D_t^{\alpha/2}(u_k - u)\right\|_{L^2(0,T;H)} \left\|{}_tD_T^{\alpha/2} u\right\|_{L^2(0,T;H)} +$$
$$+ \left\|{}_0D_t^{\alpha/2} u_k\right\|_{L^2(0,T;H)} \left\|{}_tD_T^{\alpha/2}(u_k - u)\right\|_{L^2(0,T;H)} \to 0$$
$$(k \to +\infty)$$

holds. And we have from [3, Lemma 4.2] that

$$\int_0^T ({}_0D_t^\alpha u_k(t),\ u_k(t))\,dt =$$
$$\int_0^T \left({}_0D_t^{\alpha/2} u_k(t),\ {}_tD_T^{\alpha/2} u_k(t)\right) dt.$$

From [6, Theorem 11.1 of Chapter I],

$$C_0^\infty(0,T) \subset \mathcal{W}^{\alpha/2}(0,T) :=$$
$$\left\{v|_{(0,T)};\ \left(1 + |\tau|^{\alpha/2}\right)\hat{v}(\tau) \in L^2(-\infty,+\infty)\right\}$$

is density. This lemma is proved.

3. Existence of the Weak Solutions

In this section, we see that Theorem 1.1 and Theorem 1.2.

Proof (Theorem 1.1).

We assume that $\{w_j\}$ is the proper base of $H^1(\Omega)$, i.e. it satisfies

$$(w_j, v)_{H^1(\Omega)} = \lambda_j (w_j, v)_{L^2(\Omega)} \quad (\forall v \in H^1(\Omega)),$$

and

$$y_{1m} = \sum_{j=1}^m \alpha_{mj}(t) w_j,\ y_{2m} = \sum_{j=1}^m \beta_{mj}(t) w_j.$$

Now we suppose that

$$u_{1m} \in [w_1, ..., w_m],\ u_{2m} \in [w_1, ..., w_m],$$
$$u_{im} \to u_i \ \ (i = 1, 2; m \to \infty, in\ L^2(\Omega)),$$

where $[w_1, ..., w_m]$ is a set of all linear combinations of $\{w_j\}_{j=1}^m$. Then there exist solutions of following initial value problem in $[0, t_m]$ by [13, Theorem 2.1].

$$\int_\Omega \left(\frac{\partial^\alpha y_{1m}}{\partial t^\alpha} w_j(x) + \nabla y_{1m} \nabla w_j(x) + a_1 y_{1m}^3 w_j(x)\right.$$
$$\left. - a_2 y_{1m} w_j(x) + a_3 y_{2m} w_j(x)\right) dx = \quad (14)$$
$$= \int_\Omega f_1 w_j dx \quad (1 \le j \le m),$$

$$\int_\Omega \left(\frac{\partial^\alpha y_{2m}}{\partial t^\alpha} w_j(x) + b_1 y_{2m} w_j(x) - b_2 y_{1m} w_j(x)\right) dx =$$
$$\int_\Omega f_2 w_j dx \quad (1 \le j \le m), \quad (15)$$

$$y_{1m}|_{t=0} = (u_1)_m,\ y_{2m}|_{t=0} = (u_2)_m, \quad (16)$$

where $t_m \le T$.

Multiplying (14) by $\alpha_{mj}(t)$, adding for $j (j = 1, ..., m)$, we get

$$\left(\frac{\partial^\alpha y_{1m}(t)}{\partial t^\alpha}, y_{1m}(t)\right)_{L^2(\Omega)} +$$
$$\int_\Omega \left(|\nabla y_{1m}|^2 + a_1 |y_{1m}|^4\right) dx \le$$
$$\int_\Omega \left(a_2 |y_{1m}|^2 + a_3 |y_{1m}| \cdot |y_{2m}|\right) dx +$$
$$+ \int_\Omega f_1 \cdot y_{1m} dx \le \frac{1}{2}\left(\int_\Omega |f_1|^2 dx + \int_\Omega |y_{1m}|^2 dx\right) +$$
$$a_2 \int_\Omega |y_{1m}|^2 dx + \frac{a_3}{2}\int_\Omega |y_{2m}|^2 dx + \frac{a_3}{2}\int_\Omega |y_{1m}|^2 dx. \quad (17)$$

And applying the fractional integral to it in $(0, t)$ when $t \in (0, T)$, we have from Lemma 2.1,

$$\frac{1}{2}\|y_{1m}(t)\|^2_{L^2(\Omega)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}$$
$$\left(\int_\Omega (|\nabla y_{1m}|)^2 dx + a_1 \int_\Omega |y_{1m}|^4 dx\right) d\tau \le$$
$$\le \frac{1}{2\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\int_\Omega |f_1|^2 dx + \int_\Omega |y_{1m}|^2 dx\right) d\tau +$$
$$\frac{1}{2}\|u_{1m}\|^2_{L^2(\Omega)} +$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}$$
$$\left(\frac{a_3}{2}\int_\Omega |y_{2m}|^2dx + \left(a_2+\frac{a_3}{2}\right)\int_\Omega |y_{1m}|^2dx\right)d\tau. \tag{18}$$

And since

$$(t-\tau)^{\alpha-1} \ge (T-\tau)^{\alpha-1} \ge T^{\alpha-1} \quad (\tau \le t \le T),$$

we have

$$\frac{1}{2}\|y_{1m}(t)\|^2_{L^2(\Omega)} + \frac{T^{\alpha-1}a_1}{\Gamma(\alpha)}\int_0^t\left(\int_\Omega |y_{1m}|^4dx\right)dt$$
$$+\frac{T^{\alpha-1}}{\Gamma(\alpha)}\int_0^t\left(\int_\Omega |\nabla y_{1m}|^2dx\right)dt \le \frac{1}{2}\|u_{1m}\|^2_{L^2(\Omega)}+$$
$$+\frac{1}{\Gamma(\alpha)}\left(\frac{1}{2}\int_0^t (t-\tau)^{\alpha-1}\left(\int_\Omega |f_1|^2dx + a_3\int_\Omega |y_{2m}|^2dx+\right.\right.$$
$$\left.\left.\left(2a_2+a_3+\frac{1}{2}\right)\int_\Omega |y_{1m}|^2dx\right)\right)d\tau \le$$
$$\le \frac{1}{2}\frac{T^\alpha}{\alpha\Gamma(\alpha)}\|f_1\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{1}{2}\|u_{1m}\|^2_{L^2(\Omega)}+$$
$$+\frac{1}{\Gamma(\alpha)}\left(\frac{1}{2}\int_0^t (t-\tau)^{\alpha-1}\left(\int_\Omega |f_1|^2dx+\right.\right.$$
$$\left.\left.a_3\int_\Omega |y_{2m}|^2dx + \left(2a_2+a_3+\frac{1}{2}\right)\int_\Omega |y_{1m}|^2dx\right)\right)d\tau. \tag{19}$$

And also, multiplying (15) by $\beta_{mj}(t)$, adding for $j(j = 1,...,m)$, we get

$$\left(\frac{\partial^\alpha y_{2m}(t)}{\partial t^\alpha}, y_{2m}(t)\right)_{L^2(\Omega)} + \int_\Omega b_1|y_{2m}|^2dx =$$
$$\int_\Omega f_2\cdot u_{2m}dx + \int_\Omega b_2|y_{1m}|\cdot|y_{2m}|dx \le$$
$$\le \frac{1}{2}\left(\int_\Omega |f_2|^2dx + \int_\Omega |y_{2m}|^2dx + b_2\int_\Omega |y_{1m}|^2dx+\right.$$
$$\left.b_2\int_\Omega |y_{2m}|^2dx\right).$$

Applying the fractional integral to it in $(0,t)$ we get,

$$\frac{1}{2}\|y_{1m}(t)\|^2_{L^2(\Omega)}+$$
$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\int_\Omega b_1|y_{2m}|^2dx\right)d\tau \le$$
$$\le \frac{1}{2}\|u_{2m}\|^2_{L^2(\Omega)} + \frac{1}{2\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}$$
$$\left(\int_\Omega |f_2|^2dx + \int_\Omega |y_{2m}|^2dx\right)d\tau+$$
$$+\frac{b_2}{2\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\int_\Omega |y_{1m}|^2dx + \int_\Omega |y_{2m}|^2dx\right)d\tau \le$$

$$\frac{T^\alpha}{2\alpha\Gamma(\alpha)}\|f_2\|^2_{L^\infty(0,T;L^2(\Omega))}+$$
$$+\frac{1}{2}\|u_{2m}\|^2_{L^2(\Omega)} + \frac{1}{2\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} \tag{20}$$
$$\left(b_2\int_\Omega |y_{1m}|^2dx + (1+b_2)\int_\Omega |y_{2m}|^2dx\right)d\tau.$$

By (19) and (20), we have

$$\|y_{1m}(t)\|^2_{L^2(\Omega)} + \|y_{2m}(t)\|^2_{L^2(\Omega)} \le$$
$$c_1\left(\|u_{1m}\|^2_{L^2(\Omega)} + \|u_{2m}\|^2_{L^2(\Omega)} + \|f_1\|^2_{L^\infty(0,T;L^2(\Omega))}+\right.$$
$$+\|f_2\|^2_{L^\infty(0,T;L^2(\Omega))} + \int_0^t (t-\tau)^{\alpha-1}\left(\|y_{1m}(\tau)\|^2_{L^2(\Omega)}+\right.$$
$$\left.\left.\|y_{2m}(\tau)\|^2_{L^2(\Omega)}\right)d\tau\right).$$

Thus, by singular Gronwall inequality ([12, Theorem 1.2]),

$$\|y_{1m}(t)\|^2_{L^2(\Omega)} + \|y_{2m}(t)\|^2_{L^2(\Omega)} \le$$
$$c_2\sum_{i=1}^2\left(\|u_i\|^2_{L^2(\Omega)} + \|f_i\|^2_{L^\infty(0,T;L^2(\Omega))}\right) \tag{21}$$
$$(for\ \ a.e.\ t\in\ (0,T)),$$

where, $c_2 > 0$ independent on $y_i (i = 1,2)$.

Therefore, there exist solutions of the initial value problem (14)-(16) in $(0,T)$.

Also, from (19),

$$\|y_{1m}\|^4_{L^4(Q)} + \|y_{1m}\|^2_{L^2(0,T;H^1(\Omega))} \le$$
$$c_3\sum_{i=1}^2\left(\|u_i\|^2_{L^2(\Omega)} + \|f_i\|^2_{L^\infty(0,T;L^2(\Omega))}\right), \tag{22}$$

where, $c_3 > 0$ independent on $y_i (i = 1,2)$.

Now let $\tilde{y}_{1m} : R \to H^1(\Omega)$ be given as

$$\tilde{y}_{1m}(t) = \begin{cases} y_{1m}(t), & t\in[0,T], \\ 0, & t\in R\backslash[0,T], \end{cases}$$

and $g_{1m} : (0,T) \to H^1(\Omega)^*$ be given as

$$\langle g_{1m}, w_j\rangle = \int_\Omega f_1 w_j dx - \int_\Omega \left(\nabla y_{1m}\nabla w_j - a_1 y_{1m}^3 w_j +\right.$$
$$\left. a_2 y_{1m} w_j + a_3 y_{1m} y_{2m} w_j\right)dx+$$
$$+\frac{t^{\alpha-1}}{\Gamma(\alpha)}(u_{1m}, w_j),\ w_j \in H^1(\Omega).$$

Since

$${}^c_0D^\alpha_t y_{1m}(t) = {}_0D^\alpha_t(y_{1m}(t) - y_{1m}(0)) =$$
$$\frac{d}{dt}[{}_0I^{1-\alpha}_t((y_{1m}(t) - u_{1m}))],$$

then (14) can be written as

$$\left\langle {}_0D_t^\alpha y_{1m}, w_j\right\rangle = \left\langle g_{1m}, w_j\right\rangle.$$

Let $\hat{y}_{1m}$ denote the FT of $\tilde{y}_{1m}$. Now we prove the sequence $\{\hat{y}_{1m}\}$ is bounded in $\mathcal{W}^\gamma(R : H^1(\Omega), L^2(\Omega))$. So, we have to see that

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma}\|\hat{y}_{1m}(\tau)\|^2_{L^2(\Omega)} \leq const, \;\; \exists\gamma\in(0,1), \tag{23}$$

where,

$$\mathcal{W}^\gamma(R\,; H^1(\Omega), L^2(\Omega)) =$$

$$\left\{v \in L^2(R\,; H^1(\Omega)) :\; (1+|\tau|^\gamma)\hat{v}(\tau) \in L^2(R\,; L^2(\Omega))\right\},$$

$$\|v\|_{\mathcal{W}^\gamma} = \left\{\|v\|^2_{L^2(R;H^1(\Omega))} + \||\tau|^\gamma \hat{v}(\tau)\|^2_{L^2(R;L^2(\Omega))}\right\}^{\frac{1}{2}}.$$

Now we prove (23).

$$\left\langle {}_{-\infty}D_t^\alpha \tilde{y}_{1m}, w_j\right\rangle = \left\langle \frac{d}{dt}[{}_{-\infty}I_t^{1-\alpha}\tilde{y}_{1m}], w_j\right\rangle =$$

$$\frac{1}{\Gamma(1-\alpha)}\left\langle \frac{d}{dt}\int_{-\infty}^t \frac{\tilde{y}_{1m}(\tau)}{(t-\tau)^\alpha}d\tau, w_j\right\rangle =$$

$$= \begin{cases} 0, & (t<0) \\ \left\langle {}_0D_t^\alpha y_{1m}, w_j\right\rangle = \left\langle g_{1m}(t), w_j\right\rangle, & (0\leq t\leq T) \\ \frac{1}{\Gamma(1-\alpha)}\left\langle \frac{d}{dt}\int_0^T \frac{y_{1m}(\tau)}{(t-\tau)^\alpha}d\tau, w_j\right\rangle, & (t>T) \end{cases}$$

holds. Expressing the right-hand side as $\left\langle \tilde{g}_{1m}, w_j\right\rangle$, we have

$$\left\langle {}_{-\infty}D_t^\alpha \tilde{y}_{1m}, w_j\right\rangle = \left\langle \tilde{g}_{1m}, w_j\right\rangle. \tag{24}$$

By the property of the FT ([4, Property 2.15]), (24) yields

$$(i\tau)^\alpha\left(\hat{y}_{1m}, w_j\right)_{L^2(\Omega)} = \left\langle \hat{g}_{1m}, w_j\right\rangle, \tag{25}$$

here $\hat{y}_{1m}$ and $\hat{g}_{1m}$ denote the FT of $\tilde{y}_{1m}$ and $\tilde{g}_{1m}$, respectively.

Now, we have

$$\sup_{\tau\in R}\|\hat{g}_{1m}(\tau)\|_{(H^1(\Omega))^*} =$$

$$\sup_{\tau\in R}\left\|\int_{-\infty}^{+\infty}\tilde{g}_{1m}(t)e^{-i\tau t}dt\right\|_{(H^1(\Omega))^*} \leq$$

$$\leq \int_0^T \|g_{1m}(t)\|_{(H^1(\Omega))^*}dt +$$

$$\int_T^{+\infty}\left\|\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^T\frac{y_{1m}(s)}{(t-s)^\alpha}ds\right\|_{(H^1(\Omega))^*}dt$$

and

$$\int_0^T \|g_{1m}(t)\|_{(H^1(\Omega))^*}dt = \int_0^T \sup_{z\in H^1(\Omega)}\frac{|\langle g_{1m}(t), z\rangle|}{\|z\|_{H^1(\Omega)}}dt \leq$$

$$\leq c_4\int_0^T\Big(\|f_1(t)\|_{L^2(\Omega)} + \|y_{1m}(t)\|_{H^1(\Omega)} + \|y_{1m}(t)\|^3_{L^4(\Omega)} +$$

$$\|y_{2m}(t)\|_{L^2(\Omega)} + \frac{t^{\alpha-1}\|u_{1m}\|_{L^2(\Omega)}}{\Gamma(\alpha)}\Big)dt.$$

Also,

$$\int_T^{+\infty}\left\|\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^T\frac{y_{1m}(s)}{(t-s)^\alpha}ds\right\|_{(H^1(\Omega))^*}dt \leq$$

$$c_6\int_T^{+\infty}\left\|\frac{d}{dt}\int_0^T\frac{y_{1m}(s)}{(t-s)^\alpha}ds\right\|_{L^2(\Omega)}dt \leq$$

$$\leq c_7\int_T^{+\infty}\int_0^T\frac{\|y_{1m}(s)\|_{L^2(\Omega)}}{(t-s)^{\alpha+1}}dsdt \leq$$

$$c_8\|y_{1m}(s)\|_{L^\infty(0,T;L^2(\Omega))}\int_0^T\frac{1}{(T-s)^\alpha}ds$$

hold. Multiplying (25) by $\hat{\alpha}_{mj}(\tau)$ and adding for $j(j=1,...,m)$, we get

$$(i\tau)^\alpha\|\hat{y}_{1m}(\tau)\|^2_{L^2(\Omega)} = \left\langle \hat{g}_{1m}, \hat{y}_{1m}(\tau)\right\rangle. \tag{26}$$

Then, since

$$\sup_{\tau\in R}\|\hat{g}_{1m}(\tau)\|_{(H^1(\Omega))^*} \leq$$

$$c_4\int_0^T\Big(\|f_1(t)\|_{L^2(\Omega)} + \|y_{1m}(t)\|_{H^1(\Omega)} +$$

$$\|y_{1m}(t)\|^3_{L^4(\Omega)} + \|y_{2m}(t)\|_{L^2(\Omega)} +$$

$$+\frac{t^{\alpha-1}\|u_{1m}\|_{L^2(\Omega)}}{\Gamma(\alpha)}\Big)dt +$$

$$c_8\|y_{1m}(s)\|_{L^\infty(0,T;L^2(\Omega))}\int_0^T\frac{1}{(T-s)^\alpha}ds.$$

we take

$$\sup_{\tau\in R}\|\hat{g}_{1m}(\tau)\|_{(H^1(\Omega))^*} \leq const, \;\; \forall m\geq 1$$

by (21) and (22).

From (26),

$$|\tau|^\alpha\|\hat{y}_{1m}(\tau)\|^2_{L^2(\Omega)} \leq c_9\|\hat{y}_{1m}(\tau)\|_{H^1(\Omega)}. \tag{27}$$

For γ fixed, $0<\gamma<\frac{\alpha}{2}-\frac{1}{4}$, then,

$$h(|\tau|) = |\tau|^{2\gamma}\frac{1+|\tau|^{\alpha-2\gamma}}{1+|\tau|^\alpha}$$

is continuous in $[0,+\infty)$, $h(|\tau|)\geq 0$ and

$$h(0)=0, \;\; h(|\tau|)\to 1 \;\; (|\tau|\to+\infty).$$

Now putting the supremum of $h(|\tau|)\geq 0$ in $[0,+\infty)$ as $c_{10}(\gamma)$, we have

$$|\tau|^{2\gamma} \leq c_{10}(\gamma)\frac{1+|\tau|^\alpha}{1+|\tau|^{\alpha-2\gamma}}.$$

Accordingly,

$$\begin{aligned}\int_{-\infty}^{+\infty}|\tau|^{2\gamma}\|\hat{y}_{1m}(\tau)\|^2_{L^2(\Omega)}d\tau \le\\ c_{10}(\gamma)\int_{-\infty}^{+\infty}\frac{1+|\tau|^{\alpha}}{1+|\tau|^{\alpha-2\gamma}}\|\hat{y}_{1m}(\tau)\|^2_{L^2(\Omega)}d\tau \le\\ \le c_{10}(\gamma)\int_{-\infty}^{+\infty}\frac{1}{1+|\tau|^{\alpha-2\gamma}}\|\hat{y}_{1m}(\tau)\|^2_{L^2(\Omega)}d\tau +\\ c_{10}(\gamma)\int_{-\infty}^{+\infty}\frac{|\tau|^{\alpha}\|\hat{y}_{1m}(\tau)\|^2_{L^2(\Omega)}}{1+|\tau|^{\alpha-2\gamma}}d\tau \le\\ \le c_{10}(\gamma)\int_{-\infty}^{+\infty}\|\hat{y}_{1m}(\tau)\|^2_{L^2(\Omega)}d\tau +\\ c_{10}(\gamma)\int_{-\infty}^{+\infty}\frac{|\tau|^{\alpha}\|\hat{y}_{1m}(\tau)\|^2_{L^2(\Omega)}}{1+|\tau|^{\alpha-2\gamma}}d\tau.\end{aligned} \tag{28}$$

And using (27), we can see that

$$\int_{-\infty}^{+\infty}\frac{|\tau|^{\alpha}\|\hat{y}_{1m}(\tau)\|^2_{L^2(\Omega)}}{1+|\tau|^{\alpha-2\gamma}}d\tau \le c_9\int_{-\infty}^{+\infty}\frac{\|\hat{y}_{1m}(\tau)\|_{H^1(\Omega)}}{1+|\tau|^{\alpha-2\gamma}}d\tau.$$

Thus, (23) follows by proving that

$$\int_{-\infty}^{+\infty}\frac{\|\hat{y}_{1m}(\tau)\|_{H^1(\Omega)}}{1+|\tau|^{\alpha-2\gamma}}d\tau \le const.$$

Using Schwarz inequality, we can estimate this integral as follows.

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{\|\hat{y}_{1m}(\tau)\|_{H^1(\Omega)}}{1+|\tau|^{\alpha-2\gamma}}d\tau \le\\ \left(\int_{-\infty}^{+\infty}\frac{d\tau}{(1+|\tau|^{\alpha-2\gamma})^2}\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}\|\hat{y}_{1m}(\tau)\|^2_{H^1(\Omega)}d\tau\right)^{\frac{1}{2}}.\end{aligned}$$

Because $\gamma < \frac{\alpha}{2}-\frac{1}{4}$, the first integral on the right-hand side to inequality is finite. Therefore, from (22) and (28), we have

$$\int_{-\infty}^{+\infty}|\tau|^{2\gamma}\|\hat{y}_{1m}(\tau)\|^2_{L^2(\Omega)}d\tau \le const.$$

Hence, (23) is proved.

Therefore, from (21), (22) and (23), there exist two elements and two subsequences in $L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))$ (relabeled $\{y_{1m}\}$, $\{y_{2m}\}$) such that

$$\begin{aligned}y_{1m}\to y_1 \ weakly\ in\ L^2(0,T;H^1(\Omega))\ and\\ weak-star\ in\ L^\infty(0,T;L^2(\Omega)),\\ y_{2m}\to y_2\ weak-star\ in\ L^\infty(0,T;L^2(\Omega)).\end{aligned}$$

Using compact embedding ([5, Theorem 5.2]), we get

$$y_{1m}\to y_1\ strongly\ in\ L^2(0,T;L^2(\Omega)).$$

Also, by [5, Lemma 1.3],

$$y_{1m}^3\to y_1^3\ weakly\ in\ L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\Omega))$$

hold.

And we obtain

$$\begin{aligned}\int_0^T\int_\Omega\frac{\partial^\alpha y_{1m}}{\partial t^\alpha}w_j(x)\psi(t)dxdt =\\ \int_0^T\int_\Omega\frac{d}{dt}\left(\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(y_{1m}(\tau)-y_{1m}(0)\right)d\tau\right)\\ w_j(x)\psi(t)dxdt =\\ =-\int_0^T\int_\Omega\left(\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(y_{1m}(\tau)-u_{1m}\right)d\tau\right)\\ w_j(x)\psi'(t)dxdt\to\\ \to-\int_0^T\int_\Omega\left(\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(y_1(\tau)-u_1\right)d\tau\right)\\ w_j(x)\psi'(t)dxdt,\end{aligned}$$

$$\begin{aligned}-\int_0^T\left[\int_\Omega(\nabla y_{1m}\nabla w_j(x)+\right.\\ \left.a_1y_{1m}^3w_j(x)-a_2y_{1m}w_j(x)+a_3y_{2m}w_j(x)\big)\,dx\right]\psi(t)dt\to\\ \to-\int_0^T\left[\int_\Omega(\nabla y_1\nabla w_j(x)+\right.\\ \left.a_1y_1^3w_j(x)-a_2y_1w_j(x)+a_3y_2w_j(x)\big)\,dx\right]\psi(t)dt.\end{aligned}$$

for any $\psi(t)\in C_0^\infty(0,T)$. Therefore, by (14), we obtain

$$\begin{aligned}-\int_0^T\int_\Omega\left(\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}(y_1(\tau)-u_1)d\tau\right)\\ w_j(x)\psi'(t)dxdt =\\ \int_0^T\left(\int_\Omega f_1(t,x)w_jdx\right)\psi(t)dt-\\ -\int_0^T\left[\int_\Omega(\nabla y_1\nabla w_j(x)+\right.\\ \left.a_1y_1^3w_j(x)-a_2y_1w_j(x)+a_3y_2w_j(x)\big)\,dx\right]\psi(t)dt.\end{aligned} \tag{29}$$

Now, if we put

$$\begin{aligned}\langle F_1(t),\tilde{z}\rangle = \int_0^T\left[\int_\Omega f_1(t,x)\tilde{z}(t,x)dx-\right.\\ \left.\int_\Omega\left(\nabla y_1\nabla\tilde{z}+a_1y_1^3\tilde{z}-a_2y_1\tilde{z}+a_3y_2\tilde{z}\right)dx\right]dt,\end{aligned}$$

when

$$\tilde{z}\in X = L^2(0,T;H^1(\Omega))\cap L^4(Q),$$

we can get

$$F_1\in L^2(0,T;H^1(\Omega)^*)+L^{\frac{4}{3}}(Q).$$

Thus, from (29),

$$ {}_0D_t^{\alpha}(y_1(t)-u_1) \in L^2(0,T;H^1(\Omega)^*)+L^{\frac{4}{3}}(Q) $$

holds. When $z_1 \in H^1(\Omega)$,

$$ \begin{aligned} &{}_0D_t^{\alpha}\left(y_1(t)-u_1,z_1\right)+\\ &\int_{\Omega}\left(\nabla y_1\nabla z_1(x)+a_1y_1^3z_1(x)-\right.\\ &\left.a_2y_1z_1(x)+a_3y_2z_1(x)\right)dx=\\ &=\int_{\Omega}f_1(t,x)z_1(x)dx. \end{aligned} \tag{30} $$

holds. Integrating (30) (with order α) in $(0,t)$, we get

$$ \begin{aligned} &(y_1(t),z_1)-(u_1,z_1)=\\ &\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left[\int_{\Omega}f_1(\tau,x)z_1(x)dx-\right.\\ &-\int_{\Omega}\left(\nabla y_1\nabla z_1(x)dx+\right.\\ &\left.\left.a_1y_1^3z_1(x)-a_2y_1z_1(x)+a_3y_2z_1(x)\right)dx\right]d\tau, \end{aligned} \tag{31} $$

Now, we show that the right-hand side to (31) is continuous at all points of the interval $[0,T]$. When $t,t_0 \in [0,T]$ and $t>t_0$, we obtain

$$ \begin{aligned} &{}_0I_t^{\alpha}\langle F_1(\cdot),z_1\rangle-{}_0I_{t_0}^{\alpha}\langle F_1(\cdot),z_1\rangle=\\ &\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left[\int_{\Omega}f_1(\tau,x)z_1dx-\right.\\ &-\int_{\Omega}\left(\nabla y_1\nabla z_1(x)+a_1y_1^3z_1(x)-a_2y_1z_1(x)+\right.\\ &\left.\left.a_3y_2z_1(x)\right)dx\right]d\tau-\\ &-\frac{1}{\Gamma(\alpha)}\int_0^{t_0}(t_0-\tau)^{\alpha-1}\left[\int_{\Omega}f_1(\tau,x)z_1dx-\right.\\ &-\int_{\Omega}\left(\nabla y_1\nabla z_1(x)+a_1y_1^3z_1(x)-\right.\\ &\left.\left.a_2y_1z_1(x)+a_3y_2z_1(x)\right)dx\right]d\tau=\\ &=\frac{1}{\Gamma(\alpha)}\int_0^{t_0}\left((t-\tau)^{\alpha-1}-(t_0-\tau)^{\alpha-1}\right)\\ &\left[\int_{\Omega}f_1(\tau,x)z_1dx-\right.\\ &-\int_{\Omega}\left(\nabla y_1\nabla z_1(x)+a_1y_1^3z_1(x)-\right.\\ &\left.\left.a_2y_1z_1(x)+a_3y_2z_1(x)\right)dx\right]d\tau+\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_0}^t(t-\tau)^{\alpha-1}\left[\int_{\Omega}f_1(\tau,x)z_1dx-\right.\\ &-\int_{\Omega}\left(\nabla y_1\nabla z_1(x)+a_1y_1^3z_1(x)-\right.\\ &\left.\left.a_2y_1z_1(x)+a_3y_2z_1(x)\right)dx\right]d\tau\to 0\\ &(t\to t_0+0) \end{aligned} $$

Similarly, we can show that

$$ -{}_0I_t^{\alpha}\langle F_1(\cdot),z_1\rangle+{}_0I_{t_0}^{\alpha}\langle F_1(\cdot),z_1\rangle\to 0\ (t\to t_0-0) $$

when $t,t_0\in[0,T]$ and $t<t_0$. Thus, the right-hand side to (31) is continuous in $[0,T]$.

Therefore, $(y_1(t),z_1)_{L^2(\Omega)}$ is continuous in $[0,T]$. Since $H^1(\Omega)\hookrightarrow L^2(\Omega)$ is density, $y_1\in C_w(0,T;L^2(\Omega))$ hold. Thus, from (31), we have

$$ y_1(0)=u_1\ (in\ L^2(\Omega)) $$

Finally, we have from (31),

$$ \left\langle\frac{\partial^{\alpha}y_1}{\partial t^{\alpha}},z_1\right\rangle+\int_{\Omega}(\nabla y_1\nabla z_1(x)+y_1z_1(x)+y_1(y_2)^2z_1(x))dx = \int_{\Omega}f_1(t,x)z_1dx. $$

While, for any $\psi(t)\in C_0^{\infty}(0,T)$, we obtain

$$ \begin{aligned} &\int_0^T\int_{\Omega}\frac{\partial^{\alpha}y_{2m}}{\partial t^{\alpha}}w_j(x)\psi(t)dxdt=\\ &\int_0^T\int_{\Omega}\frac{d}{dt}\left(\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(y_{2m}(\tau)-y_{2m}(0)\right)d\tau\right)\\ &w_j(x)\psi(t)dxdt=\\ &=-\int_0^T\int_{\Omega}\left(\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(y_{2m}(\tau)-u_{2m}\right)d\tau\right)\\ &w_j(x)\psi'(t)dxdt\to\\ &\to-\int_0^T\int_{\Omega}\left(\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(y_2(\tau)-u_2\right)d\tau\right)\\ &w_j(x)\psi'(t)dxdt, \end{aligned} $$

$$ \begin{aligned} &\int_0^T\left[\int_{\Omega}f_2(t,x)w_jdx-\int_{\Omega}\left(b_1y_{2m}w_j(x)-\right.\right.\\ &\left.\left.b_2y_{1m}w_j(x)\right)dx\right]\psi(t)dt\to\\ &\to\int_0^T\left[\int_{\Omega}f_2(t,x)w_jdx-\int_{\Omega}\left(b_1y_{2m}w_j(x)-\right.\right.\\ &\left.\left.b_2y_{1m}w_j(x)\right)dx\right]\\ &\psi(t)dt. \end{aligned} $$

Therefore, by (15), we get

$$ \begin{aligned} &-\int_0^T\int_{\Omega}\left(\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}(y_2(\tau)-u_2)d\tau\right)w_j(x)\psi'(t)dxdt\\ &=\int_0^T\left(\int_{\Omega}f_2(t,x)w_jdx\right)\psi(t)dt-\\ &-\int_0^T\left[\int_{\Omega}\left(b_1y_{2m}w_j(x)-b_2y_{1m}w_j(x)\right)dx\right]\psi(t)dt. \end{aligned} \tag{32} $$

Now, if we put

$$ \langle F_2(t),\tilde{z}\rangle=\int_0^T\left[\int_{\Omega}f_2(t,x)\tilde{z}(t,x)dx-\int_{\Omega}(b_1y_2\tilde{z}(t,x)-\right. $$

$$b_2 y_1 \tilde{z}(t,x))\,dx]\,dt,$$

when

$$\tilde{z} \in X = L^2(0,T;L^2(\Omega)),$$

we have

$$F_2 \in L^2(0,T;L^2(\Omega)).$$

Thus, from (32),

$${}_0D_t^\alpha(y_2(t)-u_2) \in L^2(0,T;L^2(\Omega))$$

holds. When $z_2 \in L^2(\Omega)$,

$${}_0D_t^\alpha\left(y_2(t,x)-u_2, z_2(x)\right) + \int_\Omega \left(b_1 y_2 z_2(x) - b_2 y_1 z_2(x)\right) dx = \int_\Omega f_2(t,x) z_2(x) dx. \tag{33}$$

holds. Integrating (33) (with order α) in $(0,t)$, we obtain

$$(y_2(t,x), z_2(x)) - (u_2(x), z_2(x)) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[\int_\Omega f_2(\tau,x) z_2(x) dx - \right. \\ \left. - \int_\Omega \left(b_1 y_2(t,x) z_2(x) - b_2 y_1(t,x) z_2(x)\right) dx\right] d\tau, \tag{34}$$

Now, we show that the right-hand side to (34) is continuous at all points of the interval $[0,T]$. When $t, t_0 \in [0,T]$ and $t > t_0$, we obtain

$$\begin{aligned}
&{}_0I_t^\alpha\langle F_2(\cdot), z_2\rangle - {}_0I_{t_0}^\alpha\langle F_2(\cdot), z_2\rangle = \\
&\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[\int_\Omega f_2(\tau,x) z_2(x) dx - \right. \\
&\left. - \int_\Omega \left(b_1 y_2 z_2(x) - b_2 y_1 z_2(x)\right) dx\right] d\tau - \\
&\frac{1}{\Gamma(\alpha)}\int_0^{t_0} (t_0-\tau)^{\alpha-1}\left[\int_\Omega f_2(\tau,x) z_2 dx - \right. \\
&\left. - \int_\Omega \left(b_1 y_2 z_2(x) - b_2 y_1 z_2(x)\right) dx\right] d\tau = \\
&\frac{1}{\Gamma(\alpha)}\int_0^{t_0} \left((t-\tau)^{\alpha-1} - (t_0-\tau)^{\alpha-1}\right)\left[\int_\Omega f_2(\tau,x) z_2 dx - \right. \\
&\left. - \int_\Omega \left(b_1 y_2 z_2(x) - b_2 y_1 z_2(x)\right) dx\right] d\tau + \\
&\frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-\tau)^{\alpha-1}\left[\int_\Omega f_2(\tau,x) z_2 dx - \right. \\
&\left. - \int_\Omega \left(b_1 y_2 z_2(x) - b_2 y_1 z_2(x)\right) dx\right] d\tau \to 0 \ (t \to t_0 + 0)
\end{aligned}$$

Similarly, we can show that

$$- {}_0I_t^\alpha\langle F_2(\cdot), z_2\rangle + {}_0I_{t_0}^\alpha\langle F_2(\cdot), z_2\rangle \to 0 \ (t \to t_0 - 0)$$

when $t, t_0 \in [0,T]$ and $t < t_0$. Thus, the right-hand side to (34) is continuous in $[0,T]$.
Therefore, $(y_2(t), z_2) = (y_2(t), z_2)_{L^2(\Omega)}$ is continuous in $[0,T]$ and $y_2 \in C_w(0,T;L^2(\Omega))$ hold. Thus, from (34), we have

$$y_2(0) = u_2 \ (in\ L^2(\Omega))$$

Finally, we have from (34),

$$\int_\Omega \left(\frac{\partial^\alpha y_2}{\partial t^\alpha} z_2 + b_1 y_2 z_2(x) - b_2 y_1 z_2(x)\right) dx = \int_\Omega f_2(t,x) z_2 dx.$$

Result of Theorem 1.1 is proved.
Proof(Theorem 1.2.)

Let $\{w_j\}$ be the system of the eigenfunctions of a eigenvalue problem

$$(w_j, v)_{H^2(\Omega)} = \lambda_j (w_j, v)_{L^2(\Omega)} \quad (\forall v \in H^2(\Omega)).$$

and

$$y_{1m} = \sum_{j=1}^m \alpha_{mj}(t) w_j,\ y_{2m} = \sum_{j=1}^m \beta_{mj}(t) w_j.$$

Now we suppose that

$$u_{1m} \in [w_1, ..., w_m],\ u_{2m} \in [w_1, ..., w_m],$$

$$u_{im} \to u_i \ \ (i = 1,2; m \to \infty, in\ H^2(\Omega)),$$

where $[w_1, ..., w_m]$ is a set of all linear combinations of $\{w_j\}_{j=1}^m$.
Then, similarly to Theorem 1.1, there exist solutions y_{1m}, y_{2m} to the Galerkin system (14)-(16). Therefore, since

$$\begin{aligned}
&the\ sequence\ \{y_{1m}\}\ is\ bounded\ in \\
&L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap L^4(Q), \\
&the\ sequence\ \{y_{2m}\}\ is\ bounded\ in\ L^\infty(0,T;L^2(\Omega)),
\end{aligned}$$

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} \|\hat{y}_{1m}(\tau)\|^2_{L^2(\Omega)} d\tau \le const,$$

there exist two elements and two subsequences (relabeled $\{y_{1m}\}$, $\{y_{2m}\}$) such that

$$\begin{aligned}
&y_{1m} \to y_1\ weakly\ in\ L^2(0,T;H^1(\Omega))\ and \\
&weak - star\ in\ L^\infty(0,T;L^2(\Omega)), \\
&y_{2m} \to y_2\ weak - star\ in\ L^\infty(0,T;L^2(\Omega)).
\end{aligned}$$

By (8)-(10),

$$\int_\Omega \left(\left|\frac{\partial^\alpha y_{1m}}{\partial t^\alpha}\right|^2 + \nabla y_{1m}\nabla(\frac{\partial^\alpha y_{1m}}{\partial t^\alpha}) + a_1 y_{1m}^3 \frac{\partial^\alpha y_{1m}}{\partial t^\alpha}\right) dx = \\ = \int_\Omega \left(a_2 y_{1m}\frac{\partial^\alpha y_{1m}}{\partial t^\alpha} - a_3 y_{2m}\frac{\partial^\alpha y_{1m}}{\partial t^\alpha} + f_1 \frac{\partial^\alpha y_{1m}}{\partial t^\alpha}\right) dx \tag{35}$$

follow and

$$\int_0^T \int_\Omega \nabla y_{1m} \frac{\partial^\alpha \nabla y_{1m}}{\partial t^\alpha} dtdx =$$

$$\int_0^T \int_\Omega \nabla (y_{1m} - u_{1m}) \frac{\partial^\alpha \nabla y_{1m}}{\partial t^\alpha} dtdx +$$

$$\int_0^T \int_\Omega \nabla u_{1m} \frac{\partial^\alpha \nabla y_{1m}}{\partial t^\alpha} dtdx,$$

$$\int_0^T \int_\Omega \nabla u_{1m} \frac{\partial^\alpha \nabla y_{1m}}{\partial t^\alpha} dtdx =$$

$$- \int_0^T \int_\Omega \Delta u_{1m} \frac{\partial^\alpha y_{1m}}{\partial t^\alpha} dtdx$$

hold. By Lemma 2.2, we obtain

$$\int_0^T \int_\Omega \nabla (y_{1m} - u_{1m}) \frac{\partial^\alpha \nabla y_{1m}}{\partial t^\alpha} dtdx =$$

$$\int_0^T \int_\Omega {}_0D_t^{\alpha/2} \nabla (y_{1m} - u_{1m}) \cdot$$

$${}_tD_T^{\alpha/2} \nabla (y_{1m} - u_{1m}) dtdx \geq 0.$$

Also,

$$\int_\Omega a_1 y_{1m}^3 \frac{\partial^\alpha y_{1m}}{\partial t^\alpha} dx \geq \frac{a_1}{2} \int_\Omega y_{1m}^2 \frac{\partial^\alpha y_{1m}^2}{\partial t^\alpha} dx \geq$$

$$\frac{a_1}{4} \int_\Omega \frac{\partial^\alpha y_{1m}^4}{\partial t^\alpha} dx = \frac{a_1}{2} \frac{\partial^\alpha}{\partial t^\alpha} \int_\Omega |y_{1m}|^4 dx$$

holds. Hence, we have from (35) that

$$\begin{aligned} &\int_0^T \left(\int_\Omega \left(\left| \frac{\partial^\alpha y_{1m}}{\partial t^\alpha} \right|^2 + \frac{a_1}{4} \frac{\partial^\alpha y_{1m}^4}{\partial t^\alpha} \right) dx \right) dt = \\ &= \int_0^T \left(\int_\Omega \left(a_2 y_{1m} \frac{\partial^\alpha y_{1m}}{\partial t^\alpha} - a_3 y_{2m} \frac{\partial^\alpha y_{1m}}{\partial t^\alpha} + \right.\right. \\ &\left.\left. f_1 \frac{\partial^\alpha y_{1m}}{\partial t^\alpha} + \Delta u_{1m} \frac{\partial^\alpha y_{1m}}{\partial t^\alpha} \right) dx \right) dt. \end{aligned} \tag{36}$$

At this time, we get

$$\frac{a_1}{4} \int_0^T \left(\int_\Omega \frac{\partial^\alpha y_{1m}^4}{\partial t^\alpha} dx \right) dt =$$

$$= \frac{a_1}{4} \frac{1}{\Gamma(\alpha)} \int_0^T \left(\frac{d}{dt} \int_0^t \left(\frac{1}{(t-\tau)^{1-\alpha}} \right.\right.$$

$$\left.\left. \int_\Omega \left(y_{1m}^4(\tau, x) - u_{1m}^4(\tau, x) \right) dx \right) d\tau \right) dt =$$

$$= \frac{a_1}{4} \frac{1}{\Gamma(\alpha)} \int_0^T \frac{1}{(T-\tau)^{1-\alpha}}$$

$$\left(\int_\Omega \left(y_{1m}^4(\tau, x) - u_{1m}^4 \right) dx \right) d\tau \geq$$

$$\geq \frac{a_1}{4} \frac{1}{\Gamma(\alpha)} T^{\alpha-1} \int_0^T \left(\int_\Omega \left(y_{1m}^4(\tau, x) - u_{1m}^4 \right) dx \right) d\tau.$$

Thus, by (36), Hölder inequality and Young inequality, we obtain

$$\left\| \frac{\partial^\alpha y_{1m}}{\partial t^\alpha} \right\|_{L^2(Q)}^2 + \|y_{1m}\|_{L^4(Q)}^4 \leq$$

$$\leq c_1 \left\| \frac{\partial^\alpha y_{1m}}{\partial t^\alpha} \right\|_{L^2(Q)} \left(\|y_{1m}\|_{L^2(Q)} + \|y_{2m}\|_{L^2(Q)} + \right.$$

$$\left. \|f_1\|_{L^2(Q)} + \|u_{1m}\|_{H^2(\Omega)} \right) + c_2 \|u_{1m}\|_{L^4(\Omega)}^4 \leq$$

$$\leq c_1(\varepsilon) \left(\|y_{1m}\|_{L^2(Q)} + \|y_{2m}\|_{L^2(Q)} + \right.$$

$$\left. \|f_1\|_{L^2(Q)} + \|u_{1m}\|_{H^2(\Omega)} \right)^2 +$$

$$\varepsilon \left\| \frac{\partial^\alpha y_{1m}}{\partial t^\alpha} \right\|_{L^2(Q)}^2 + c_2 \|u_{1m}\|_{L^4(\Omega)}^4,$$

where $0 < \varepsilon < 1$. Therefore, we obtain

$$\begin{aligned} \left\| \frac{\partial^\alpha y_{1m}}{\partial t^\alpha} \right\|_{L^2(Q)}^2 &\leq c_3 \left(\|y_{1m}\|_{L^2(Q)}^2 + \|y_{2m}\|_{L^2(Q)}^2 \right. \\ &\left. + \|f_1\|_{L^2(Q)}^2 + \|u_{1m}\|_{H^2(\Omega)}^2 + \|u_{1m}\|_{L^2(\Omega)}^4 \right) \end{aligned} \tag{37}$$

Then, similarly to Theorem 1.1, for any $\psi(t) \in C_0^\infty(0, T)$,

$$\int_0^T \left(\int_\Omega \left(\nabla y_{1m} \nabla w_j(x) + a_1 y_{1m}^3 w_j(x) - \right.\right.$$

$$\left.\left. a_2 y_{1m} w_j(x) + a_3 y_{2m} w_j(x) \right) dx \right) \psi(t) dt \to$$

$$\to \int_0^T \left(\int_\Omega \left(\nabla y_1 \nabla w_j(x) + a_1 y_1^3 w_j(x) - \right.\right.$$

$$\left.\left. a_2 y_1 w_j(x) + a_3 y_2 w_j(x) \right) dx \right) \psi(t) dt$$

and

$$\int_0^T \left(\int_\Omega \frac{\partial^\alpha y_{1m}}{\partial t^\alpha} w_j(x) dx \right) \psi(t) dt =$$

$$- \int_0^T \left({}_0I_t^{1-\alpha} \left(\int_\Omega (y_{1m}(t, x) - \right.\right.$$

$$\left.\left. y_{1m}(0, x)) w_j(x) dx \right) \psi'(t) \right) dt =$$

$$= - \int_0^T \left(\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \right.$$

$$\left. \left(\int_\Omega (y_{1m}(\tau, x) - y_{1m}(0, x)) w_j(x) dx \right) d\tau \right) \psi'(t) dt =$$

$$= - \frac{1}{\Gamma(\alpha)} \int_0^T \int_\Omega (y_{1m}(\tau, x) - y_{1m}(0, x))$$

$$\left(\int_\tau^T (t-\tau)^{\alpha-1} \psi'(t) dt \right) w_j(x) dx d\tau \to$$

$$\to - \frac{1}{\Gamma(\alpha)} \int_0^T \int_\Omega (y_1(\tau, x) - y_1(0, x))$$

$$\left(\int_\tau^T (t-\tau)^{\alpha-1} \psi'(t) dt \right) w_j(x) dx d\tau =$$

$$= -\int_0^T \left({}_0I_t^{1-\alpha} \left(\int_\Omega (y_1(t,x) - y_1(0,x)) \, w_j(x) dx \right) \psi'(t) \right) dt.$$

Hence, from (37),

$$\int_0^T \left(\int_\Omega \frac{\partial^\alpha y_{1m}}{\partial t^\alpha} w_j(x) dx \right) \psi(t) dt \to \int_0^T \left(\int_\Omega \chi_1(t,x) w_j(x) dx \right) \psi(t) dt$$

holds for some $\chi_1 \in L^2(Q)$. Thus, we have $\chi_1 = \frac{\partial^\alpha y_1}{\partial t^\alpha} \in L^2(Q)$.
Finally, (8) holds by (14). In a similar way, we get (9).

The result of theorem is proved.

4. Uniqueness of the Weak Solutions

In this section, we see that Theorem 1.3.

Proof(Theorem 1.3).

Assume that (y_1, y_2), $(\bar{y}_1, \bar{y}_2)$ are two solutions of (1)-(4) and let

$$z_1 = y_1 - \bar{y}_1, \; z_2 = y_2 - \bar{y}_2.$$

Then from Definition 1.2, we obtain

$$\int_\Omega \left(\frac{\partial^\alpha z_1}{\partial t^\alpha} z_1 + \nabla z_1 \nabla z_1 + a_1(y_1^3 - \bar{y}_1^3) z_1 - a_2 z_1 z_1 + a_3 z_2 z_1 \right) dx = 0, \tag{38}$$

$$\int_\Omega \left(\frac{\partial^\alpha z_2}{\partial t^\alpha} z_2 + b_1 z_2 z_2 - b_2 z_1 z_2 \right) dx = 0, \tag{39}$$

$$z_1(0,x) = 0, \; z_2(0,x) = 0, \tag{40}$$

where

$$\int_\Omega a_1(y_1^3 - \bar{y}_1^3) z_1 dx \geq 0.$$

Hence, applying the fractional integral to the both sides of (38), (39) and summing two equations, we have

$$\|z_1(t)\|^2_{L^2(\Omega)} + \|z_2(t)\|^2_{L^2(\Omega)} \leq \leq c \int_0^t (t-\tau)^{\alpha-1} \left(\|z_1(\tau)\|^2_{L^2(\Omega)} + \|z_2(\tau)\|^2_{L^2(\Omega)} + \|z_1(\tau)\|_{L^2(\Omega)} \|z_2(\tau)\|_{L^2(\Omega)} \right) d\tau \leq \leq \frac{3}{2} c \int_0^t (t-\tau)^{(\alpha-1)} \left(\|z_1(\tau)\|^2_{L^2(\Omega)} + \|z_2(\tau)\|^2_{L^2(\Omega)} \right) d\tau.$$

Thus, by singular Gronwall inequality ([12, Theorem 1.2]),

$$\|z_1(t)\|^2_{L^2(\Omega)} + \|z_2(t)\|^2_{L^2(\Omega)} = 0 \;\; (for \;\; a.e. \; t \in \; (0,T)).$$

The proof is complete.

Abbreviations

FNRDS	Fitzhugh-Nagumo Reaction-Diffusion System
CFD	Caputo Fractional Derivative
RL	Riemann-Liouville
FT	Fourier Transform

Acknowledgments

The authors would like to thanks to the anonymous reviewers and editors for their valuable comments and suggestions which led to the improvement of the original manuscript.

Conflicts of Interest

The authors declare that they have no conflict of interests.

References

[1] Alikhanov, A. A., A priori estimates for solutions of boundary value problems for fractional-order equations: Differential Equations, 46(5), 660-666(2010). https://doi.org/10.1134/S0012266110050058

[2] Armanyos, M., Radwan, A. G., Fractional-Order Fitzhugh-Nagumo and Izhikevich Neuron Models: 978-1-4673-9749-0/16/$31.00 (2016) IEEE.

[3] Ford, N. J., Xiao, J. Y., Yan, Y. B., A Finite Element Method for Time Fractional Partial Differential Equations: Fract. Calc. Appl. Anal., 14, 454-474. (2011).

[4] Kilbas, A. A., Srivastava, H. M., Trujllo, J. J., Theory and Applications of Fractional Differential Equations: ELSEVIER, North-Holland, 2006. ISBN-13: 978-0-444-51832-3

[5] Lions, J. L., Quelques méthodes de résolution des problémes aux limites non linéaire: DUNOD GAUTHIER-VILLARS, Paris, 1969.

[6] Lions, J. L., Magenes, E., Problémes aux limites non homogénes et applications: DUNOD, Paris, 1968.

[7] Li, X. J., Xu, C. J., A space-time spectral method for the time fractional diffusion equation: SIAM J. NUMER. ANAL., 47(3), 2108-2131(2009). https://doi.org/10.1137/080718942

[8] Plotnikov, S. A., Alexander, L. F., Controlled synchronization in two hybrid FitzHugh-Nagumo systems: IFAC-PapersOnLine, 49-14, 137-141(2016). https://doi.org/10.1016/j.ifacol.2016.07.998

[9] Qiao, Q., Zhang, X., Spectrum and stability of travelling pulses in a coupled Fitzhugh-Nagumo equation: arXiv: 2302.06110V1 [math.DS] 13 Feb 2023, 1-44.

[10] Taki-Eddine, O., Abdelfatah, B., A priori estimates for weak solution for a time-fractional nonlinear reaction-diffusion equations with an integral condition: Chaos, Solitons and Fractals, 103, 79-89(2017). https://doi.org/10.1016/j.chaos.2017.05.035

[11] Ucar, A., Lonngren, K. E., Bai, E. W., Synchronization of the coupled FitzHugh-Nagumo systems: Chaos, Solitons and Fractals, 20, 1085-1090(2004). https://doi.org/10.1016/j.chaos.2003.09.039

[12] Webb, J. R. L., Weakly singular Gronwall inequalities and applications to fractional differential equations: J. Math. Anal. Appl., 2018 :30939-9. https://doi.org/10.1016/j.jmaa.2018.11.004

[13] Wei, L., Global existence theory and chaos control of fractional differential equations: J.Math.Anal.Appl., 332, 709-726(2007). https://doi.org/10.1016/j.jmaa.2006.10.040

[14] Zhang, C., Ke, A., Zheng, B. D., Patterns of interaction of coupled reaction-diffusion systems of the FitzHugh-Nagumo type: Nonlinear Dyn., https://doi.org/10.1007/s11071-2019-05065-8

[15] Zhou, Y., Peng, L., Weak solutions of the time-fractional Navier- Stokes equations and optimal control: Computers and Mathematics with Applications, 2016: 0898-1221. https://doi.org/10.1016/j.camwa.2016.7.007