

Research Article

Interacting Multi-Model Strong Robust Adaptive Unscented Kalman Filter to Bearing only Tracking of Underwater Vehicle Approaching the Observer

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Abstract

We have designed an interacting multi-model strong robust adaptive unscented Kalman filter for bearing only tracking of an underwater vehicle approaching the observer. To solve the problem of tracking an approaching underwater vehicle to the observer based on only its bearing, an interactive multi-model robust adaptive unscented Kalman filter is proposed in this paper. First, a new model of the bearing sense motion towards the observer is proposed to construct a set of realistic target motion modes consisting of linear and curved motion modes. In addition, to account for the influence of outliers in the target bearing measurements, the distribution of measurement noise is assumed to have a Student's t-distribution, and the probability distribution of the degree of function and the scaling matrix of this distribution is assumed to have a gamma distribution and an inverse Wishart distribution. Thus, the model interaction step is to factorize the mixed probability density function using variational Bayesian method and, based on this, a predictive update method is proposed. In the measurement update phase, the posterior probability density function is obtained in factorization form using variational Bayesian method, and based on this, a posteriori mode probability calculation method is proposed. Simulation results show that our proposed method greatly improves the convergence rate of target tracking error.

Keywords

Bearing only Tracking, Unscented Kalman Filter, Inverse Wishart Distribution, Interactive Multi-Model

1. Introduction

For its wide application to civilian and military fields, target tracking has been deeply studied in the last decades. [1-5]. In particular, the problem of increasing the convergence rate of target tracking is considered very difficult due to the uncertainty of the motion model of target and the non-Gaussianity of measurement noise [6-8]. So far, many methods have been proposed for target tracking. According to the structure of the motion model, the methods for target tracking can be classified into

multi-segment analysis [9], reinforcement learning [10], data-driven method [11], filter-based estimation method [12-16], Singer model-based method [17], multi-model-based method, interactive multi-model-based method [18, 19], model-switching method [19], and modern statistical model-based method [20], etc. Especially, the most remarkable method is to use various updates of the Unscented Kalman Filter (UKF) and Cubature Kalman Filter (CKF) which are designed based on

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nonlinear motion models and observation models [21-27, 31, 32]. However, many methods are based on the simultaneous measurement of range as well as the bearing of the target, or on the use of more than two sensors. In general, it is an important issue to develop a method that guarantees the desired tracking accuracy within a minimum time in the problem of tracking the state of a target (range, magnitude and direction of velocity) with only bearing measurements by a single sensor. An important factor affecting the tracking accuracy is the uncertainty of the motion model and the non-Gaussianity of the measurement noise. In water, the non-Gaussianity of the measurement noise is much stronger than in the air. In underwater target tracking, outliers have a great impact. The uncertainty of the motion model has the greatest influence on tracking accuracy.

Various motion models have been proposed in [1], and two-dimensional underwater target tracking methods based on these models [2, 4, 5, 13, 14] with only bearing have been proposed. However, the simulation results in these methods do not have enough convergence rate of tracking error.

In real situations, underwater targets approach observers with various tactical maneuvers. One way to ensure the convergence rate of state tracking in such environments is to construct a set of motion modes as possible motion models and to perform state tracking based on the Interacting Multiple Models (IMM). In many literatures, IMM-based state tracking methods have been introduced [12, 23, 28, 29]. Considering the simulation results of [12] which introduced IMM-based two-dimensional (2D) underwater target tracking method, the error convergence time is too long and the simulation conditions in [29] did not fully reflect the realistic condition because the measurement noise was set as zero-mean Gaussian white noise. In [3, 29], Kramo-Rao bounds are analyzed according to the characteristics of measurement noise in target tracking with only bearing. In real environments, we often know all possible modes of motion of a 2D underwater target. If this condition is assumed to hold, a state tracking method with sufficient error convergence rate should be proposed, taking into account the more realistic noise of the target bearing measurements.

In real environments, the measurement data of underwater targets include outliers. The outliers have the characteristic of thickening the tail of the probability density function (PDF) of the measurement noise. In [30], a method to improve the robustness of IMM filter is proposed. To do this, we assume the measurement noise as a Student's t-distributed random variable and apply the IMM method and Variational Bayesian (VB) approximation to estimate the state of the target.

However, this filter is not suitable for tracking 2D underwater target with nonlinear measurement equations, since it is designed for linear systems. Therefore, we aimed to design a robust IMM filter with fast error convergence considering the measurement noise with outliers and the nonlinearity of the system. For this purpose, we assume that the measurement noise is a Student's t-distribution, and the unknown degrees of freedom and scale matrix of this distribution are assumed to follow

the gamma distribution and inverse Wishart distribution as in [30]. To address the nonlinearity of the system, an UKF using the Unscented transform (UT) is used, and to enable state estimation, we obtain the factorization form of the mixed PDF and the posterior PDF using VB approximation. In addition, we have proposed a new method to approximate computational expression of Gaussian integral using the UT to simplify the expression, and using constants, such as Euler constant, during the expression derivation to improve the approximation accuracy. Simulation results confirm that the convergence rate of the model tracking error increases remarkably, provided that the entire motion mode of the underwater target is known.

The paper is organized as follows;

Section 2 gives the basic concepts of the method of the UT and VB approximation, and Section 3 presents the design of an Interacting Multi-Model Strong Robust Adaptive Unscented Kalman filter (IMM-SRAUKF) over a set of modes consisting of two motion modes, newly constructing the motion equations for the approaching target, always with the direction of the observer. Section 4 shows the simulation analysis and Section 5 concludes.

2. Basic Concepts

2.1. UT and Gaussian Integral Approximation Functions

Consider the following nonlinear dynamic system.

$$\begin{aligned} \mathbf{x}_{k+1} &= F(\mathbf{x}_k) + \mathbf{w}_k \\ \mathbf{z}_{k+1} &= H(\mathbf{x}_{k+1}) + \mathbf{\varepsilon}_{k+1} \end{aligned} \quad (1)$$

where k represents the $k\Delta t$ moment (simply called k moment), $\mathbf{x}_k \in R^n, \mathbf{z}_{k+1} \in R$ is the state vector and the measurement of the system. Δt is the sampling period, $\mathbf{w}_k, \mathbf{\varepsilon}_{k+1}$ is the independent process noise vector and measurement noise.

Given a possible non-Gaussian filter density function $\rho(\mathbf{x}_k | \mathbf{z}_{0:k})$ with mean $\hat{\mathbf{x}}_k$ and co-variance P_k , UKF approximates $\rho(\mathbf{x}_k | \mathbf{z}_{0:k})$ by samples of the Gaussian density function $N(\mathbf{x}_k; \hat{\mathbf{x}}_k, P_k)$ respectively to perform a one-step prediction of the state quantity $\mathbf{x}_k$ and co-variance P_k , respectively, as follows [33].

$$\rho_x(\mathbf{x}_k | \mathbf{z}_{0:k}) \approx \sum_{i=0}^{2n+1} w_{mi} \delta(\mathbf{x}_k - \chi_k^{(i)}) \quad (2)$$

$$\rho_p(\mathbf{x}_k | \mathbf{z}_{0:k}) \approx \sum_{i=0}^{2n+1} w_{ci} \delta(\mathbf{x}_k - \chi_k^{(i)}) \quad (3)$$

where

$$w_{m0} = \lambda / (n + \lambda), w_{c0} = \lambda / (n + \lambda) + (1 - \alpha^2 + \beta), w_{mi} = w_{ci} = 0.5 / (n + \lambda), (i = 1, \dots, 2n)$$

$\lambda = \alpha^2(n + \kappa) - n$ is the scale parameter, α controls the deviation of sigma points and is usually set to a small parameter. κ is usually set to zero as a second-order scale parameter.

$\delta(\cdot)$ is a dirac-delta function. For a Gaussian distribution, $\beta = 2$ is optimal.

$\chi_k^{(i)}, (i = 0, 1, \dots, 2n+1)$ is obtained as follows.

$$\begin{cases} \chi_k^{(0)} = \hat{x}_k \\ \chi_k^{(i)} = \hat{x}_k + \left(\sqrt{(n + \lambda) \hat{P}_k} \right)_i, i = 1, \dots, n \\ \chi_k^{(i)} = \hat{x}_k - \left(\sqrt{(n + \lambda) \hat{P}_k} \right)_i, i = n+1, \dots, 2n \end{cases} \quad (4)$$

where $\left(\sqrt{(n + \lambda) \hat{P}_k} \right)_i$ represents the i -th column vector of the square root matrix of the matrix $(n + \lambda) \hat{P}_k$. Equation (4) is called the UT for computing the sigma points of $2n+1$

from $\hat{x}_k$. $\chi_k^{(i)}, (i = 0, 1, \dots, 2n)$ is called a sigma point.

Then, the posterior density function for the state quantity and co-variance prediction is respectively described as follows.

$$\begin{aligned} \rho_x(x_{k+1} | z_{0:k}) &\approx \int \rho(x_{k+1} | x_k) \rho_x(x_k | z_{0:k}) dx_k = \sum_{i=0}^{2n} w_{mi} \rho(x_{k+1} | x_k) \\ \rho_p(x_{k+1} | z_{0:k}) &\approx \int \rho(x_{k+1} | x_k) \rho_p(x_k | z_{0:k}) dx_k = \sum_{i=0}^{2n} w_{ci} \rho(x_{k+1} | x_k) \end{aligned}$$

Approximating the state transition density function as $\rho(x_{k+1} | \chi_k^{(i)}) \approx \delta(x_{k+1} - F(\chi_k^{(i)}))$, the predicted state becomes

$$\hat{x}_{k+1|k} = \int x_{k+1} \rho(x_{k+1} | z_{0:k}) dx_{k+1} = \sum_{i=0}^{2n} w_{mi} F(\chi_k^{(i)}) \quad (5)$$

Likewise, the predicted co-variance becomes

$$\begin{aligned} \hat{P}_{k+1} &= \int (x_{k+1} - \hat{x}_{k+1|k}) (x_{k+1} - \hat{x}_{k+1|k})^T \rho_p(x_{k+1} | z_{0:k}) dx_{k+1} \\ &= \sum_{i=0}^{2n+1} w_{ci} [H(\chi_k^{(i)}) - \hat{x}_{k+1|k}] [H(\chi_k^{(i)}) - \hat{x}_{k+1|k}]^T + Q_{k+1} \end{aligned} \quad (6)$$

where $Q_{k+1} = \text{cov}[w_k]$

For convenience, we predefine the following notation.

$$GU(f(x) | x; \mu, P, w) = \sum_{i=0}^{2n} w_i f(\chi^{(i)}) \quad (7)$$

where $\chi^{(i)}$ is the sigma points distributed with co-variance P centered around μ and w is the weight vector. This function is called the Gaussian integral approximation function using the UT and denoted by GU.

Using Eq. (7), Eqs. (5) and (6) can be written as follows:

$$\begin{aligned} \hat{x}_{k+1|k} &= GU(F(x_{k+1}) | x_{k+1}; \hat{x}_k, \hat{P}_k, w_m) \\ \hat{P}_{k+1} &= GU \left[(x_{k+1} - \hat{x}_{k+1|k}) (x_{k+1} - \hat{x}_{k+1|k})^T | x_{k+1}; \hat{x}_k, \hat{P}_k, w_c \right] + Q_{k+1} \end{aligned} \quad (8)$$

where w_m, w_c is a vector with $w_{mi}, w_{ci}, (i = 0, 1, \dots, 2n)$ as a component.

2.2. Kullback-Leibler (KL) Deviation

The KL deviation can be used as a similarity measure of two PDFs [26].

Definition 1. Assuming $f_1(x), f_2(x)$ to be a PDF on R^n , the KL deviation of $f_2(x)$ from $f_1(x)$ is defined as follows.

$$D_{KL}(f_1 \| f_2) = \int f_1(x) \log \frac{f_1(x)}{f_2(x)} dx \quad (9)$$

Some authors call the KL range, but since $D_{KL}(f_1 \| f_2) + D_{KL}(f_2 \| f_3) \neq D_{KL}(f_1 \| f_3)$, in fact, KL deviation does not satisfy the range axiom.

Also, some authors associate Eq. (9) with $D_{KL}(f_1 \| f_2) + D_{KL}(f_2 \| f_1)$ to call KL divergence.

Lemma 1 (Theorem 2.1 of [37])

Assuming $p(x; m^*, P^*)$ in the probability space P is a PDF on R^n with mean m^* and co-variance P^* , $N(x; \mu, P)$ is a Gaussian PDF, we have

$$N(x; m^*, P^*) = \arg \min_{N(x; \mu, P) \in P} D_{KL}[p(x; m^*, P^*) \| N(x; \mu, P)] \quad (10)$$

(The proof is omitted).

The lemma above proves that the Gaussian PDF with the smallest KL deviation from the PDF with some mean and co-variance is a Gaussian PDF with the same mean and co-variance.

Definition 2 [33]

For N PDFs $p_i(x) \in P$ defined in the probability space P , we define the weighted KL deviation as follows:

$$\bar{p}(x) = \arg \inf_{p \in P} \sum_{i=1}^N \lambda_i D_{KL}(p \| p_i) \quad (11)$$

where $\sum_{i=1}^N \lambda_i = 1, \lambda_i \geq 0$

Lemma 2 [33]

The weighted KL deviation (11) can be written as follows:

$$\bar{p}(x) = \frac{\prod_{i=1}^N p_i^{\lambda_i}(x)}{\int \prod_{i=1}^N p_i^{\lambda_i}(x) dx} \propto \prod_{i=1}^N p_i^{\lambda_i}(x) \quad (12)$$

3. Interacting Multi-Model Strong Robust Adaptive Unscented Kalman Filter Design

3.1. Construction and Assumptions of Motion Sets

To construct the set of motion of underwater vehicles, we make the following assumptions.

Assumption 1. The magnitude of the underwater vehicle velocity is constant and the direction of motion of the underwater vehicle at any time during tactical maneuvers is directed towards the observer.

Assumption 1 captures the characteristics of the underwater vehicle controlled by sensing the direction of the observer's acoustic signal generation.

3.1.1. Non-Maneuvering Motion Model

In the uncontrolled region of underwater vehicles, the non-maneuvering motion model is assumed to follow a white noise acceleration model moving at constant velocity in a 2D plane [1].

$$x_{k+1} = F^{(1)} x_k + G^{(1)} w_k^{(1)} \quad (13)$$

where $F^{(1)}$ is the state transition matrix and $G^{(1)}$ is the noise gain matrix.

The components of the state vector are the position and the velocity vector components of target in the 2D Cartesian coordinate system XOY ; $x_k = [x_k, y_k, \dot{x}_k, \dot{y}_k]^T$.

$$F^{(1)} = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad G^{(1)} = \begin{bmatrix} \Delta t^2 / 2 & 0 \\ 0 & \Delta t^2 / 2 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \quad (14)$$

The components of the noise vector $w_k^{(1)} = [w_{x_k}^{(1)}, w_{y_k}^{(1)}]^T$ are mutually independent zero-mean white noises whose variances are all $\sigma_{1x}^2 = \sigma_{1y}^2 = \sigma_1^2$.

3.1.2. Maneuvering Motion Model

In this paper, according to the motion characteristics of vehicles such as underwater vehicles approaching the observer at a constant speed, the motion model is newly modeled based on Figure 1 as follows:

In Figure 1, $B \in [-\pi/2, \pi/2]$ is the bearing, and v_t, v_o are the constant velocity of the underwater vehicle and the

observer, respectively.

When the vehicle and observer are placed in the same position as Figure 1, $B < 0$.

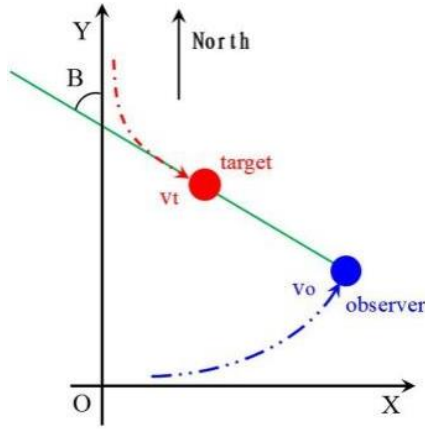


Figure 1. Motion Characteristics of Underwater Vehicle Approaching the Observer.

According to Assumption 1, the velocity component of the underwater vehicle without noise at any time t can be written as follows:

$$\begin{cases} \dot{x}(t) = -v_t \sin B(t) \\ \dot{y}(t) = -v_t \cos B(t) \end{cases}$$

In the absence of noise, if the observer is stationary, $B(t) = \text{const}$, and if it is moving, $B(t) \neq \text{const}$.

When $B(t) \neq \text{const}$, the vehicle does not generally have a straight line motion, so that all components of acceleration are nonzero. Thus

$$F^{(2)} = \begin{bmatrix} 1 & 0 & \Delta t \sin^2 B_k & \Delta t \sin 2B_k / 2 \\ 0 & 1 & \Delta t \sin 2B_k / 2 & \Delta t \cos^2 B_k \\ 0 & 0 & -\cos 2B_k & \sin 2B_k \\ 0 & 0 & \sin 2B_k & \cos 2B_k \end{bmatrix}, \quad G^{(2)} = \begin{bmatrix} \Delta t^3 / 6 & 0 \\ 0 & \Delta t^3 / 6 \\ \Delta t^2 / 2 & 0 \\ 0 & \Delta t^2 / 2 \end{bmatrix} \quad (18)$$

The measurement process can be modeled as:

$$B_k = \arctan[(x_k - x_{ok}) / (y_k - y_{ok})] \quad (19)$$

Remark 1. Since the bearing B_k in Eq. (16) varies with the position of the observer, as in Eq. (19), by means of the position vector $[x_{ok}, y_{ok}]^T$ of the observer, we can treat the position of the observer as the optimal state estimation problem of the underwater vehicle with the control signal.

However, the tactical maneuvering models, such as the rotational motion models and the curve motion models, pre-

$$\begin{cases} \ddot{x}(t) = -v_t \dot{B}(t) \cos B(t) \\ \ddot{y}(t) = v_t \dot{B}(t) \sin B(t) \end{cases} \quad (15)$$

The corresponding discrete model of Eq. (15) is

$$\begin{cases} \dot{x}_{k+1} = \dot{x}_k - \Delta t v_t \dot{B}_k \cos B_k \\ \dot{y}_{k+1} = \dot{y}_k + \Delta t v_t \dot{B}_k \sin B_k \end{cases}$$

Assumption 1, which is constant in magnitude at any time, gives the following expression;

$$\begin{bmatrix} \dot{x}_{k+1} \\ \dot{y}_{k+1} \end{bmatrix} = \begin{bmatrix} -\cos 2B_k & \sin 2B_k \\ \sin 2B_k & \cos 2B_k \end{bmatrix} \begin{bmatrix} \dot{x}_k \\ \dot{y}_k \end{bmatrix} \quad (16)$$

Thus, we can see that at any time the velocity vector of the vehicle moving toward the observer at a constant velocity rotates by $2B_k$ during Δt .

Then, by the acceleration vector $a_k = [\ddot{x}_k, \ddot{y}_k]^T$ obtained from Eq. (15) at time $k\Delta t$, the equation of motion of the underwater vehicle can be expressed as follows:

$$x_{k+1} = F^{(2)} x_k + G^{(2)} w_k^{(2)} \quad (17)$$

Here, the components of the noise vector $w_k^{(2)} = [w_{x_k}^{(2)}, w_{y_k}^{(2)}]^T$ are mutually independent zero-mean white noises whose variances are all $\sigma_{2x}^2 = \sigma_{2y}^2 = \sigma_2^2$, and the state transition matrix $F^{(2)}$ is obtained by $F^{(2)} x_k = F^{(1)} x_k + G^{(1)} a_k$.

The results are as follows;

sented in [1], do not present such a control signal. The model uncertainty in the state estimation problem of the plant under study is only represented by the model selection uncertainty in a given set of motion models, unlike the uncontrolled underwater vehicle state estimation problem. In this paper, Interacting multi-model strong robust adaptive unscented Kalman filter is designed to solve the optimal state estimation problem of underwater vehicles with model selection uncertainty.

3.1.3. Assumptions on Measurement Noise

We can summarize Eqs.(13) and (17) and (19) as follows:

$$\left. \begin{aligned} \mathbf{x}_{k+1} &= F(\mathbf{x}_k, m_k) + G(m_k) \mathbf{w}_k(m_k) \\ \mathbf{z}_{k+1} &= H(\mathbf{x}_{k+1}) + \varepsilon_{k+1} \end{aligned} \right\} \quad (20)$$

where $F(\mathbf{x}_k, 1) = F^{(1)}(\mathbf{x}_k)$, $F(\mathbf{x}_k, 2) = F^{(2)}(\mathbf{x}_k)$, $G(1) = G^{(1)}$, $G(2) = G^{(2)}$, $\mathbf{w}_k(1) = \mathbf{w}_k^{(1)}$, $\mathbf{w}_k(2) = \mathbf{w}_k^{(2)}$ and $Q_k^{(i)} = E \left[\left(G^{(i)} \mathbf{w}_k^{(i)} \right) \left(G^{(i)} \mathbf{w}_k^{(i)} \right)^T \right]$ is the process noise co-variance.

Remark 2. To reflect the non-Gaussian character of the

$$\begin{aligned} p(\mathbf{z}_{k+1} | \mathbf{x}_{k+1}, m_{k+1} = i) &= std(\mathbf{z}_{k+1} : H(\mathbf{x}_{k+1}), S_{k+1}, n_{k+1}) \\ &= \int_0^{+\infty} N(\mathbf{z}_{k+1} : H(\mathbf{x}_{k+1}), S_{k+1} / c_{k+1}) G(c_{k+1} : n_{k+1} / 2, n_{k+1} / 2) dc_{k+1} \end{aligned} \quad (21)$$

Where $G(x; a, b) = b^a / \Gamma(a) x^{a-1} e^{-bx}$, ($x \geq 0$) is the gamma PDF. Where c_{k+1} is an auxiliary parameter.

According to [36], for the parameter c_{k+1} , we assume as follows:

Assumption 3. The posterior distribution of the parameter c_{k+1} is gamma distributed.

$$p(c_{k+1} | m_{k+1} = i, \mathbf{z}_{1:k+1}) = G(c_{k+1} : \hat{a}_{k+1}^{(i)}, \hat{b}_{k+1}^{(i)}) \quad (22)$$

Assumption 4. To overcome the thickening of the tail of the Student's t-PDF when the DOF n_k has a large value, the posterior distribution of DOF n_{k+1} is assumed to be gamma distributed.

thickening of the tail of the PDF due to outliers in the measurement sequence, we assume that the measurement noise ε_{k+1} follows the Student's t-distribution.

That is, we assume that the following is true for $\mathbf{z}_{k+1} \sim std(\mathbf{z}_{k+1} : \mu_{k+1}, S_{k+1}, n_{k+1})$.

Here μ_{k+1} , S_{k+1} , n_{k+1} is the mean, scale matrix and the degree of function (DOF), respectively.

Since the Student's t-distribution is expressed by a Gaussian PDF, the likelihood PDF under the condition that it is in the motion mode $m_{k+1} = i$ can be written as follows [34]:

$$p(n_{k+1} | m_k = i, \mathbf{z}_{1:k+1}) = G(n_{k+1} : \hat{\alpha}_{k+1}^{(i)}, \hat{\beta}_{k+1}^{(i)}) \quad (23)$$

We also assume the posterior distribution of the scaling matrix S_{k+1} as follows.

Assumption 5. The posterior distribution of the scaling matrix S_{k+1} is the inverse Wishart distribution.

$$p(S_{k+1} | m_{k+1} = i, \mathbf{z}_{1:k+1}) = IW(S_{k+1} : \hat{d}_{k+1}^{(i)}, D_{k+1}^{(i)})$$

where $IW(S_{k+1} : \hat{d}_{k+1}^{(i)}, D_{k+1}^{(i)})$ is the inverse Wishart PDF,

$\hat{d}_{k+1}^{(i)}$ is the DOF, and $D_{k+1}^{(i)}$ is the inverse scaling matrix.

The inverse Wishart PDF is

$$IW(S : d, D) = \frac{|D|^{0.5d}}{2^{0.5dm} \Gamma_m(0.5d)} |S|^{-0.5(d+m+1)} \exp(-0.5 \text{tr}(DS^{-1})) \quad (24)$$

where $\Gamma_m(x) = \pi^{0.25m(m-1)} \prod_{i=0}^{m-1} \Gamma(x - 0.5i)$ is the Di-gamma function and $m(=1)$ is the output number.

Assumption 6. Given the mode of motion at time k , the state quantities $\mathbf{x}_k$, the scaling matrix S_k , and the DOF n_k are independent of each other.

This assumption is set by the condition that the measurement noise at time $k+1$ in Eq. (1) is independent of the state quantity at time k , enabling the design of the new IMM filter we are going to propose.

3.2. Model Interaction

The posterior mode probability $r_{k+1}^{(i)} = p(m_{k+1} = i | \mathbf{z}_{1:k+1})$, the predicted mode probability $r_{k+1|k}^{(i)} = p(m_{k+1} = i | \mathbf{z}_{1:k})$ and the mixed probability $r_k^{(i,j)} = p(m_k = i | m_{k+1} = j, \mathbf{z}_{1:k})$ have the following relations:

$$r_{k+1|k}^{(j)} = \sum_{i=1}^2 p(m_{k+1} = j | m_k = i, \mathbf{z}_{1:k}) p(m_k = i | \mathbf{z}_{1:k}) = \sum_{i=1}^2 \lambda_{ij} r_{k+1}^{(i)} \quad (25)$$

$$\begin{aligned}
r_k^{(i,j)} &= p(m_{k+1} = j | m_k = i, z_{1:k}) p(m_k = i | z_{1:k}) / p(m_{k+1} = j | z_{1:k}) \\
&= \lambda_{ij} r_k^{(i)} / r_{k+1|k}^{(j)}
\end{aligned} \tag{26}$$

Here λ_{ij} is the probability that the i -th motion is converted to the j -th motion [32]. By Assumption 6, the mixed PDF of x_k, S_k and n_k becomes as follows:

$$\begin{aligned}
p(x_k, S_k, n_k | m_{k+1} = i, z_{1:k}) &= \sum_{j=1}^2 p(m_k = j | m_{k+1} = i, z_{1:k}) p(x_k, S_k, n_k | m_k = j, z_{1:k}) \\
&= \sum_{j=1}^2 r_k^{(j,i)} N(x_k : \hat{x}_k^{(j)}, \hat{P}_k^{(j)}) IW(S_k : \hat{d}_k^{(j)}, D_k^{(j)}) G(n_k : \hat{\alpha}_k^{(j)}, \hat{\beta}_k^{(j)})
\end{aligned} \tag{27}$$

The mixed PDF (27) can be approximated in a more convenient form based on the VB approximation.

The following theorem is given for this.

Theorem 1. The Gaussian PDF minimizing the KL deviation from the marginal density function for x_k in the mixed PDF (27) is

$$N(x_k : \hat{\bar{x}}_k^{(i)}, \bar{P}_k^{(i)}) = N\left(x_k : \sum_{j=1}^2 r_k^{(j,i)} \hat{x}_k^{(j)}, \sum_{j=1}^2 r_k^{(j,i)} \left[\hat{P}_k^{(j)} + \left(\hat{x}_k^{(j)} - \hat{\bar{x}}_k^{(j)} \right) \left(\hat{x}_k^{(j)} - \hat{\bar{x}}_k^{(j)} \right)^T \right] \right) \tag{28}$$

Proof.

The marginal density function for x_k is

$$\begin{aligned}
&\int p(x_k, S_k, n_k | m_{k+1} = i, z_{1:k}) dS_k dn_k \\
&= \sum_{j=1}^2 r_k^{(j,i)} N(x_k : \hat{x}_k^{(j)}, \hat{P}_k^{(j)}) \int IW(S_k : \hat{d}_k^{(j)}, D_k^{(j)}) dS_k \int G(n_k : \hat{\alpha}_k^{(j)}, \hat{\beta}_k^{(j)}) dn_k \\
&= \sum_{j=1}^2 r_k^{(j,i)} N(x_k : \hat{x}_k^{(j)}, \hat{P}_k^{(j)})
\end{aligned} \tag{29}$$

Then the average is

$$\hat{\bar{x}}_k^{(j)} = \sum_{i=1}^2 r_k^{(j,i)} \int x_k N(x_k : \hat{x}_k^{(i)}, \hat{P}_k^{(i)}) dx_k = \sum_{i=1}^2 r_k^{(j,i)} \hat{x}_k^{(i)} \tag{30}$$

Also, using $\int (x - \mu)(x - \mu)^T N(x; \mu, P) dx = P + (\mu - \mu)(\mu - \mu)^T$, the co-variance becomes

$$\begin{aligned}
\bar{P}_k^{(j)} &= \sum_{i=1}^2 r_k^{(j,i)} \int \left(x_k - \hat{\bar{x}}_k^{(j)} \right) \left(x_k - \hat{\bar{x}}_k^{(j)} \right)^T N(x_k : \hat{x}_k^{(i)}, \hat{P}_k^{(i)}) dx_k \\
&= \sum_{i=1}^2 r_k^{(j,i)} \left[\hat{P}_k^{(i)} + \left(\hat{x}_k^{(i)} - \hat{\bar{x}}_k^{(j)} \right) \left(\hat{x}_k^{(i)} - \hat{\bar{x}}_k^{(j)} \right)^T \right]
\end{aligned} \tag{31}$$

And by Lemma 1, it is proved that the Gaussian PDF (28) with the smallest KL deviation from the marginal PDF (29) for x_k with the mean (30) and co-variance (31).

Theorem 2

The weighted KL deviations from the marginal density functions for S_k and n_k in the mixed PDF (27) are ob-

tained, respectively, as follows:

$$IW\left(S_k : \hat{d}_k^{(i)}, \bar{D}_k^{(i)}\right) = IW\left(S_k : \sum_{j=1}^2 r_k^{(j,i)} \hat{d}_k^{(j)}, \sum_{i=1}^2 r_k^{(j,i)} D_k^{(j)}\right) \quad (32)$$

$$G\left(n_k : \hat{\alpha}_k^{(i)}, \hat{\beta}_k^{(i)}\right) = G\left(n_k : \sum_{j=1}^2 r_k^{(j,i)} \hat{\alpha}_k^{(j)}, \sum_{j=1}^2 r_k^{(j,i)} \hat{\beta}_k^{(j)}\right) \quad (33)$$

Then the mean and co-variance of n_k are respectively

$$E\left[n_k | m_{k+1} = i, z_{1:k}\right] = \sum_{j=1}^2 r_k^{(j,i)} \hat{\alpha}_k^{(j)} / \hat{\beta}_k^{(j)} \quad (34)$$

$$\text{Cov}\left[n_k | m_{k+1} = i, z_{1:k}\right] = \sum_{j=1}^2 r_k^{(j,i)} \hat{\alpha}_k^{(j)} / \left[\hat{\beta}_k^{(j)}\right]^2 + \sum_{j=1}^M r_k^{(j,i)} \left(\hat{\alpha}_k^{(j)} / \hat{\beta}_k^{(j)} - E\left(n_k | m_{k+1} = i, z_{1:k}\right)\right)^2 \quad (35)$$

Proof.

First we prove (32). The marginal density function for S_k is

$$\begin{aligned} & \int p\left(x_k, S_k, n_k | m_{k+1} = i, z_{1:k}\right) dx_k dn_k \\ &= \sum_{j=1}^2 r_k^{(j,i)} IW\left(S_k : \hat{d}_k^{(j)}, D_k^{(j)}\right) \int N\left(x_k : \hat{x}_k^{(j)}, \hat{P}_k^{(j)}\right) dx_k \int G\left(n_k : \hat{\alpha}_k^{(j)}, \hat{\beta}_k^{(j)}\right) dn_k \\ &= \sum_{j=1}^2 r_k^{(j,i)} IW\left(S_k : \hat{d}_k^{(j)}, D_k^{(j)}\right) \end{aligned} \quad (36)$$

Then, since $\sum_{i=1}^2 r_k^{(j,i)} = 1, r_k^{(j,i)} \geq 0$, the weighted KL deviation for two inverse Wishart PDFs $IW\left(S_k : \hat{d}_k^{(j)}, D_k^{(j)}\right), j = 1, 2$ with weight $r_k^{(1,i)}, r_k^{(2,i)}$ is given by Definition 2 and Lemma 2 as follows.

$$\begin{aligned} & \prod_{j=1}^2 \left[IW\left(S_k : \hat{d}_k^{(j)}, D_k^{(j)}\right) \right]^{r_k^{(j,i)}} \propto \prod_{j=1}^2 |S_k|^{-\left(r_k^{(j,i)} \hat{d}_k^{(j)} / 2\right)} \exp\left\{-0.5 \text{tr}\left[r_k^{(j,i)} S_k^{-1} D_k^{(j)}\right]\right\} \\ & \propto |S_k|^{-\left(\sum_{i=1}^2 r_k^{(j,i)} \hat{d}_k^{(j)} / 2\right)} \exp\left\{-0.5 \text{tr}\left[S_k^{-1} \sum_{i=1}^2 r_k^{(j,i)} D_k^{(j)}\right]\right\} \\ & \propto IW\left(S_k : \sum_{i=1}^2 r_k^{(j,i)} \hat{d}_k^{(j)}, \sum_{i=1}^2 r_k^{(j,i)} D_k^{(j)}\right) \end{aligned} \quad (37)$$

Thus, Eq. (32) is proved.

Next we prove (33).

The marginal density function for n_k is

$$\int p\left(x_k, S_k, n_k | m_{k+1} = i, z_{1:k}\right) dx_k dn_k = \sum_{j=1}^2 r_k^{(j,i)} G\left(n_k : \hat{\alpha}_k^{(j)}, \hat{\beta}_k^{(j)}\right) \quad (38)$$

Since

$$\begin{aligned} \prod_{j=1}^2 \left[G\left(n_k : \hat{\alpha}_k^{(j)}, \hat{\beta}_k^{(j)}\right) \right]^{r_k^{(j,i)}} &\propto x^{\sum_{j=1}^i r_k^{(j,i)} [\hat{\alpha}_k^{(j)} - 1]} \exp \left\{ - \left[\sum_{j=1}^i r_k^{(j,i)} \hat{\beta}_k^{(j)} \right] x \right\} \\ &= x^{\sum_{j=1}^i r_k^{(j,i)} [\hat{\alpha}_k^{(j)} - 1]} \exp \left\{ - \left[\sum_{j=1}^i r_k^{(j,i)} \hat{\beta}_k^{(j)} \right] x \right\} \end{aligned} \quad (39)$$

as in Eq. (37), Eq. (33) is proved.

Equations (34) and (35) can be proved as follows considering that the mean and co-variance of the gamma distribu-

tion PDF $G(\cdot; a, b)$ are a/b , a/b^2 respectively.

By Eqs. (38) and (39), we have

$$\begin{aligned} E[n_k | m_{k+1} = i, z_{1:k}] &= \int n_k \sum_{j=1}^2 r_k^{(j,i)} G(n_k : \hat{\alpha}_k^{(j)}, \hat{\beta}_k^{(j)}) dn_k \\ &= \sum_{j=1}^M u_k^{ji} \int n_k G(n_k : \hat{\alpha}_k^{(j)}, \hat{\beta}_k^{(j)}) dn_{k-1} = \sum_{j=1}^M u_{k-1}^{ji} \hat{a}_{k-1}^j / \hat{b}_{k-1}^j \end{aligned} \quad (40)$$

$$\begin{aligned} \text{Cov}[n_k | m_{k+1} = i, z_{1:k}] &= \sum_{j=1}^2 r_k^{(j,i)} \int \left(n_k - E(n_k | m_{k+1} = i, z_{1:k}) \right)^2 G(n_k : \hat{\alpha}_k^{(j)}, \hat{\beta}_k^{(j)}) dn_k \\ &= \sum_{j=1}^2 r_k^{(j,i)} \int \left(n_k - \hat{\alpha}_k^{(j)} / \hat{\beta}_k^{(j)} \right)^2 G(n_k : \hat{\alpha}_k^{(j)}, \hat{\beta}_k^{(j)}) dn_k \\ &= \sum_{j=1}^2 r_k^{(j,i)} \hat{\alpha}_k^{(j)} / \left[\hat{\beta}_k^{(j)} \right]^2 + \sum_{j=1}^2 r_k^{(j,i)} \left(\hat{\alpha}_k^{(j)} / \hat{\beta}_k^{(j)} - E(n_k | m_{k+1} = i, z_{1:k}) \right)^2 \end{aligned} \quad (41)$$

The weighted KL deviation from the peripheral density function for n_k by Eq. (39) has a gamma distribution, and hence from Eqs. (40) and (41) we obtain

$$\hat{\alpha}_k^{(i)} = \frac{\{E[n_k | m_{k+1} = i, z_{1:k}]\}^2}{\text{Cov}[n_k | m_{k+1} = i, z_{1:k}]}, \quad \hat{\beta}_k^{(i)} = \frac{\{E[n_k | m_{k+1} = i, z_{1:k}]\}}{\text{Cov}[n_k | m_{k+1} = i, z_{1:k}]} \quad (42)$$

Proof. End.

By Theorems 1 and 2, the mixed PDF (27) of x_k, S_k and n_k is approximated as follows.

$$p(x_k, S_k, n_k | m_{k+1} = i, z_{1:k}) = N(x_k : \hat{x}_k^{(j)}, \bar{P}_k^{(j)}) IW(S_k : \hat{d}_k^{(j)}, \bar{D}_k^{(j)}) G(n_k : \hat{\alpha}_k^{(j)}, \hat{\beta}_k^{(j)}) \quad (43)$$

3.3. Prediction Step

In the prediction step, the state and co-variance are predicted on the condition given the mode of motion. The predictions in the filters are obtained by the predictive PDF

$p(x_{k+1}, S_{k+1}, n_{k+1} | m_{k+1} = i, z_{1:k})$ at time $k+1$, and this predictive PDF is obtained by the Chapman-Kolmogorov equation [35], the propagation equation of the posterior state PDF at k moment.

$$\begin{aligned} p(x_{k+1}, S_{k+1}, n_{k+1} | m_{k+1} = i, z_{1:k}) \\ = \int p(x_{k+1}, S_{k+1}, n_{k+1} | x_k, S_k, n_k, m_{k+1} = i) p(x_k, S_k, n_k | m_{k+1} = i, z_{1:k}) dx_k dS_k dn_k \end{aligned} \quad (44)$$

Here, $p(x_{k+1}, S_{k+1}, n_{k+1} | x_k, S_k, n_k, m_{k+1} = i)$ is a state transition PDF, and the distribution of the dynamic model, scale matrix and DOF that give the state transition can be

considered independent of each other. Also, since the posterior PDF $p(x_k, S_k, n_k | m_{k+1} = i, z_{1:k})$ in (44) is a mixed PDF at the time k shown in (43), we can write (44) as follows.

$$p(x_{k+1}, S_{k+1}, n_{k+1} | m_{k+1} = i, z_{1:k}) = \int N(x_{k+1}; F_k^{(i)}(x_k), Q_k^{(i)}) N(x_k; \hat{x}_k^{(i)}, \bar{P}_k^{(i)}) dx_k \quad (45)$$

$$\times \int p(S_{k+1} | S_k, m_{k+1} = i) IW(S_k; \hat{d}_k^{(j)}, \bar{D}_k^{(j)}) dS_k \times \int p(n_{k+1} | n_k, m_k = i) G(n_k; \hat{\alpha}_k^{(j)}, \hat{\beta}_k^{(j)}) dn_k$$

This indicates that, given the mode of motion at the time $k+1$, the predicted state, the scaling matrix and the probability distribution of DOF are independent of each other and

maintain the previous distribution patterns.

Thus, Eq. (45) can be written as

$$p(x_{k+1}, S_{k+1}, n_{k+1} | m_{k+1} = i, z_{1:k}) = N(x_{k+1}; \hat{x}_{k+1|k}^{(i)}, P_{k+1|k}^{(i)}) IW(S_k; \hat{d}_{k+1|k}^{(i)}, D_{k+1|k}^{(i)}) G(n_k; \hat{\alpha}_{k+1|k}^{(i)}, \hat{\beta}_{k+1|k}^{(i)}) \quad (46)$$

Using Eqs. (45) and (46) and Gaussian integral approximation function (7) using the unscented transformation, we obtain

$$\hat{x}_{k+1|k}^{(i)} = \int F_k^{(i)}(x_k) N(x_k; \hat{x}_k^{(i)}, \bar{P}_k^{(i)}) dx_k \approx GU(F_k^{(i)}(x_k) | x_k; \hat{x}_k^{(i)}, \bar{P}_k^{(i)}, w_m) \quad (47)$$

$$P_{k+1|k}^{(i)} = \int (F_k^{(i)}(x_k) - \hat{x}_{k+1|k}^{(i)}) (F_k^{(i)}(x_k) - \hat{x}_{k+1|k}^{(i)})^T N(x_k; \hat{x}_k^{(i)}, \bar{P}_k^{(i)}) dx_k + Q_k^{(i)} \quad (48)$$

$$\approx GU \left[(F_k^{(i)}(x_k) - \hat{x}_{k+1|k}^{(i)}) (F_k^{(i)}(x_k) - \hat{x}_{k+1|k}^{(i)})^T | x_k; \hat{x}_k^{(i)}, \bar{P}_k^{(i)}, w_c \right] + Q_k^{(i)}$$

In the champhman-Kolmogorov model (46), the prediction parameters $\hat{d}_{k+1|k}^{(i)}, D_{k+1|k}^{(i)}$ and $\hat{\alpha}_{k+1|k}^{(i)}, \hat{\beta}_{k+1|k}^{(i)}$ of the scaling matrix S_k and the DOF n_k are difficult to find directly by Eq. (45). This is because the dynamic model of the scale matrix and the DOF is usually unknown in practice, and the dynamics $p(S_{k+1} | S_k, m_{k+1} = i)$ of the scale matrix and the DOF $p(n_{k+1} | n_k, m_k = i)$ in Eq. (45) are not known exactly. Therefore, we propose heuristic dynamics for scaling matrices S_k and DOF n_k . This is the dynamics that simply spreads the previous approximation posterior of the scaling matrix S_k and the DOF n_k [34]. Based on Eq. (43) which is obtained based on the VB approximation, the prediction parameters $\hat{d}_{k+1|k}^{(i)}, D_{k+1|k}^{(i)}, \hat{\alpha}_{k+1|k}^{(i)}, \hat{\beta}_{k+1|k}^{(i)}$ that must be determined in the prediction step are determined by the following discovery dynamics such that in the measurement update step the posterior parameters $\hat{d}_{k+1}^{(i)}, D_{k+1}^{(i)}, \hat{\alpha}_{k+1}^{(i)}, \hat{\beta}_{k+1}^{(i)}$ can be determined accurately by the fixed point iteration method. (see the IMM-SRAUKF algorithm)

$$\hat{d}_{k+1|k}^{(i)} = \rho(\hat{d}_k^{(i)} - 2) + 2, \quad D_{k+1|k}^{(i)} = \rho \bar{D}_k^{(j)} \quad (49)$$

$$\hat{\alpha}_{k+1|k}^{(i)} = \rho \hat{\alpha}_k^{(j)}, \quad \hat{\beta}_{k+1|k}^{(i)} = \rho \hat{\beta}_k^{(j)}$$

where $\rho \in (0, 1]$. We usually set $\rho = 1 - \exp(-4)$. The IMM-SRAUKF cannot estimate the process noise

co-variance $Q_k^{(i)}$ shown in Eq. (20) as the total mean of $(G^{(i)} w_k^{(i)}) (G^{(i)} w_k^{(i)})^T$. One method is to dynamically adjust $Q_{k+1}^{(i)}$ by fusing the past and the current of the co-variance matrix through the forgetting factor γ [36].

$$\hat{Q}_{k+1}^{(i)} = (1 - \gamma) \hat{Q}_k^{(i)} + \gamma (G^{(i)} w_k^{(i)}) (G^{(i)} w_k^{(i)})^T \quad (50)$$

3.4. Measurement Update Step

The aim of the measurement update step is to obtain the posterior PDF on the condition that the new momentary motion mode and the measurement is given, and based on this, to find the state quantity and co-variance matrix of the new moments. IMM filters must first obtain the posterior PDF and the posterior mode probability based on it at this stage. First, we obtain the posterior PDF

$$p(x_{k+1}, S_{k+1}, c_{k+1}, n_{k+1} | m_{k+1} = i, z_{1:k+1}).$$

Since the motion mode of the time $k+1$ and the measurement are difficult to obtain analytically the posterior PDF, the measurement update step is to approximate the posterior PDF $p(x_{k+1}, S_{k+1}, c_{k+1}, n_{k+1} | m_{k+1} = i, z_{1:k+1})$ by using the VB approximation in the following factorized form, given the motion mode $m_{k+1} = i$ of the time $k+1$ and the new measurement z_{k+1} .

$$p(x_{k+1}, S_{k+1}, c_{k+1}, n_{k+1} | m_{k+1} = i, z_{1:k+1}) \approx p(x_{k+1} | m_{k+1} = i, z_{1:k+1}) p^{(i)}(S_{k+1} | m_{k+1} = i, z_{1:k+1}) p^{(i)}(c_{k+1} | m_{k+1} = i, z_{1:k+1}) p^{(i)}(n_{k+1} | m_{k+1} = i, z_{1:k+1}) \quad (51)$$

For convenience, we will have the following notation.

$$\begin{aligned} p_{x_{k+1}}^{(i)} &= p(x_{k+1} | m_{k+1} = i, z_{1:k+1}), & p_{S_{k+1}}^{(i)} &= p^{(i)}(S_{k+1} | m_{k+1} = i, z_{1:k+1}) \\ p_{c_{k+1}}^{(i)} &= p^{(i)}(c_{k+1} | m_{k+1} = i, z_{1:k+1}), & p_{n_{k+1}}^{(i)} &= p^{(i)}(n_{k+1} | m_{k+1} = i, z_{1:k+1}) \end{aligned} \quad (52)$$

The posterior PDFs $p_{x_{k+1}}^{(i)}$, $p_{S_{k+1}}^{(i)}$, $p_{c_{k+1}}^{(i)}$, $p_{n_{k+1}}^{(i)}$ are obtained by the VB approximation as the PDFs that minimize the KL deviation as follows:

$$\min_{p_{x_{k+1}}^{(i)}, p_{S_{k+1}}^{(i)}, p_{c_{k+1}}^{(i)}, p_{n_{k+1}}^{(i)}} KL \left[p_{x_{k+1}}^{(i)} p_{S_{k+1}}^{(i)} p_{c_{k+1}}^{(i)} p_{n_{k+1}}^{(i)} \parallel p(x_{k+1}, S_{k+1}, c_{k+1}, n_{k+1} | m_{k+1} = i, z_{1:k+1}) \right]$$

The VB marginal density functions $p_{x_{k+1}}^{(i)}$, $p_{S_{k+1}}^{(i)}$, $p_{c_{k+1}}^{(i)}$, $p_{n_{k+1}}^{(i)}$ satisfying the above equation have the following forms:

$$p_{x_{k+1}}^{(i)} \propto \exp \left\{ E_{p_{S_{k+1}}^{(i)}, p_{c_{k+1}}^{(i)}, p_{n_{k+1}}^{(i)}} \left[\ln p(x_{k+1}, S_{k+1}, c_{k+1}, n_{k+1}, z_{1:k+1} | m_{k+1} = i) \right] \right\} \quad (53)$$

$$p_{S_{k+1}}^{(i)} \propto \exp \left\{ E_{p_{x_{k+1}}^{(i)}, p_{c_{k+1}}^{(i)}, p_{n_{k+1}}^{(i)}} \left[\ln p(x_{k+1}, S_{k+1}, c_{k+1}, n_{k+1}, z_{1:k+1} | m_{k+1} = i) \right] \right\} \quad (54)$$

$$p_{c_{k+1}}^{(i)} \propto \exp \left\{ E_{p_{x_{k+1}}^{(i)}, p_{S_{k+1}}^{(i)}, p_{n_{k+1}}^{(i)}} \left[\ln p(x_{k+1}, S_{k+1}, c_{k+1}, n_{k+1}, z_{1:k+1} | m_{k+1} = i) \right] \right\} \quad (55)$$

$$p_{n_{k+1}}^{(i)} \propto \exp \left\{ E_{p_{x_{k+1}}^{(i)}, p_{S_{k+1}}^{(i)}, p_{c_{k+1}}^{(i)}} \left[\ln p(x_{k+1}, S_{k+1}, c_{k+1}, n_{k+1}, z_{1:k+1} | m_{k+1} = i) \right] \right\} \quad (56)$$

To derive a method for calculating Eqs. (53)-(56), we can first write the following expression using Eqs. (21) and (51):

$$\begin{aligned} p(x_{k+1}, S_{k+1}, c_{k+1}, n_{k+1}, z_{1:k+1} | m_{k+1} = i) &= C_1 p(x_{k+1}, S_{k+1}, c_{k+1}, n_{k+1}, z_{k+1} | z_{1:k}, m_{k+1} = i) \\ &= C_1 p(x_{k+1}, S_{k+1}, z_{k+1} | z_{1:k}, m_{k+1} = i) p(c_{k+1}, n_{k+1} | z_{1:k}, m_{k+1} = i) \\ &= C_1 p(x_{k+1}, S_{k+1} | z_{1:k}, m_{k+1} = i) p(z_{k+1} | x_{k+1}, S_{k+1}, z_{1:k}, m_{k+1} = i) \\ &\quad \times p(n_{k+1} | z_{1:k}, m_{k+1} = i) p(c_{k+1} | n_{k+1}, z_{1:k}, m_{k+1} = i) \\ &= C_1 p(x_{k+1} | z_{1:k}, m_{k+1} = i) p(S_{k+1} | z_{1:k}, m_{k+1} = i) p(z_{k+1} | x_{k+1}, S_{k+1}, z_{1:k}, m_{k+1} = i) \\ &\quad \times p(c_{k+1} | n_{k+1}, z_{1:k}, m_{k+1} = i) p(n_{k+1} | z_{1:k}, m_{k+1} = i) \\ &= C_1 N(x_{k+1} : \hat{x}_{k+1|k}^{(i)}, P_{k+1|k}^{(i)}) IW(S_{k+1} : \hat{d}_{k+1|k}^{(i)}, D_{k+1|k}^{(i)}) N(z_{k+1} : H^{(i)}(x_{k+1}), S_{k+1} / c_{k+1}) \\ &\quad \times G(c_{k+1} : n_{k+1} / 2, n_{k+1} / 2) G(n_{k+1}, \hat{\alpha}_{k+1|k}^{(i)}, \hat{\beta}_{k+1|k}^{(i)}) \end{aligned} \quad (57)$$

where $C_1 = p(z_k | m_{k+1} = i)$.

Using Eq. (24), Eq. (57) becomes as follows:

$$\begin{aligned}
p(\mathbf{x}_{k+1}, S_{k+1}, c_{k+1}, n_{k+1}, z_{1:k+1} | m_{k+1} = i) \propto \exp \left\{ -0.5 \left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}^{(i)} \right)^T \left(P_{k+1|k}^{(i)} \right)^{-1} \left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}^{(i)} \right) \right. \\
- 0.5 c_{k+1} \left(z_{k+1} - H^{(i)}(\mathbf{x}_{k+1}) \right)^T S_{k+1}^{-1} \left(z_{k+1} - H^{(i)}(\mathbf{x}_{k+1}) \right) - 0.5 \left(\hat{d}_{k+1|k}^{(i)} + 3 \right) \ln |S_{k+1}| \\
- 0.5 \text{tr} \left(D_{k+1|k}^{(i)} S_{k+1}^{-1} \right) - 0.5 n_{k+1} c_{k+1} + 0.5 (n_{k+1} - 1) \ln c_{k+1} \\
\left. + \left(\hat{\alpha}_{k+1|k}^{(i)} - 1 \right) \ln n_{k+1} - \hat{\beta}_{k+1|k}^{(i)} n_{k+1} + 0.5 n_{k+1} \ln (0.5 n_{k+1}) - \ln \Gamma (0.5 n_{k+1}) \right\}
\end{aligned} \quad (58)$$

First, let us look at the corrective form of Eq. (53).

Substituting Eq. (58) into Eq. (53), we obtain

$$p_{x_{k+1}}^{(i)} \propto \exp \left\{ -0.5 \left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}^{(i)} \right)^T \left(P_{k+1|k}^{(i)} \right)^{-1} \left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}^{(i)} \right) - 0.5 c_{k+1} \left(z_{k+1} - H^{(i)}(\mathbf{x}_{k+1}) \right)^T \bar{S}_{k+1}^{-1} \left(z_{k+1} - H^{(i)}(\mathbf{x}_{k+1}) \right) \right\} \quad (59)$$

where

$$\bar{S}_{k+1} = \left(E_{p_{S_{k+1}}^{(i)}} \left[S_{k+1}^{-1} \right] E_{p_{c_{k+1}}^{(i)}} \left[c_{k+1} \right] \right)^{-1} \quad (60)$$

$\bar{S}_{k+1}$ is the modified co-variance matrix. By Eq. (59), we can write the following relation:

$$p_{x_{k+1}}^{(i)} \propto N \left(\mathbf{x}_{k+1}; \hat{\mathbf{x}}_{k+1|k}^{(i)}, P_{k+1|k}^{(i)} \right) N \left(z_{k+1}; H^{(i)}(\mathbf{x}_{k+1}), \bar{S}_{k+1} \right)$$

Thus, $p_{x_{k+1}}^{(i)}$ can be written in the form of Gaussian PDF as

$$p_{x_{k+1}}^{(i)} = N \left(\mathbf{x}_{k+1}; \hat{\mathbf{x}}_{k+1}^{(i)}, P_{k+1}^{(i)} \right) \quad (61)$$

where

$$\hat{\mathbf{x}}_{k+1}^{(i)} = \hat{\mathbf{x}}_{k+1|k}^{(i)} \left(P_{zz,k+1|k}^{(i)} \right)^{-1} \left(z_{k+1} - \hat{z}_{k+1|k} \right) \quad (62)$$

$$P_{k+1}^{(i)} = P_{k+1|k}^{(i)} - P_{xz,k+1|k}^{(i)} \left(P_{zz,k+1|k}^{(i)} \right)^{-1} \left(P_{xz,k+1|k}^{(i)} \right)^T \quad (63)$$

The $\hat{z}_{k+1|k}^{(i)}, P_{zz,k}^{(i)}, P_{xz,k}^{(i)}$ in Eqs. (61) and (62) are obtained as follows:

$$\begin{aligned}
\hat{z}_{k+1|k}^{(i)} &= \int H^{(i)}(\mathbf{x}_{k+1}) N \left(\mathbf{x}_{k+1}; \hat{\mathbf{x}}_{k+1|k}^{(i)}, P_{k+1|k}^{(i)} \right) d\mathbf{x}_{k+1} \\
&\approx GU \left(H^{(i)}(\mathbf{x}_{k+1}) \middle| \mathbf{x}_{k+1}; \hat{\mathbf{x}}_{k+1|k}^{(i)}, P_{k+1|k}^{(i)}, \mathbf{w}_m \right)
\end{aligned} \quad (64)$$

$$\begin{aligned}
P_{zz,k+1|k}^{(i)} &= \int \left(H^{(i)}(\mathbf{x}_{k+1}) - \hat{z}_{k+1|k}^{(i)} \right) \left(H^{(i)}(\mathbf{x}_{k+1}) - \hat{z}_{k+1|k}^{(i)} \right)^T N \left(\mathbf{x}_{k+1}; \hat{\mathbf{x}}_{k+1|k}^{(i)}, P_{k+1|k}^{(i)} \right) d\mathbf{x}_{k+1} + \bar{S}_{k+1} \\
&\approx GU \left[\left(H^{(i)}(\mathbf{x}_{k+1}) - \hat{z}_{k+1|k}^{(i)} \right) \left(H^{(i)}(\mathbf{x}_{k+1}) - \hat{z}_{k+1|k}^{(i)} \right)^T \middle| \mathbf{x}_{k+1}; \hat{\mathbf{x}}_{k+1|k}^{(i)}, P_{k+1|k}^{(i)}, \mathbf{w}_c \right] + \bar{S}_{k+1}
\end{aligned} \quad (65)$$

$$\begin{aligned}
P_{xz,k+1|k}^{(i)} &= \int \left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}^{(i)} \right) \left(H^{(i)}(\mathbf{x}_{k+1}) - \hat{z}_{k+1|k}^{(i)} \right)^T N \left(\mathbf{x}_{k+1}; \hat{\mathbf{x}}_{k+1|k}^{(i)}, P_{k+1|k}^{(i)} \right) d\mathbf{x}_k \\
&\approx GU \left[\left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}^{(i)} \right) \left(H^{(i)}(\mathbf{x}_{k+1}) - \hat{z}_{k+1|k}^{(i)} \right)^T \middle| \mathbf{x}_{k+1}; \hat{\mathbf{x}}_{k+1|k}^{(i)}, P_{k+1|k}^{(i)}, \mathbf{w}_c \right]
\end{aligned} \quad (66)$$

Next, we consider the corrective form of Eq. (54). Substituting Eq. (58) into Eq. (54), we obtain

$$p_{S_{k+1}}^{(i)} \propto \exp \left\{ -0.5 \text{tr} \left(\left[E_{p_{c_{k+1}}^{(i)}} [c_{k+1}] \right] M_{k+1}^{(i)} + D_{k+1|k}^{(i)} \right) S_{k+1}^{-1} \right\} - 0.5 \left(\hat{d}_{k+1|k}^{(i)} + 3 \right) \ln |S_{k+1}| \} \quad (67)$$

where

$$M_{k+1}^{(i)} = E_{p_{z_{k+1}}^{(i)}} \left[\left(z_{k+1} - H^{(i)}(\mathbf{x}_{k+1}) \right) \left(z_{k+1} - H^{(i)}(\mathbf{x}_{k+1}) \right)^T \right] \quad (68)$$

By Assumption 5, $p_{S_{k+1}}^{(i)}$ is an inverse Wishart PDF because S_{k+1} has an inverse Wishart distribution. Thus, we have the following form of PDF:

$$p_{S_{k+1}}^{(i)} = IW \left(S_{k+1}; \hat{d}_{k+1}^{(i)}, D_{k+1}^{(i)} \right) \quad (69)$$

Comparing Eqs. (68) and (66) by (24), the following relation is obtained:

$$\hat{d}_{k+1}^{(i)} = \hat{d}_{k+1|k}^{(i)} + 1 \quad (70)$$

$$D_{k+1}^{(i)} = E_{p_{c_{k+1}}^{(i)}} [c_{k+1}] M_{k+1}^{(i)} + D_{k+1|k}^{(i)} \quad (71)$$

Substituting these results into Eq. (66), we obtain

$$p_{S_{k+1}}^{(i)} \propto \exp \left\{ -0.5 \left(\text{tr} \left(E_{p_{S_{k+1}}^{(i)}} [S_{k+1}^{-1}] M_{k+1}^{(i)} \right) + E_{p_{n_{k+1}}^{(i)}} [n_{k+1}] \right) c_{k+1} + 0.5 \left(E_{p_{c_{k+1}}^{(i)}} [c_{k+1}] - 1 \right) \ln c_{k+1} \right\} \quad (72)$$

Next, we consider the corrective form of Eq. (55). By Assumption 3, c_{k+1} has a gamma distribution, so $p_{c_{k+1}}^{(i)}$ is a gamma PDF as follows:

$$p_{c_{k+1}}^{(i)} = G \left(c_{k+1}; \hat{a}_{k+1}^{(i)}, \hat{b}_{k+1}^{(i)} \right) \quad (73)$$

Substituting Eq. (58) into Eq. (55) and comparing with Eq. (72), the following relationship is obtained:

$$\hat{a}_{k+1}^{(i)} = 0.5 \left(1 + E_{p_{n_{k+1}}^{(i)}} [n_{k+1}] \right) \quad (74)$$

$$\hat{b}_{k+1}^{(i)} = 0.5 \left\{ \text{tr} \left(E_{p_{S_{k+1}}^{(i)}} [S_{k+1}^{-1}] M_{k+1}^{(i)} \right) + E_{p_{n_{k+1}}^{(i)}} [n_{k+1}] \right\} \quad (75)$$

Next, we consider the corrective form of Eq. (56). Substituting Eq. (58) into Eq. (56), we use the following approximation [21].

$$\begin{aligned}\ln \Gamma(0.5n_{k+1}) &\approx -\ln(0.5n_{k+1}) - \gamma(0.5n_{k+1}) \\ n_{k+1} \ln n_{k+1} &\approx e_1 n_{k+1} + e_2 \ln n_{k+1}\end{aligned}\quad (76)$$

Here $\gamma \approx 0.577215$ is the Euler gamma constant and e_1, e_2 is the least squares estimate. For example, for $n_{k+1} = 2, \dots, 50$, $e_1 = 4.7421$, $e_2 = -11.3466$. Then Eq. (56) is obtained as follows:

$$p_{n_{k+1}}^{(i)} \propto \exp \left\{ \left(\hat{\alpha}_{k+1|k}^{(i)} + 0.5e_2 \right) \ln n_{k+1} - \left(\hat{\beta}_{k+1|k}^{(i)} + 0.5E_{p_{c_{k+1}}^{(i)}}[c_{k+1}] - 0.5E_{p_{c_{k+1}}^{(i)}}[\ln c_{k+1}] - 0.5 \ln 0.5 - 0.5e_1 - 0.5\gamma \right) n_{k+1} \right\} \quad (77)$$

By Assumption 4, n_{k+1} has the gamma PDF, so let us write this density function as

$$p_{n_{k+1}}^{(i)} = G(n_{k+1}; \hat{\alpha}_{k+1}^{(i)}, \hat{\beta}_{k+1}^{(i)}) \quad (78)$$

Comparing Eqs. (78) and (77), we have the following relation:

$$\begin{aligned}\hat{\alpha}_{k+1}^{(i)} &= \hat{\alpha}_{k+1|k}^{(i)} + 0.5e_2 \\ \hat{\beta}_{k+1}^{(i)} &= \hat{\beta}_{k+1|k}^{(i)} + 0.5E_{p_{c_{k+1}}^{(i)}}[c_{k+1}] - 0.5E_{p_{c_{k+1}}^{(i)}}[\ln c_{k+1}] - 0.5(\ln 0.5 + e_1 + \gamma)\end{aligned}\quad (79)$$

By Eqs. (61), (69), (73), and (79), the unknown quantities $E_{p_{S_{k+1}}^{(i)}}[S_{k+1}^{-1}]$, $E_{p_{c_{k+1}}^{(i)}}[c_{k+1}]$, $M_{k+1}^{(i)}$, $\hat{z}_{k+1}^i$, $E_{p_{n_{k+1}}^{(i)}}[n_{k+1}]$ present in Eqs. (60), (68), (71), (74), and (75) are determined as follows:

$$E_{p_{S_{k+1}}^{(i)}}[S_{k+1}^{-1}] = \left(\hat{d}_{k+1}^{(i)} - 2 \right) \left(D_{k+1}^{(i)} \right)^{-1} \quad (80)$$

$$E_{p_{c_{k+1}}^{(i)}}[c_{k+1}] = \hat{a}_{k+1}^{(i)} / \hat{b}_{k+1}^{(i)} \quad (81)$$

$$E_{p_{c_{k+1}}^{(i)}}[\ln c_{k+1}] = \psi \left(\hat{a}_{k+1}^{(i)} \right) - \ln \hat{b}_{k+1}^{(i)} \quad (82)$$

$$M_{k+1}^{(i)} \approx \left(z_{k+1} - \hat{z}_{k+1}^{(i)} \right) \left(z_{k+1} - \hat{z}_{k+1}^{(i)} \right)^T + GU \left[\left(H^{(i)}(x_{k+1}) - \hat{z}_{k+1}^{(i)} \right) \left(H^{(i)}(x_{k+1}) - \hat{z}_{k+1}^{(i)} \right)^T \middle| x_{k+1}; \hat{x}_{k+1}^{(i)}, P_{k+1}^{(i)}, w_c \right] \quad (83)$$

$$\hat{z}_{k+1}^i = \int H^{(i)}(x_{k+1}) N(x_{k+1}; \hat{x}_{k+1}^{(i)}, P_{k+1}^{(i)}) dx_{k+1} \approx GU \left[H^{(i)}(x_{k+1}) \middle| x_{k+1}; \hat{x}_{k+1}^{(i)}, P_{k+1}^{(i)}, w_m \right] \quad (84)$$

$$E_{p_{n_{k+1}}^{(i)}}[n_{k+1}] = \hat{\alpha}_{k+1}^{(i)} / \hat{\beta}_{k+1}^{(i)} \quad (85)$$

In Eq. (86), $\psi(\cdot)$ is a di-gamma function.

When using the results obtained above, Eqs. (62)-(66),

(70), (71), (74), (75), (79)-(85), except for the methods of finding $\hat{d}_{k+1}^{(i)}$ and $\hat{\alpha}_{k+1}^{(i)}$ in Eqs. (70) and (80), the problem of finding $\hat{x}_{k+1}^{(i)}, P_{k+1}^{(i)}, \hat{a}_{k+1}^{(i)}, D_{k+1}^{(i)}, \hat{\beta}_{k+1}^{(i)}$ can be solved using the fixed point iteration method.

Figure 2 shows the fixed point iterative scheme.

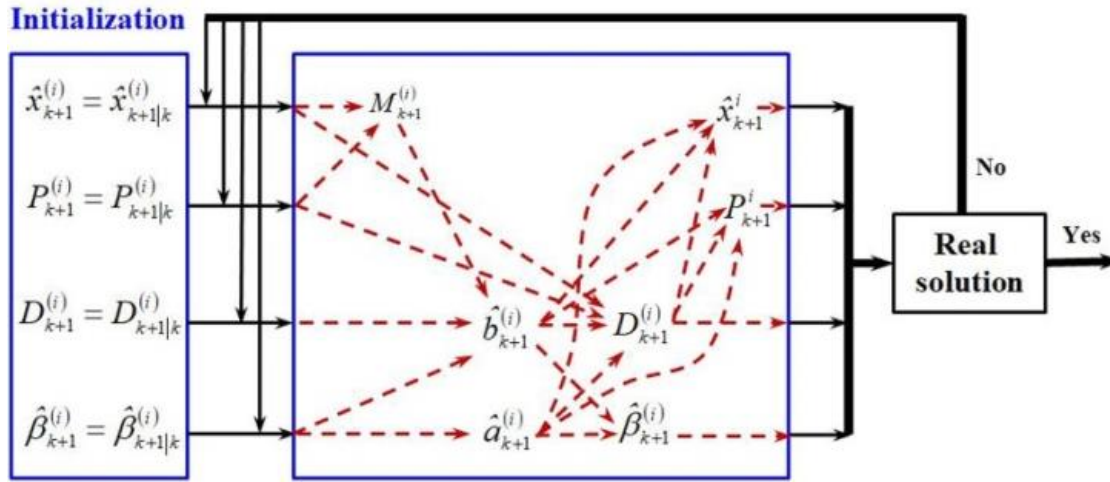


Figure 2. A Fixed Point Iterative Scheme for Measurement Update Parameter Determination.

The posterior mode probability is calculated as

$$r_{k+1}^{(i)} = \frac{p(m_{k+1} = i | z_{1:k}) p(z_{k+1} | m_{k+1} = i, z_{1:k})}{\sum_{i=1}^M r_{k+1|k}^i p(z_{k+1} | m_{k+1} = i, z_{1:k})} \quad (86)$$

Therefore, to calculate the posterior mode probability, we must determine the predictive likelihood function

$$p(z_{k+1} | m_k = i, z_{1:k}) = \int p(x_{k+1}, S_{k+1}, c_{k+1}, n_{k+1}, z_{k+1} | m_{k+1} = i, z_{1:k}) dx_{k+1} dS_{k+1} dc_{k+1} dn_{k+1}$$

However, due to (57), we cannot find the analytical solution of the above integral.

To overcome this problem, we can approximate the predicted likelihood function using the VB approximation as shown in [30] as follows:

$$\begin{aligned} \ln p(z_{k+1} | m_{k+1} = i, z_{1:k}) &= \int p_{x_{k+1}}^{(i)} p_{S_{k+1}}^{(i)} p_{c_{k+1}}^{(i)} p_{n_{k+1}}^{(i)} \ln \frac{p(x_{k+1}, S_{k+1}, c_{k+1}, n_{k+1}, z_{k+1} | m_{k+1} = i, z_{1:k})}{p_{x_{k+1}}^{(i)} p_{S_{k+1}}^{(i)} p_{c_{k+1}}^{(i)} p_{n_{k+1}}^{(i)}} dx_{k+1} dS_{k+1} dc_{k+1} dn_{k+1} \\ &+ KL \left(p_{x_{k+1}}^{(i)} p_{S_{k+1}}^{(i)} p_{c_{k+1}}^{(i)} p_{n_{k+1}}^{(i)} \parallel p(x_{k+1}, S_{k+1}, c_{k+1}, n_{k+1} | m_{k+1} = i, z_{1:k+1}) \right) \end{aligned}$$

Since the second term on the right-hand side of the above equation is minimized by the VB approximation, we can write the above equation as

$$\ln p(z_{k+1} | m_{k+1} = i, z_{1:k}) \approx \int p_{x_{k+1}}^{(i)} p_{S_{k+1}}^{(i)} p_{c_{k+1}}^{(i)} p_{n_{k+1}}^{(i)} \ln \frac{p(x_{k+1}, S_{k+1}, c_{k+1}, n_{k+1}, z_{k+1} | m_{k+1} = i, z_{1:k})}{p_{x_{k+1}}^{(i)} p_{S_{k+1}}^{(i)} p_{c_{k+1}}^{(i)} p_{n_{k+1}}^{(i)}} dx_{k+1} dS_{k+1} dc_{k+1} dn_{k+1} \quad (87)$$

Substituting Eqs. (58) and (61), (69), (73), and (78) in Eq. (87), we obtain the following expression:

$$\begin{aligned} \ln p(z_{k+1} | m_{k+1} = i, z_{1:k}) &\approx C_{k+1} + \frac{1}{2} \frac{\hat{\alpha}_{k+1}^{(i)}}{\hat{\beta}_{k+1}^{(i)}} \left[-\frac{\hat{a}_{k+1}^{(i)}}{\hat{b}_{k+1}^{(i)}} + 2 \left(\hat{\beta}_{k+1}^{(i)} - \hat{\beta}_{k+1|k}^{(i)} \right) + e_1 - \ln 2 + \gamma \right] + \left[\psi \left(\hat{\alpha}_{k+1}^{(i)} \right) - \ln \hat{\beta}_{k+1}^{(i)} \right] \\ &+ \frac{1}{2} \left[\psi \left(\hat{a}_{k+1}^{(i)} \right) - \ln \hat{b}_{k+1}^{(i)} \right] \left[\frac{\hat{\alpha}_{k+1}^{(i)}}{\hat{\beta}_{k+1}^{(i)}} - 2\hat{a}_{k+1}^{(i)} - 1 \right] + \hat{a}_{k+1}^{(i)} + \ln \Gamma \left(\hat{a}_{k+1}^{(i)} \right) - \hat{a}_{k+1}^{(i)} \ln \hat{b}_{k+1}^{(i)} - \ln 2 + \frac{n}{2} - \frac{m}{2} \ln \pi \end{aligned} \quad (88)$$

where

$$C_{k+1} = -\frac{1}{2} \text{tr} \left[\left(P_{k+1|k}^{(i)} \right)^{-1} \left(P_{k+1}^{(i)} + \left(\hat{x}_{k+1}^{(i)} - \hat{x}_{k+1|k}^{(i)} \right) \left(\hat{x}_{k+1}^{(i)} - \hat{x}_{k+1|k}^{(i)} \right)^T \right) \right] + \frac{1}{2} \left(\ln |P_{k+1}^{(i)}| - \ln |P_{k+1|k}^{(i)}| \right) \\ - \frac{1}{2} \left(\ln |D_{k+1}^{(i)}| - \ln |D_{k+1|k}^{(i)}| \right) - \left[\ln \left(\hat{\beta}_{k+1}^{(i)} \right)^{\hat{\alpha}_{k+1}^{(i)}} - \ln \left(\hat{\beta}_{k+1|k}^{(i)} \right)^{\hat{\alpha}_{k+1|k}^{(i)}} \right] \\ + \left[\ln \Gamma_m \left(\frac{1}{2} \hat{d}_{k+1}^{(i)} \right) - \ln \Gamma_m \left(\frac{1}{2} \hat{d}_{k+1|k}^{(i)} \right) \right] + \left[\ln \Gamma \left(\hat{\alpha}_{k+1}^{(i)} \right) - \ln \Gamma \left(\hat{\alpha}_{k+1|k}^{(i)} \right) \right]$$

By Eq. (88), we can determine the posterior mode probability $r_{k+1}^{(i)}$ shown in Eq. (88). Thus, the final state estimator and co-variance matrix are determined as follows:

$$\hat{x}_{k+1} = \sum_{i=1}^2 r_{k+1}^{(i)} \hat{x}_{k+1}^{(i)} \quad (89)$$

$$P_{k+1} = \sum_{i=1}^M r_{k+1}^{(i)} \left[P_{k+1}^{(i)} + \left(\hat{x}_{k+1}^{(i)} - \hat{x}_{k+1} \right) \left(\hat{x}_{k+1}^{(i)} - \hat{x}_{k+1} \right)^T \right] \quad (90)$$

Summarizing the above results, Figure 3 shows the block diagram of the IMM-SRAUKF algorithm.

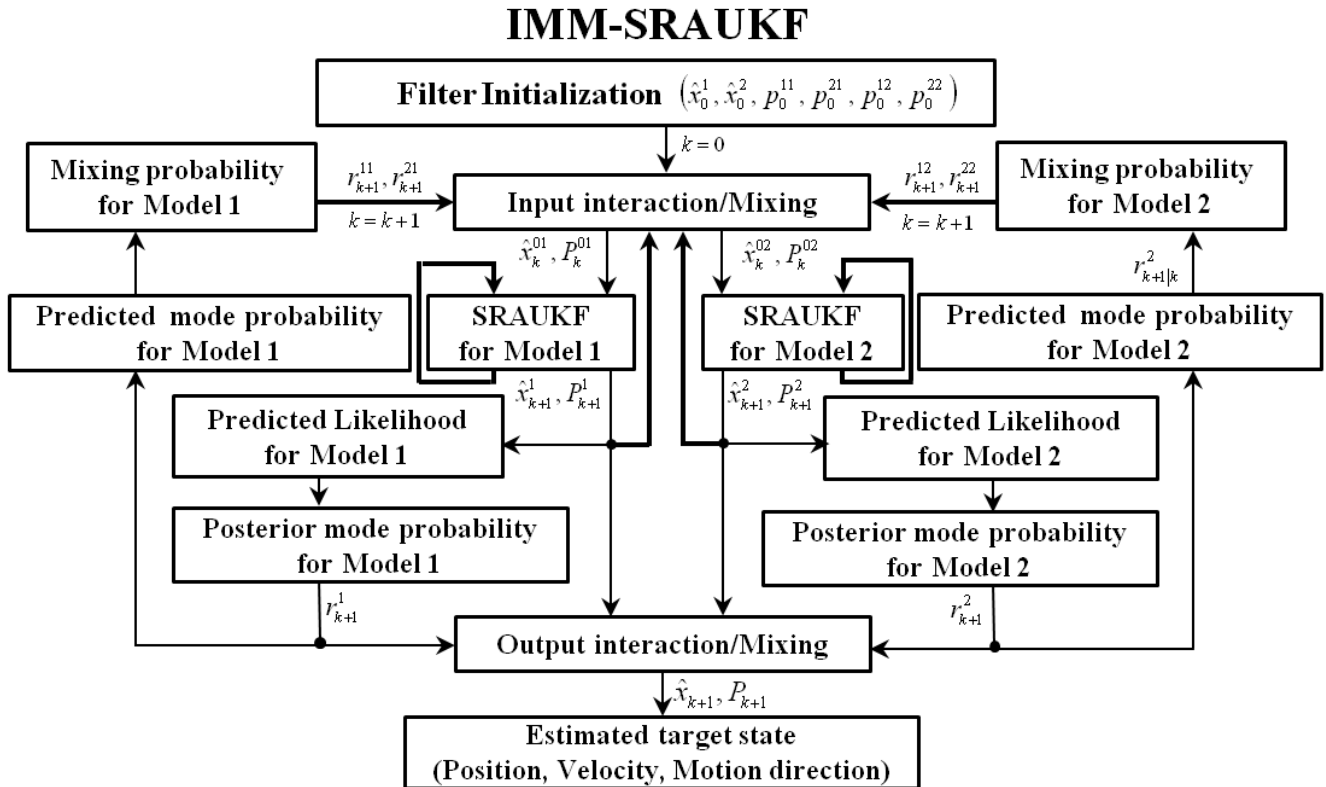


Figure 3. Schematic Diagram of IMM-SRAUKF Algorithm.

The IMM-SRAUKF algorithm is as follows.

IMM-SRAUKF algorithm

Step 1: We set the initial estimates

$\hat{x}_0^{(i)}, P_0^{(i)}, r_0^{(i)}, \hat{\alpha}_0^{(i)}, \hat{\beta}_0^{(i)}, \hat{d}_0^{(i)}, \hat{D}_0^{(i)}$ for each mode of motion i and the forgetting factor ρ, γ of Eqs. (49) and (50). Let

$k = k + 1$ and go to step 2.

Step 2: Using Eqs. (25) and (26), the prediction mode probability $r_{k+1|k}^{(j)}$ and the mixed mode probability $r_k^{(i,j)}$ are calculated. Then, the mixing quantities $\hat{\mathbf{x}}_k^{(i)}, \bar{P}_k^{(i)}, \hat{d}_k^{(i)}, \bar{D}_k^{(i)}, \hat{\alpha}_k^{(i)}, \hat{\beta}_k^{(i)}$ are calculated using Eqs. (30)-(32), (42).

Step 3: Using Eqs. (47)-(50), we calculate the following predictions for each mode.

$$\hat{\mathbf{x}}_{k+1|k}^{(i)}, P_{k+1|k}^{(i)}, \hat{\alpha}_{k+1|k}^{(i)}, \hat{\beta}_{k+1|k}^{(i)}, \hat{d}_{k+1|k}^{(i)}, D_{k+1|k}^{(i)}$$

Step 4: We update the posterior quantities $\hat{\mathbf{x}}_{k+1}^{(i)}, P_{k+1}^{(i)}, \hat{\alpha}_{k+1}^{(i)}, \hat{\beta}_{k+1}^{(i)}, \hat{d}_{k+1}^{(i)}, D_{k+1}^{(i)}$ in each mode using the fixed point iteration method shown in Figure 2.

Step 5: We update the posterior probability $r_{k+1}^{(i)}$ using Eqs. (86) and (88).

Step 6: Calculate $\hat{\mathbf{x}}_{k+1}, P_{k+1}$ using Eqs. (89) and (90). Go to step 2.

4. Simulation and Analysis

To evaluate the performance of the proposed filter in tracking an underwater vehicle approaching the observer in

two dimensions, the root mean square errors (RMSE) is used as follow:

$$RMSE_{pos} = \sqrt{\frac{1}{N} \sum_{k=1}^N \left[(x_k - \hat{x}_k)^2 + (y_k - \hat{y}_k)^2 \right]} \quad (91)$$

$$RMSE_{vel} = \sqrt{\frac{1}{N} \sum_{k=1}^N \left[(\dot{x}_k - \hat{\dot{x}}_k)^2 + (\dot{y}_k - \hat{\dot{y}}_k)^2 \right]} \quad (92)$$

The underwater vehicle is assumed to approach the observer as two modes of motion shown in Section 2. It is assumed that the underwater vehicle approach observer by the linear motion modes with constant velocity for the non-maneuvering motion model and the bearing sense motion modes for maneuvering motion model. In the time $t = k\Delta t$, the linear motion mode $m_k = 1$, and the Bearing sense motion mode $m_k = 2$. The sampling period $\Delta t = 0.1ms$. The simulation time interval is 0~200s. Figure 4 shows the motion trajectory of the observer and the target. The sight range of the day (SRD) was given as 6000 m.

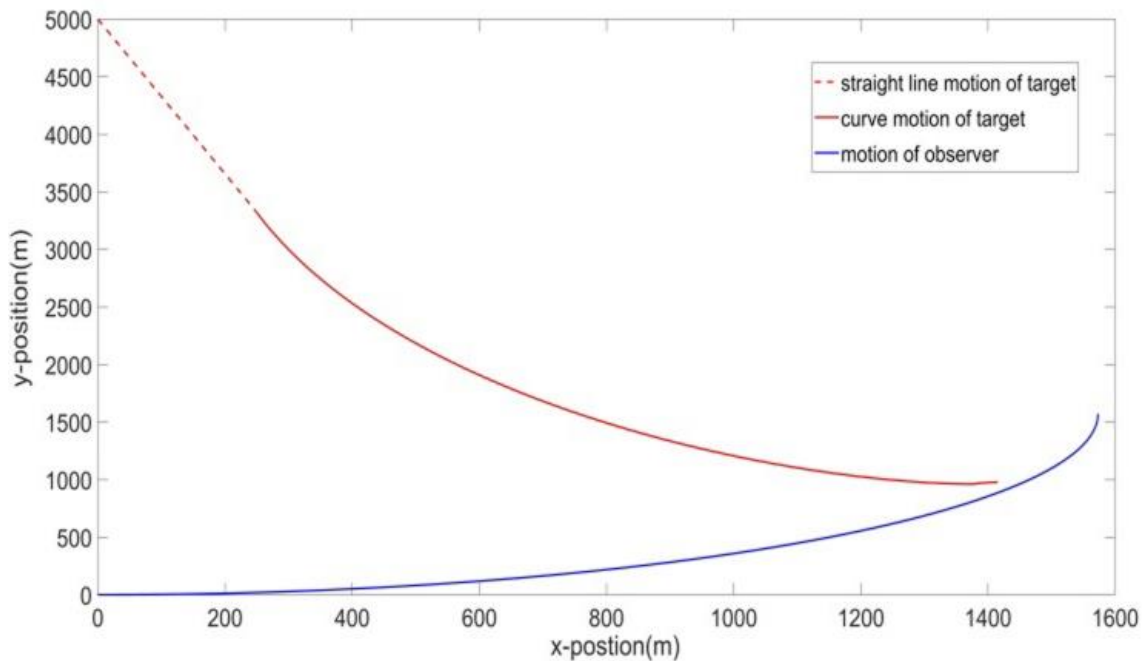


Figure 4. Target Motion to the Observer's Motion.

The tactical maneuver of the observer is assumed to be circular with center $(0, 6000)$ and radius SRD in the coor-

dinate system shown in Figure 4, and the target is assumed to have a linear motion for 100 s and then a bearing sense mo-

tion towards the observer. To evaluate the performance of the proposed filter in tracking an underwater vehicle approaching the observer in two dimensions, we compare the tracking performance in three measurement noises.

The modal transition probability is given by $\lambda_{ij} = \begin{bmatrix} 0.9997 & 0.01 \\ 0.0003 & 0.09 \end{bmatrix}$ and the initial posterior mode probability by $r_0^{(1)} = 0.9, r_0^{(2)} = 0.1$ respectively. The initial state and covariance matrix are given as follows:

$$\hat{x}_0 = [10^{-2}, SRD, 10^{-2}, 10^{-2}]^T, \\ P_0 = \text{diag}[10^{-4}, SRD^2, 10^{-4}, 10^{-4}]$$

In the Eqs. (13) and (17), acceleration noise $w_k^{(i)} = [w_{k1}^{(i)}, w_k^{(i)}]^T, (i=1,2)$ for process noise is the Gaussian noise with zero and co-variance $10^{-1} \text{diag}\{1,1\}$. In the simulation, The Co-variance is assumed unknown. The

forgetting factor of Eqs. (49) and (50) is given by $\rho = 1 - e^{-4}, \gamma = 0.99$, respectively, the initial parameter of the inverse Wishart distribution is given by $\hat{d}_0^{(1)} = \hat{d}_0^{(2)} = 7$, respectively, and all other initial parameters are given by 0.5.

The observer velocity is assumed to be constant $7m/s$ and the target velocity is constant $17m/s$.

To analyze the effectiveness of the proposed method, it is compared with Interacting Multi-Model Unscented Kalman filter (IMM-UKF) [25], Interacting Multi-Model Cubature Kalman Filter (IMM-CKF) [19], and Variational Bayesian-based Interacting Multi-Model Filter (IMM-VBF) [30].

First, to see if the real covariance is different from the standard covariance, we simulated Gaussian noise with the following time-varying co-variance:

$$\varepsilon_k \sim N[0, (3 + 0.5 \cos(0.1k\pi/100))\pi/180\sigma^2]$$

Here $\sigma = 5^\circ$. In Figure 5, the modal probabilities for the linear and bearing sense motions obtained by the proposed method are shown.

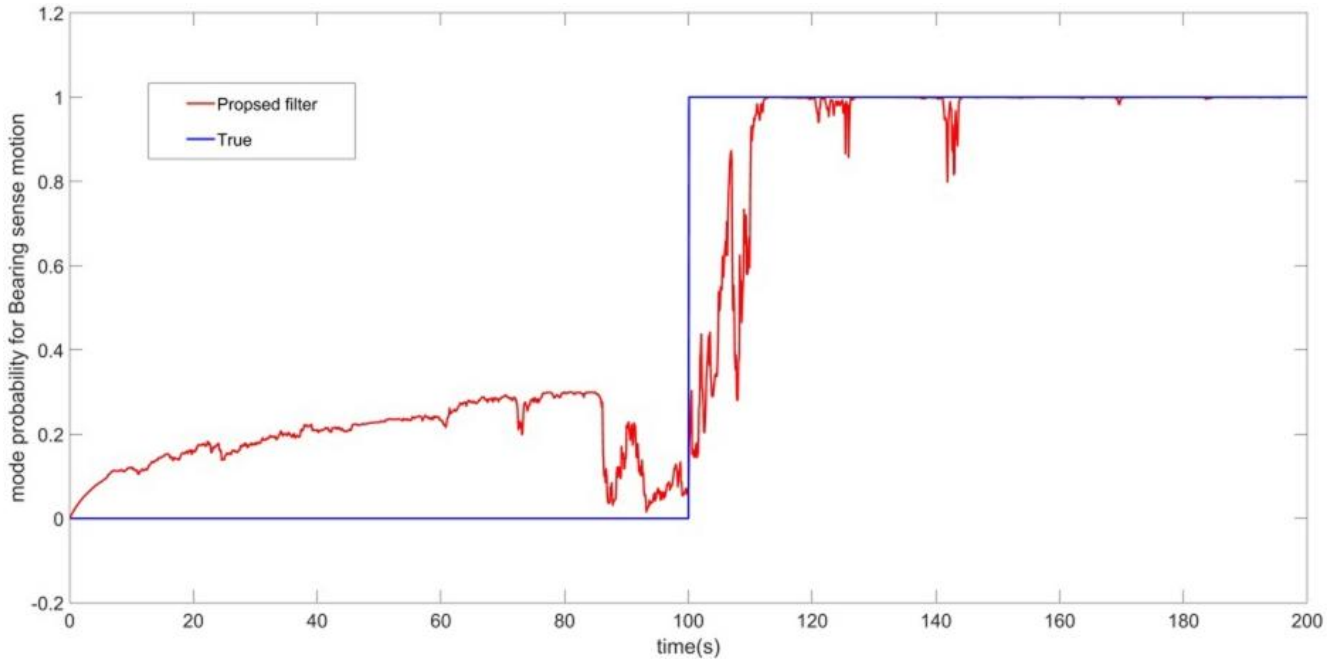


Figure 5. Mode Probabilities for Constant Velocity Models.

As shown in Figure 5, it is shown that the linear motion with constant velocity between 0 and 100 s and the bearing sense motion toward the observer between 100 and 200 s are almost tracked. In Figure 6, the performance of the proposed filter is shown by comparison with IMM-CKF, IMM-UKF and IMM-VBF. As shown in the figure, the proposed method can track the position of the real target accurately compared

to other methods.

Figure 7 shows the reduction of the range measurement error with time. It can also be seen in Figure 7 that the proposed method has the highest reduction rate of range tracking error compared to other methods. The detailed numerical data are shown in Table 1.

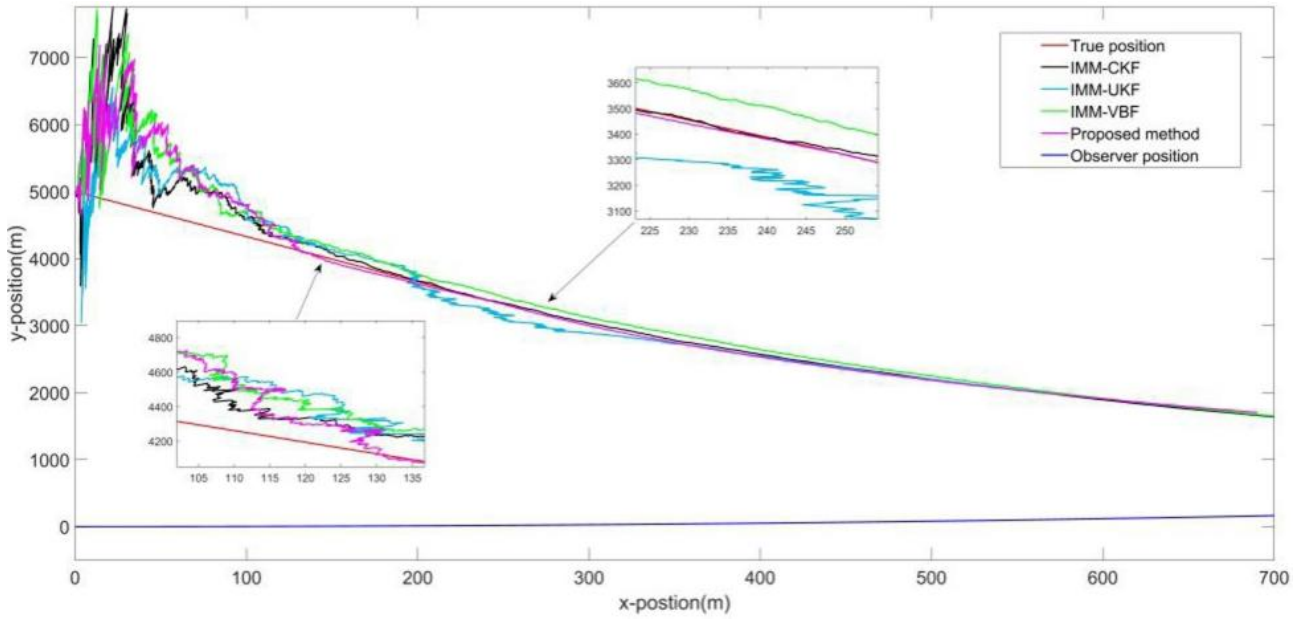


Figure 6. Comparison with Various Methods for Target Tracking.

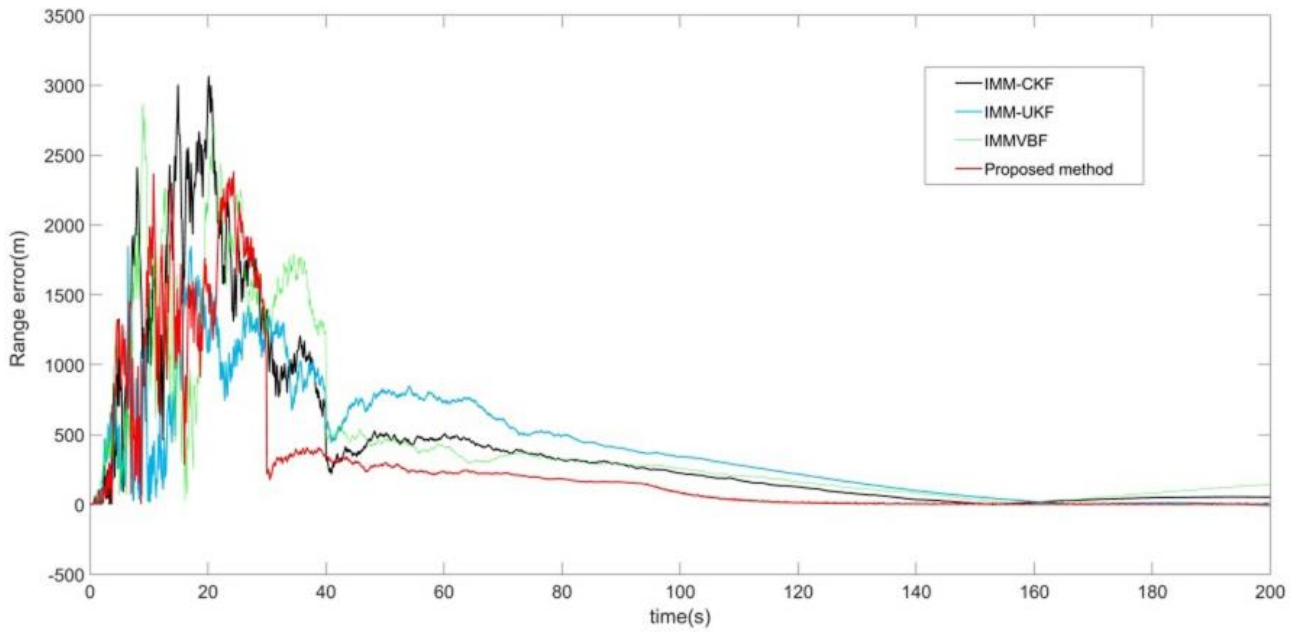


Figure 7. Comparison of Range Measurement Error of Various Methods for Target Tracking.

Table 1 shows the mean values of the RMSE with respect to the position of the measurement noise for the cases.

$$(a) \quad \varepsilon_k \sim N\left[0, \left(3 + 0.5 \cos(0.1k\pi/100)\right) \pi / 180 \sigma^2\right], (\sigma = 3^\circ)$$

$$(b) \quad \varepsilon_k \sim 0.3N\left[0, \sigma_1^2\right] + 0.7N\left[0, \sigma_2^2\right], (\sigma_1 = 3^\circ, \sigma_2 = 5^\circ)$$

$$(c) \quad \varepsilon_k \sim 0.7N\left[0, \left(3 + 0.5 \cos(0.1k\pi/100)\right) \pi / 180 \sigma_1^2\right] + 0.3N\left[0, \sigma_2^2\right], (\sigma_1 = 3^\circ, \sigma_2 = 5^\circ)$$

respectively, averaged through 500 Monte Carlo simulations.

Table 1. Root Mean Square Error of the Position.

Case	IMM-CKF	IMM-UKF	IMM-VBF	Proposed method
a	16.3853	12.5860	15.7735	6.7743
b	15.1209	10.9820	15.0893	5.0908
c	19.4562	13.8467	17.3328	7.0835

As shown in Table 1, the proposed method can see a significant reduction in RMSE compared to other methods.

Compared with the case with time-varying co-variance (a) and the case with Gaussian hybrid noise (b), the RMSE is large in the case of (c), and the case of (b) with Gaussian hybrid noise is the smallest.

Table 2 shows the mean values of the RMSE with respect to the magnitude of velocity.

Table 2. Root Mean Square Error of the Magnitude of Velocity.

Case	IMM-CKF	IMM-UKF	IMM-VBF	Proposed method
a	0.1092	0.1078	0.1001	0.0733
b	0.1352	0.9677	0.0992	0.0912
c	0.2921	0.8110	0.1191	0.1164

As shown in Table 2, also the proposed method can see a significant reduction in RMSE compared to other methods.

Simulation results show that our method is robust against measurement noise with outliers and the target tracking performance is very effective compared to other methods.

5. Conclusions

We design and validate the effectiveness of a IMM-SRAUKF for tracking of the bearing of an underwater vehicle approaching the observer only with a tactical maneuver. In particular, given the definite set of motion model of the underwater vehicle approaching the observer, the tracking performance is significantly higher when using our proposed model than when using a conventional model such as the past constant turn models or the constant speed curvature model, especially when the IMM-SRAUKF designed in this paper is very robust to the outliers which are included in the measurements. The reason for the high performance of the proposed filter, even for very severe measurement noise, is that it has the effective application of various techniques that can significantly improve the approximation accuracy in the calculation of the posterior probability density by VB

approximation in the measurement updating step of the filter design, and the new motion equation that accurately reflects the motion characteristics of the underwater vehicle approaching the observer. In general, the problem of accurately and quickly tracking the behavior of underwater vehicles with bearing only measurements is very difficult, and it is expected that in the future, it will be important to improve the target tracking performance in real environments, taking into account the effect of process noise in the equations of motion caused by more severe measurement noise, measurement noise of devices such as GPS on the observer, and measurement delay due to the speed of sound waves in the fluid.

Abbreviations

UKF	Unscented Kalman Filter
CKF	Cubature Kalman Filter
IMM	Interacting Multiple Models
2D	Two-Dimensional
PDF	Probability Density Function
VB	Variational Bayesian
UT	Unscented Transform
IMM-SRAUKF	Interacting Multi-Model Strong Robust Adaptive Unscented Kalman Filter
KL	Kullback-Leibler
DOF	Degree of Function
RMSE	Root Mean Square Errors
SRD	Sight Range of the Day
IMM-UKF	Interacting Multi-Model Unscented Kalman Filter
IMM-CKF	Interacting Multi-Model Cubature Kalman Filter
IMM-VBF	Variational Bayesian-based IMM Filter

Conflicts of Interest

The author declares no conflicts of interest.

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